

UNIT-REGULAR MODULES

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(Received 2 January 2016; accepted 12 July 2016; first published online 23 February 2017)

Abstract. In 2014, the first two authors proved an extension to modules of a theorem of Camillo and Yu that an exchange ring has stable range 1 if and only if every regular element is unit-regular. Here, we give a Morita context version of a stronger theorem. The definition of regular elements in a module goes back to Zelmanowitz in 1972, but the notion of a unit-regular element in a module is new. In this paper, we study unit-regular elements and give several characterizations of them in terms of “stable” elements and “lifting” elements. Along the way, we give natural extensions to the module case of many results about unit-regular rings. The paper concludes with a discussion of when the endomorphism ring of a unit-regular module is a unit-regular ring.

2000 *Mathematics Subject Classification.* 16E50.

A theorem of Camillo and Yu asserts that an exchange ring R has stable range 1 if and only if every regular element of R is unit-regular [4, Theorem 3]. In [5], we defined the notion of a stable module ${}_R M$ in such a way that ${}_R R$ is stable if and only if R has stable range 1. We also defined the unit-regular elements in any module M , and called M regular-stable if every regular element is unit-regular. We then proved [5, Theorem 26] that a module with the finite exchange property is stable if and only if it is regular-stable, extending the Camillo–Yu theorem (see Theorem 19).

In the present paper, we investigate the unit-regular modules where every element is unit-regular. We show that unit-regular elements in a module are characterized by several natural module analogues of unit-regular elements of a ring. Moreover, we define the stable elements in a module, we identify a lifting property for these elements, and we show that a regular element is unit-regular if and only if it is stable or has the lifting property. These results lead to several new characterizations of unit-regular modules.

Throughout this paper, rings are associative with non-zero unity, modules are left modules unless otherwise specified, and morphisms will be written on the right of their arguments. We write $\text{end}(M)$ for the ring of all endomorphisms of a module M . If K and M are modules, the notation $K \subseteq^{\oplus} M$ means that K is a direct summand of M . We always use $M_n(R)$ to stand for the ring of all $n \times n$ matrices over a ring R , we write $U = U(R)$ for the group of units of R , and $J = J(R)$ denotes the Jacobson radical of R . The ring of integers is denoted \mathbb{Z} , and the localization of \mathbb{Z} at a prime p is written $\mathbb{Z}_{(p)}$. The term “regular ring” means “von Neumann regular”. The left and right annihilators of a set X will be written as $\mathbf{l}(X)$ and $\mathbf{r}(X)$, respectively.

1. Background. If R and S are rings, and ${}_R V_S$ and ${}_S W_R$ are bimodules, we say that the 4-tuple (R, V, W, S) is a **Morita context** (**context** for short) if there exist products

$V \times W \rightarrow R$, written $(v, w) \mapsto vw$ and $W \times V \rightarrow S$, written $(w, v) \mapsto wv$ which induce bimodule morphisms $V \otimes_S W \rightarrow R$ and $W \otimes_R V \rightarrow S$ and satisfy

$$v(wv_1) = (vw)v_1 \quad \text{and} \quad w(vw_1) = (wv)w_1 \quad \text{for all } v, v_1 \in V \text{ and } w, w_1 \in W.$$

These requirements are equivalent to asking that $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is an associative ring using “matrix” operations, called the **context ring**.¹ The images VW and WV are ideals of R and S , respectively, called the **trace ideals** of the context. Morita contexts were introduced by Bass in his Oregon lectures on the Morita theorems.

Our interest here is in a module ${}_R M$. Write $E = \text{end}({}_R M)$, so that ${}_R M_E$ is a bimodule. The **dual** of M is denoted $M^* = \text{hom}({}_R M, R)$. Then M^* is a left E -module via composition of maps, and M^* becomes a right R -module as follows: Given $\lambda \in M^*$ and $r \in R$ define $\lambda r \in M^*$ by $x(\lambda r) = (x\lambda)r$ for all $x \in M$. Thus, we have two bimodules:

$${}_R M_E \quad \text{and} \quad E(M^*)_R.$$

Moreover, if $m \in M$ and $\lambda \in M^*$, we have a product $m\lambda \in R$; dually, $\lambda m \in E$ via $x(\lambda m) = (x\lambda)m$ for all $x \in M$. Hence, we have products

$$m\lambda \in R \quad \text{and} \quad \lambda m \in E,$$

and it is routine to verify that (R, M, M^*, E) is a Morita context, called the **standard context** of the module ${}_R M$. While this context is our primary interest, we will use the language of general Morita contexts to simplify and clarify the discussion. In particular, we often formulate and prove propositions in the general context, and use them in the standard context.

While many of our results hold for any Morita context, some require that the context has a property valid in any standard context. An important example involves the following observation. If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is any Morita context and $w \in W$, the right multiplication $\cdot w : {}_R V \rightarrow {}_R R$ is R -linear.² In particular, $\{\cdot w \mid w \in W\} \subseteq {}_R V^*$. We say that a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is **W -full** if $\{\cdot w \mid w \in W\} = {}_R V^*$, that is if every R -linear map $\lambda : {}_R V \rightarrow R$ has the form $\lambda = \cdot w$ for some $w \in W$. Of course, any standard context is W -full. Here is another example.

EXAMPLE 1. If a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ satisfies $WV = S$, then $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is W -full. The converse is false even if ${}_R V$ is projective.

¹We frequently abuse the notation and refer to the context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$.

²In fact, $w \mapsto \cdot w$ is an R -linear map $W_R \rightarrow V^*_R$ with kernel $\mathbf{l}_V(w)$.

Proof. Suppose $1_S = \sum w_i v_i$ in WV . Given $\lambda \in {}_R V^*$ write $v_i \lambda = a_i \in R$ for each i , and then define $w = \sum w_i a_i \in W$. Then $\lambda = \cdot w$ because, for each $v \in V$, $v\lambda = (v1_S)\lambda = [\sum (vw_i)v_i]\lambda = \sum (vw_i)(v_i\lambda) = \sum (vw_i)a_i = \sum v(w_i a_i) = v(\sum w_i a_i) = vw$. To see that the converse is false, let $S = \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $e = (1, 0) \in S$. Then consider the context $\begin{bmatrix} R & V \\ W & S \end{bmatrix} := \begin{bmatrix} eSe & eS \\ Se & S \end{bmatrix}$. Clearly, ${}_R V = {}_R R$ is torsionless and projective, and $WV = SeS = eS \neq S$. To see that the context is W -full, let $\lambda \in {}_R V^*$. Then either $\lambda = 0$ or $\lambda = 1$. If $\lambda = 0$, we have $\lambda = \cdot 0$ where $0 \in W$; if $\lambda = 1$, then $\lambda = \cdot e$ and $e \in W$. So the context is W -full. \square

A theorem of Azumaya [2] asserts that an element $\alpha \in \text{end}({}_R M)$ is regular if and only if both $M\alpha$ and $\ker(\alpha)$ are direct summands of M . The following Proposition is the context version of Azumaya’s result. Given a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, an element $v \in V$ is called **regular** if $vvv = v$ for some $w \in W$.

PROPOSITION 2. *If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is a W -full Morita context, the following are equivalent for $v \in V$:*

- (1) v is regular.
- (2) $Rv \subseteq^{\oplus} {}_R V$ and $1_R(v) \subseteq^{\oplus} {}_R R$.

Proof. (1) \Rightarrow (2). Let $vvv = v$ and assume (by passing $w \mapsto wv$) that $vvw = w$ too. Then $1_V(w) = V(1_S - vw)$ and $V(wv) = Rv$.

Hence, $V = V(wv) \oplus V(1_S - vw) = Rv \oplus 1_V(w)$. Turning to $1_R(v)$, we have $1_R(v) = R(1_R - vw)$ so $R = Rvw \oplus R(1_R - vw) = Rvw \oplus 1_R(v)$.

(2) \Rightarrow (1). Suppose that ${}_R V = Rv \oplus P$ and ${}_R R = 1_R(v) \oplus Q$. Then $Rv = 0 + Qv$ so $V = Qv \oplus P$. With this, define $\lambda : {}_R V \rightarrow R$ by $(qv + p)\lambda = q$. This is well-defined because $qv + p = 0$ implies $qv = 0$, whence $q \in Q \cap 1_R(v) = 0$. Hence, λ is R -linear, and we have $(v\lambda)v = (1_R)v = v$. By hypothesis, $\lambda = \cdot w$ for some $w \in W$ and we obtain $vvw = (vw)v = (v\lambda)v = v$. This proves (1). \square

If R is a ring, Vaserstein’s lemma [11] shows that the following are equivalent for a and b in R :

- (1) $ab + s = 1, s \in R, \Rightarrow b + xs$ is a unit for some $x \in R$.
- (2) $ab + s = 1, s \in R, \Rightarrow a + sy$ is a unit for some $y \in R$.

When these conditions hold, R is said to have stable range 1. In [5], we defined stable Morita contexts as a generalization of rings with stable range 1. The key result was the following generalization of Vaserstein’s lemma [5, Lemma 1].

LEMMA 3. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context. The following are equivalent for $w \in W$ and $v \in V$:*

- SC1. *If $wv + s = 1_S, s \in S$, there exists $v_1 \in V$ such that $(v + v_1s)W = R$.*
- SC2. *If $wv + s = 1_S, s \in S$, there exists $w_1 \in W$ such that $V(w + sw_1) = R$.*

As in [5], we call $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ a **stable context** if SC1 and SC2 hold for all $w \in W$ and $v \in V$, and we called a module ${}_R M$ **stable** if the standard context of M is stable. The terminology comes from Bass [1] because the module ${}_R R$ is stable if and only if R has stable range 1 [5, Corollary 12].

A module ${}_R M$ is called a **regular module** [13] if each $m \in M$ is **regular**, that is $(m\lambda)m = m$ for some $\lambda \in M^*$. In [5], M was called **regular-stable** if, in the standard

context for M , we only required that SC1 and SC2 hold for all *regular* elements $m \in M$ and all $\lambda \in M^*$. Furthermore, we defined the unit-regular elements in M as follows: An element $m \in {}_R M$ was called **unit-regular** if $m = (m\gamma)m$ for some *epimorphism* $\gamma \in M^*$. With this, we showed in [5, Theorem 25] that

THEOREM 4. *A module ${}_R M$ is regular-stable if and only if every regular element of M is unit-regular.*

A module M is said to have **internal cancellation (IC)** if $M = N \oplus K = L \oplus K'$ with $K \cong K'$ implies that $N \cong L$. In 1976, Ehrlich [6] showed that ${}_R M$ has IC if and only if each regular element in $\text{end}(M)$ is unit-regular. In [8], Khurana and Lam call a ring an **IC ring** if ${}_R R$ has IC. In [5, Theorem 19], we gave a new characterization of these IC rings: R is an IC ring if and only if ${}_R R$ is a regular-stable module. If in addition ${}_R M$ has the finite exchange property, then by [5, Theorem 26] we can say more: M is stable, if and only if M is regular-stable, if and only if every regular element in M is unit-regular. This extends an important theorem of Camillo and Yu [4, Theorem 3] who proved it when $M = {}_R R$. We shall return to this below.

2. Stable elements. Having defined stable Morita contexts, we now investigate stable elements. Given a context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, an element $v \in V$ is called **stable** if SC1 and SC2 hold for all $w \in W$; that is

SC1: If $wv + s = 1_S, s \in S, w \in W$, there exists $v_1 \in V$ with $(v + v_1s)W = R$.

SC2: If $wv + s = 1_S, s \in S, w \in W$, there exists $w_1 \in W$ with $V(w + sw_1) = R$.

Hence, a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is stable if and only if every $v \in V$ is stable.

Having a stable element in a Morita context has consequences.

LEMMA 5. *The following are equivalent for any Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$:*

- (1) 0_V is stable.
- (2) The context has a stable element.
- (3) $v_0w_0 = 1_R$ for some $v_0 \in V$ and $w_0 \in W$.

Proof. (1) \Rightarrow (2). This is clear.

(2) \Rightarrow (3). If $v \in V$ is stable, $0_Wv + 1_S = 1_S$ implies $V(0_W + 1_Sw_1) = R, w_1 \in W$, and (3) follows.

(3) \Rightarrow (1). If $w0_V + s = 1_S, w \in W, s \in S$, then $(0_V + v_0s)W = v_0W = R$, proving (1). □

Returning to stable elements, we have:

LEMMA 6. *Let $v \in V$ be stable in the context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$. If $wv = 1_S$ where $w \in W$, then $vw = 1_R$.*

Proof. If $wv = 1_S$, then $wv + 0_S = 1_S$. Since v is stable, $(v + v_10_S)W = R$ by SC1, say $vv_1 = 1_R, v_1 \in W$. Then $w_1 = 1_Sw_1 = (wv)w_1 = w(vw_1) = w1_R = w$, so $vw = vv_1 = 1_R$, as required. □

COROLLARY 7. *If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is a stable context, then $wv = 1_S$ implies $vw = 1_R$.*

Lemma 6 provides a “directly finite” condition in a stable context. In particular, if a ring R has stable range 1 then R is directly finite because the context $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ is stable (by [5, Corollary 3]).

Before proceeding, we make several definitions for any Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$:
 $v \in V$ is called $\begin{cases} \text{left invertible} & \text{if } wv = 1_S \text{ for some } w \in W, \text{ equivalently if } Wv = S. \\ \text{right invertible} & \text{if } vw = 1_R \text{ for some } w \in W, \text{ equivalently if } vW = R. \end{cases}$
 $w \in W$ is called $\begin{cases} \text{left invertible} & \text{if } vw = 1_R \text{ for some } v \in V, \text{ equivalently if } Vw = R. \\ \text{right invertible} & \text{if } wv = 1_S \text{ for some } v \in V, \text{ equivalently if } wV = S. \end{cases}$

Note that the zero element in V or W is neither left nor right invertible because we are assuming that R and S are non-zero rings.

The following lemma [5, Lemma 21] provides a class of stable elements in any context. We use it repeatedly so we include a short, alternate proof for completeness.

LEMMA 8. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context. If $v \in V$ has the form $v = v_0f$ where $v_0 \in V$ is right invertible and $f^2 = f \in S$, then v is stable.*

Proof. Let $wv + s = 1_S$; we show that $v + v_0(1_S - f)s$ is right invertible (giving SCI). Compute

$$(1_S - f)wv_0f + (1_S - f)s = (1_S - f)(wv_0f + s) = (1_S - f)(wv + s) = 1_S - f.$$

Now observe that

$$f + (1_S - f)s = f + [(1_S - f) - (1_S - f)wv_0f] = 1_S - (1_S - f)wv_0f$$

is a unit in S . It follows that

$$v + v_0(1_S - f)s = v_0f + v_0(1_S - f)s = v_0[f + (1_S - f)s]$$

is right invertible because v_0 is right invertible. This is what we wanted. □

Taking $f = 1_S$ in Lemma 8 immediately leads to the following corollary.

COROLLARY 9. *In any Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, every right invertible element of V is stable.*

EXAMPLE 10. A left invertible element in a context need not be stable.

Proof. Let R be a ring with an element v such that $wv = 1_R$ but $vw \neq 1_R$ (see Example 13 below). Then $v \in {}_R R$ is left invertible in the standard context $\begin{bmatrix} R & {}_R R \\ R^* & \text{end}({}_R R) \end{bmatrix}$. But v is not stable by Lemma 6. □

The converse of Lemma 8 is not true.

EXAMPLE 11. If $R = \mathbb{Z}_{(p)}$ is the localization of \mathbb{Z} at a prime p , then the standard context $\begin{bmatrix} R & {}_R R \\ {}_R R^* & \text{end}({}_R R) \end{bmatrix}$ is stable because R has stable range 1 (it is local). In particular, $p \in {}_R R$ is stable, but p is not of the form $p = v_0f$ where v_0 is right invertible and $f^2 = f$ because R is commutative and $p \in J(R)$.

The stable elements in a context are “translation invariant” in the following sense.

LEMMA 12. *Given a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, let $a \in U(R)$ and $b \in U(S)$. If $v \in V$ is stable, so are av and vb .*

Proof. Let $w(av) + s = 1_S$ where $w \in W$ and $s \in S$. Then $(wa)v + s = 1_S$ so, as v is stable, $V(wa + sw_1) = R$ for some $w_1 \in W$. Right multiplication by a^{-1} gives $V(w + s(w_1a^{-1})) = Ra^{-1} = R$, proving that av is stable.

Turning to vb , let $w(vb) + s = 1_S$, $w \in W$, $s \in S$. If we conjugate by b , we obtain $(bw)v + bsb^{-1} = 1_S$. Since v is stable, we have $(v + v_1(bsb^{-1}))W = R$ for some $v_1 \in V$. But then

$$R = (v + v_1(bsb^{-1}))W = (vb + v_1(bs))b^{-1}W = (vb + v_1(bs))W,$$

proving that vb is stable. □

QUESTION 1. If u and v are stable in ${}_R R$, is uv also stable? [The answer is “yes” if the ring R has stable range 1 [5, Lemma 17].]

3. Unit-regular elements. If ${}_R M$ is a module, an element $m \in M$ is called unit-regular [5] if $(m\lambda)m = m$ for some epimorphism $\lambda \in M^*$. More generally:

DEFINITION. If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is any Morita context, an element $v \in V$ is called **unit-regular** if $v w v = v$ where w is left invertible (that is $V w = R$).

It is not sufficient that w is right invertible, and these notions differ in ${}_R R$ and the ring R :

EXAMPLE 13. Let $R = \text{end}(V)$ where ${}_D V$ is a vector space on basis $\{v_0, v_1, v_2, \dots\}$ over a division ring D , and let $\alpha \in R$ be the shift operator defined by $v_i \alpha = v_{i+1}$ for each i .

- (1) In the ring R , $\alpha = \alpha \beta \alpha$ where β has a left inverse, but α is not unit-regular.
- (2) In the module ${}_R R$, α is unit-regular but $\alpha \gamma \alpha \neq \alpha$ for all right invertible $\gamma \in {}_R R$.

Proof. Define $\beta \in R$ by $v_0 \beta = 0$ and $v_i \beta = v_{i-1}$ for all $i \geq 1$. Then $\alpha \beta = 1_V$ so $\alpha \beta \alpha = \alpha$ and β has a left inverse in R .

- (1) Clearly, $\alpha = \alpha \beta \alpha$ is regular in the ring R , and β has a left inverse in R . But $\ker(\alpha) = 0$ and $V \alpha = F v_1 \oplus F v_2 \oplus \dots$, so $V / V \alpha \cong F \not\cong \ker(\alpha)$. Hence, α is not unit-regular as an element of the ring R by a result of Ehrlich [6, Theorem 1]; see Proposition 27 below.
- (2) Now view R as the left module ${}_R R$, so α is unit-regular in ${}_R R$ because $\alpha \beta \alpha = \alpha$ and β is left invertible. Suppose that $\alpha \gamma \alpha = \alpha$ where $\gamma \delta = 1_V$. Right multiplying $\alpha \gamma \alpha = \alpha$ by β gives $\alpha \gamma = 1_V$ so $\alpha = \delta = \gamma^{-1}$. But then $\gamma \alpha = \gamma \delta = 1_V$ so α is onto, a contradiction. □

To relate unit-regularity to stability, we need the following technical result.

LEMMA 14. Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context, and let $v \in V$ be stable. Suppose that $v w v = v$ for some particular $w \in W$. Then there exists $v_1 \in V$ such that

$$v = v_1(wv) = (vw)v_1 \text{ and } v_1 \text{ is right invertible.}$$

Proof. We may assume that $w v w = w$ and $v w v = v$ both hold. [Replace w by $w' = w v w$.]

Write $s = 1_S - w v \in S$, that is $w v + s = 1_S$. As v is stable, we have $(v + v_2 s)W = R$ for some $v_2 \in V$ by SC1. Define $v' = v + v_2 s \in V$, so v' is right invertible. Then

$$v' w = v w \text{ (because } s w = 0), \text{ so } w v' w = w v w = w.$$

Write $a = 1_R - 2v'w \in R$ and $b = 1_S - wv - wv' \in S$. Then $a^2 = 1_R$ because $(v'w)^2 = v'w$; and

$$b^2 = 1_S - 2(wv + wv') + (wv wv + wv wv' + wv' wv + wv' wv') = 1_S.$$

Finally, define $v_1 = av'b \in V$. Then v_1 is right invertible because we have $v_1 W = av'bW = av'W = aR = R$. Moreover, one verifies that $-w = wa = bw$, so we obtain

$$wv_1 = (wa)v'b = (-w)v'b = -(wv')b = -wv' + wv'wv + wv'wv' = wv,$$

and

$$v_1 w = (av')bw = av'(-w) = -a(v'w) = -[v'w - 2(v'w)^2] = v'w = wv.$$

These give $(wv)v_1 = v(wv_1) = v(wv) = v$ and $v_1(wv) = (v_1 w)v = (wv)v = v$, respectively, as required. □

The following theorem characterizes the unit-regular elements in terms of stability. The last two conditions generalize the fact that an element of a ring is unit-regular if and only if it is the product of a unit and an idempotent (in either order).

THEOREM 15. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context. The following are equivalent for $v \in V$:*

- (1) v is unit-regular.
- (2) v is regular and $(wv)^2 = wv \in S$ for some left invertible $w \in W$.
- (3) v is regular and $v = v_0 f$ for some right invertible $v_0 \in V$ and $f^2 = f \in S$.
- (4) v is regular and stable.
- (5) There exists $w \in W$ with $v = vwv$ and $v = (vw)v_1 = v_1(wv)$ where $v_1 \in V$ is right invertible.
- (6) There exists $w \in W$ with $v = vwv$ and $v = (vw)v_1$ where $v_1 \in V$ is right invertible.

Proof. (1) \Rightarrow (2). By (1), let $v = vwv$ for some left invertible $w \in W$. Then $(wv)^2 = w(vwv) = wv$.

(2) \Rightarrow (3). If w is as in (2), set $f = wv \in S$. If $v_0 w = 1_R$ for $v_0 \in V$, then v_0 is right invertible and $v_0 f = v_0(wv) = (v_0 w)v = 1_R v = v$, as required.

(3) \Rightarrow (4). This follows from Lemma 8.

(4) \Rightarrow (5). This follows from Lemma 14.

(5) \Rightarrow (6). This is clear.

(6) \Rightarrow (1). Choose $w \in W$ and $v_1 \in V$ as in (6). As v_1 is right invertible, let $v_1 w_1 = 1_R$, $w_1 \in W$. Then w_1 is left invertible and $v w_1 = (v w v_1) w_1 = v w$, so $v w_1 v = v w v = v$, proving (1). □

EXAMPLE 16. Condition (6) implies that every unit-regular element v has the form $v = ev_1$ where $e^2 = e \in R$ and v_1 is right invertible. However, if v_1 is left invertible, then v need not be unit-regular.

Proof. Let R , α and β be as in Example 13. In the context $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$, we have $\beta = 1\beta$ where $1^2 = 1$ in R and β has a left inverse (in fact $\alpha\beta = 1$). But β is not unit-regular in ${}_R R$. For if $\beta\gamma\beta = \beta$ with γ left invertible, then $\gamma\beta = 1$ (because $\alpha\beta = 1$), so γ is invertible. It follows that $\beta = \gamma^{-1}$ is invertible, a contradiction. □

We shall need the following corollary of Theorem 15.

THEOREM 17. *Every unit-regular element in a Morita context is stable. The converse is not true.*

Proof. The first statement is by Theorem 15(3). As to the converse, if $R = \mathbb{Z}_{(p)}$ for a prime p then the standard context $\begin{bmatrix} R & R \\ R^* & \text{end}(R) \end{bmatrix}$ is stable (see Example 11), so $p \in {}_R R$ is stable. But p is not unit-regular because $p \in J(R)$. □

EXAMPLE 18. If ${}_Z M = \mathbb{Z} \oplus \mathbb{Z}$, the standard context $\begin{bmatrix} \mathbb{Z} & M \\ M^* & \text{end}(M) \end{bmatrix}$ is stable by [5, Example 13], but \mathbb{Z} does not have stable range 1.

In 1995, Camillo and Yu proved that an exchange ring R has stable range 1 if and only if every regular element in R is unit-regular [4, Theorem 3]. The next theorem is a context version of a stronger theorem.

THEOREM 19. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context. If S is an exchange ring, the following are equivalent:³*

- (1) *The context is stable (that is every element of V is stable).*
- (2) *Every regular element of V is unit-regular.*
- (3) *Every regular element of V is stable.*
- (4) *If $wv + f = 1_S$, $w \in W$, $v \in V$, $f^2 = f \in S$, then $V(w + fw_1) = R$ for some $w_1 \in W$.*
- (5) *If $wv + f = 1_S$, $w \in W$, $v \in V$, $f^2 = f \in S$, then $(v + v_1f)W = R$ for some $v_1 \in V$.*

Proof. (1) \Rightarrow (2) \Rightarrow (3) by Theorem 15.

(3) \Rightarrow (4). Let $wv + f = 1_S$ as in (4). Then $wv = 1_S - f$ is an idempotent, so $w(vwv) + f = 1_S$. But vwv is regular [in fact $(vwv)w(vwv) = vwv$], so vwv is stable by (3). Thus, Lemma 3 shows that $V(w + fw_1) = R$ for some $w_1 \in W$. This proves (4).

(4) \Rightarrow (5). This is by Lemma 3.

(5) \Rightarrow (1). If $v \in V$, we must show that v is stable. So, suppose that $wv + s = 1_S$ where $w \in W$ and $s \in S$; we prove that $(v + v_1s)W = R$ for some $v_1 \in V$.

As S is an exchange ring, [9, Theorem 2.1] asserts that:

There exists $f^2 = f \in S$ such that $f \in Ss$ and $1_S - f \in S(1_S - s)$.

Then $1_S - f \in S(1_S - s) = Swv \subseteq Wv$, say $1_S - f = w_1v$ for some $w_1 \in W$. Thus, $w_1v + f = 1_S$, so (5) implies that $(v + v_2f)W = R$ for some $v_2 \in V$. But $f \in Ss$, say $f = s_2s$ with $s_2 \in S$, so $v_2f = v_2(s_2s) = (v_2s_2)s = v_1s$ where $v_1 = v_2s_2 \in V$. Thus, $(v + v_1s)W = R$, as required. □

If ${}_R M$ has the finite exchange property, then $\text{end}(M)$ is an exchange ring by [12]. Hence, the standard context $\begin{bmatrix} R & M \\ M^* & \text{end}(M) \end{bmatrix}$ satisfies the hypotheses of Theorem 19, and we obtain immediately:

THEOREM 20. *Given ${}_R M$, write $S = \text{end}(M)$. If ${}_R M$ has the finite exchange property, the following are equivalent:*

- (1) *M is stable.*
- (2) *Every regular element of M is stable.*

³The hypothesis that S is an exchange ring is only needed for (5) \Rightarrow (1).

- (3) Every regular element of M is unit-regular.
- (4) If $\lambda m + \theta = 1_S$, $\lambda \in M^*$, $m \in M$, $\theta^2 = \theta \in S$, then $M(\lambda + \theta\lambda_1) = R$ for some $\lambda_1 \in M^*$.
- (5) If $\lambda m + \theta = 1_S$, $\lambda \in M^*$, $m \in M$, $\theta^2 = \theta \in S$, then $(m + m_1\theta)M^* = R$ for some $m_1 \in M$.

REMARK. In Theorem 20, (1) \Leftrightarrow (2) is a module version of the original Camillo–Yu theorem, and (1) \Leftrightarrow (4) extends Lemma 2 in the same paper.

The following result strengthens conditions (5) and (6) in Theorem 15, but requires that the context satisfies a weaker condition than being W -full. A Morita context $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix} \right]$ is called W -**epi-full** if every epic R -linear map $\lambda : {}_R V \rightarrow R$ has the form $\lambda = \cdot w$ for some $w \in W$. [Note $\lambda : {}_R V \rightarrow R$ is R -epic if and only if $\phi\lambda = 1_R$ for some $\phi : R \rightarrow {}_R V$.]

QUESTION 2. Is every stable Morita context W -epi-full?

THEOREM 21. Let $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix} \right]$ be a W -epi-full Morita context. The following are equivalent for $v \in V$:

- (1) v is unit-regular.
- (2) There exists $w \in W$ such that $v = (vw)v_1 = v_1(wv)$ for some right invertible $v_1 \in V$.
- (3) There exists $w \in W$ such that $v = v_1w$ and $(vw)v_1 = v_1(wv)$ for some right invertible $v_1 \in V$.

Proof. (1) \Rightarrow (2) is Lemma 14, and (2) \Rightarrow (3) is clear.

(3) \Rightarrow (1). Given (3), write $vw = e$ and $wv = f$ so $e^2 = e \in R$, $f^2 = f \in S$ and $ev = v = vf$. By (3), let $ev_1 = v_1f$ where v_1 is right invertible, say $v_1w_1 = 1_R$ for $w_1 \in W$. We have decompositions

$$V = Vf \oplus V(1_S - f) \quad \text{and} \quad R = Re \oplus R(1_R - e)$$

so we can define two R -linear maps $\lambda : V \rightarrow R$ and $\phi : R \rightarrow V$ componentwise as follows:

$$\begin{aligned} (pf + q(1_S - f))\lambda &= pfw + q(1_S - f)w_1(1_R - e) & p, q \in V, \\ (re + t(1_R - e))\phi &= rev + t(1_R - e)v_1(1_S - f) & r, t \in R. \end{aligned}$$

We claim that $\phi\lambda = 1_R$. As $ev = vf$, we have

$$(re)\phi\lambda = (rev)\lambda = (rvf)\lambda = (rv)fw = re^2 = re.$$

For the second component, observe first that

$$v_1(1_S - f)w_1 = v_1w_1 - (v_1f)w_1 = 1_R - (ev_1)w_1 = 1_R - e.$$

Then

$$[t(1_R - e)]\phi\lambda = [t(1_R - e)v_1(1_S - f)]\lambda = [q_1(1_S - f)]\lambda,$$

where $q_1 = t(1_R - e)v_1$. Hence,

$$[t(1_R - e)]\phi\lambda = t(1_R - e)v_1[(1_S - f)w_1(1_R - e)] = t(1_R - e)^3 = t(1_R - e).$$

It follows that $\phi\lambda = 1_R$.

In particular, the map $\lambda : V \rightarrow R$ is epic so, by hypothesis, $\lambda = \cdot w_0$ for some $w_0 \in W$. Then w_0 is left invertible because $1_R = (1_R)\phi\lambda = (1_R\phi)w_0$, so it remains

to show that $vw_0v = v$, that is $(v\lambda)v = v$. But $v = vf$ so $v\lambda = (vf)\lambda = vfw$ by the definition of λ . Hence, $(v\lambda)v = vfwv = v(wv)wv = v$, as required. \square

4. The lifting property. Let L be a left ideal of a ring R . We say that $a \in R$ is **left invertible modulo L** if $ba - 1 \in L$ for some $b \in R$, and that **left invertible elements lift modulo L** if, whenever $a \in R$ is left invertible modulo L , there exists a left invertible $u \in R$ such that $a - u \in L$. It is proved in [9, Theorem 2.1] that a ring R is an exchange ring if and only if idempotents lift modulo every left ideal of R . Here is an analogous description of rings with stable range 1; the equivalence (1) \Leftrightarrow (3) was proved by F. Siddique [7].

THEOREM 22. *The following are equivalent for a ring R :*

- (1) R has stable range 1.
- (2) Left invertible elements lift modulo every left ideal of R .
- (3) Left invertible elements lift modulo every principal left ideal of R .
- (4) The left–right analogues of (2) and (3).

Proof. As (1) is left–right symmetric and (2) \Rightarrow (3) is obvious, we only prove (1) \Rightarrow (2) and (3) \Rightarrow (1).

(1) \Rightarrow (2). If L is a left ideal of R , let $ba - 1 \in L$ where $b, a \in R$, say $ba - 1 = l \in L$. Then $ba - l = 1$ so, as a is stable by (1), $a - xl := u$ is a unit for some $x \in R$. Since $a - u = xl \in L$, (2) follows.

(3) \Rightarrow (1). Assume that $ba + s = 1$ in R . Then $ba - 1 \in Rs$ so, by (3), $a - u \in Rs$, u left invertible. If $a - u = ys$, $y \in R$, then $a - ys = u$; we show that u is a unit by proving that R is directly finite.

So let $pq = 1$ in R ; we must show that $qp = 1$. We have $qp - 1 \in R(qp - 1)$ so, by (3), $p - u \in R(qp - 1)$ where u is left invertible. But then $(p - u)q = 0$, that is $uq = 1$. As u is left invertible, it follows that u is a unit. Hence, $q = u^{-1}$ is a unit, and finally $p = q^{-1}$ is a unit. Thus, $qp = 1$. \square

REMARK 1. The proof of (3) \Rightarrow (1) in Theorem 22 shows that a ring R is directly finite if left invertible elements lift modulo every principal left summand Re where $e^2 = e \in R$.

We are going to show that every stable element in a Morita context has a similar “lifting” property. The following variation on Theorem 22 motivates our definition.

PROPOSITION 23. *The following are equivalent for a ring R :*

- (1) R has stable range 1.
- (2) Left invertible elements lift to right invertible elements modulo every left ideal of R .

Proof. (1) \Rightarrow (2). The proof of (1) \Rightarrow (2) in Theorem 22 goes through.

(2) \Rightarrow (1). Assume that $ba + s = 1$ in R . Then $ba - 1 \in Rs$ so, by (2), $a - v \in Rs$ where v is right invertible in R . Write $a - v = ys$, $y \in R$, so that $a - ys = v$ is right invertible. Again (1) follows if we can show that R is directly finite. To this end, assume $pq = 1$ in R . Then $pq - 1 \in \{0\}$ so, by (2), $q - u \in \{0\}$ for some right invertible $u \in R$. Hence, $q = u$ is right invertible and it follows that q is a unit (since $pq = 1$). Hence, $qp = 1$, as required. \square

DEFINITION. Given a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, we say that an element $v \in V$ is **lifting** if the following implication holds for all $s \in S$:

If $V(wv - 1_S) \subseteq V_s, w \in W$, then $v - v_0 \in V_s$ for some right invertible $v_0 \in V$.

And $v \in V$ is called **summand-lifting** if this condition holds only for $s \in S$ such that $V_s \subseteq^{\oplus} {}_R V$.

THEOREM 24. *In a Morita context $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$, an element $v \in V$ is stable if and only if it is lifting.*

Proof. Let $v \in V$ be stable, and suppose $V(wv - 1_S) \subseteq V_s, s \in S, w \in W$. Write $q = wv - 1_S$ so $wv - q = 1_S$. As v is stable, there exists $v_1 \in V$ with the property that $(v - v_1q)W = R$. Set $v_0 = v - v_1q$, so that $v_0 \in V$ is right invertible. Furthermore, $v - v_0 = v_1q \in V(wv - 1_S) \subseteq V_s$, as desired.

Conversely, assume that $v \in V$ is lifting, and write $wv + s = 1_S, s \in S, w \in W$. Then $wv - 1_S = -s$, and so $V(wv - 1_S) \subseteq V_s$. By hypothesis, $v - v_0 \in V_s$ for some right invertible $v_0 \in V$, say $v - v_0 = v_1s, v_1 \in V$. Therefore, $v - v_1s = v_0$ is right invertible, that is $(v - v_1s)W = R$. Consequently, v is stable. □

THEOREM 25. *Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be any Morita context. The following are equivalent for $v \in V$:*

- (1) v is unit-regular.
- (2) v is regular and lifting in ${}_R V$.
- (3) v is regular and summand-lifting in ${}_R V$.
- (4) There exists $w \in W$ and a right invertible $v_0 \in V$ with the property that $vwv = v$ and $v - v_0 \in \perp_V(w)$.

Proof. (1) \Rightarrow (2). By (1), v is stable in ${}_R V$ by Theorem 17, and so v is lifting by Theorem 24.

(2) \Rightarrow (3). This is clear.

(3) \Rightarrow (4). Let $vw_0v = v$ where $w_0 \in W$. Define $w = w_0vw_0 \in W$, so that $vwv = v$ and $wvw = w$.

Note that $vw = (vw_0)^2 = vw_0$ and $wv = (w_0v)^2 = w_0v$.

Write $s = 1_S - w_0v \in S$. Observe first that $V_s \subseteq^{\oplus} V$ because w_0v is an idempotent. Moreover, $V(wv - 1_S) = V(w_0v - 1_S) \subseteq V_s$, so condition (3) provides a right invertible $v_0 \in V$ such that $v - v_0 \in V_s$. Thus $(v - v_0)w \subseteq V_s w = V(w - w_0vw) = 0$, as desired.

(4) \Rightarrow (1). This follows from Theorem 15(3). □

Let $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context. Then Theorems 17 and 24 give the following for $v \in {}_R V$:

$$v \text{ is unit-regular} \quad \Rightarrow \quad v \text{ is stable} \quad \Leftrightarrow \quad v \text{ is lifting.}$$

These properties are all equivalent for regular elements v by Theorem 15; the converse of the first implication is false in general by Theorem 17.

5. Unit-regular endomorphisms. If $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is a Morita context, the multiplication map $s \mapsto \cdot s$ is a ring morphism $S \rightarrow \text{end}({}_R V)$ with kernel $\text{r}_S(V)$ and image $\{\cdot s \mid s \in S\}$.

Hence, if b is a unit in S , then $\cdot b : {}_R V \rightarrow {}_R V$ is an R -isomorphism. A context $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix} \right]$ is called S -full if $\text{end}({}_R V) = \{\cdot s \mid s \in S\}$, and we say that the context is S -iso-full if every R -isomorphism $\sigma : {}_R V \rightarrow {}_R V$ has the form $\sigma = \cdot b$ for some unit $b \in S$.

THEOREM 26. *Let $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix} \right]$ be an S -iso-full Morita context in which V_S is faithful. Then the following are equivalent for $s \in S$:⁴*

- (1) s is unit-regular in the ring S .
- (2) There is a unit $b \in S$ such that $V = V_S \oplus [1_V(s)]b$.
- (3) V_S and $1_V(s)$ are both direct summands of ${}_R V$ and $1_V(s) \cong V/V_S$.

Proof. (1) \Rightarrow (2). Let $sus = s$ where u is a unit in S . Then

$$1_V(s) = V(1_S - su) = Vu^{-1}(1_S - bs)u = V(1_S - bs)u,$$

so $V(1_S - us) = [1_V(s)]u^{-1}$. But $V_S = Vus$, so (2) follows because

$$V = Vus \oplus V(1_S - us) = V_S \oplus [1_V(s)]u^{-1}.$$

(2) \Rightarrow (3). By (2), let $V = V_S \oplus [1_V(s)]b$, b a unit in S . Then $V = Vb^{-1} = Vsb^{-1} \oplus 1_V(s)$, so both V_S and $1_V(s)$ are direct summands. Finally, $V/V_S \cong [1_V(s)]b \cong 1_V(s)$, proving (3).

(3) \Rightarrow (1). By (3), let $V = V_S \oplus K = 1_V(s) \oplus N$, so $K \cong V/V_S \cong 1_V(s)$. Let $\gamma : K \rightarrow 1_V(s)$ be an R -isomorphism. We have $V_S = [1_V(s) \oplus N]s = Ns$, so $V = Ns \oplus K$. Using this, define

$$\sigma : V = Ns \oplus K \rightarrow V \quad \text{by} \quad (ns + k)\sigma = n + k\gamma, \quad \text{where } n \in N \text{ and } k \in K.$$

This is well defined because $V_S \cap K = 0 = N \cap 1_V(s)$, and we claim that σ is an isomorphism. Indeed, σ is monic because $\ker(\sigma) = N \cap K\gamma = 0$, and σ is epic because $V\sigma = N + K\gamma = N + 1_V(s) = V$.

So, by the S -iso-full hypothesis, $\sigma = \cdot b$ for some $b \in U(S)$. Hence, to prove (1), it suffices to show that $sbs = s$. Since V_S is faithful, it is enough to show that $v(sbs) = vs$ for all $v \in V$. As $V = 1_V(s) \oplus N$, write $v = v_0 + n$ where $v_0s = 0$ and $n \in N$. Hence,

$$v(sb) = (vs)b = (ns)b = (ns)\sigma = n.$$

Finally, $v(sbs) = [v(sb)]s = ns = vs$, as required. □

If ${}_R M$ is any module, the hypotheses of Theorem 26 are satisfied for the standard context, so we obtain

PROPOSITION 27. *Given a module ${}_R M$, the following are equivalent for $\alpha \in \text{end}(M)$:*

- (1) α is unit-regular.
- (2) There is an automorphism $\sigma : M \rightarrow M$ such that $M = M\alpha \oplus (\ker \alpha)\sigma$.
- (3) $M\alpha$ and $\ker(\alpha)$ are both direct summands of M and $\ker(\alpha) \cong M/M\alpha$.

The equivalence of (1) and (3) in Proposition 27 is due to Ehrlich [6]. Of course, $M\alpha$ and $\ker(\alpha)$ are both direct summands if and only if α is regular in $\text{end}(M)$ by Proposition 2.

The following result appears in [10, Lemma 1].

⁴The hypotheses that the context is S -iso-full and that V_R is faithful are only used in (3) \Rightarrow (1).

LEMMA 28. *The following are equivalent for $\alpha \in \text{end}(M)$ where $M = {}_R M$ is a module.*

- (1) $M/M\alpha \cong \ker(\alpha)$.
- (2) $M\alpha = \ker(\beta)$ and $\ker(\alpha) = M\beta$ for some $\beta \in \text{end}(M)$.
- (3) $M\alpha = \ker(\beta)$ and $\ker(\alpha) \cong M\beta$ for some $\beta \in \text{end}(M)$.

Given a module $M = {}_R M$, an endomorphism $\alpha \in \text{end}(M)$ is called **morphic** [10] if the conditions in Lemma 28 are satisfied.

COROLLARY 29. *$\alpha \in \text{end}(M)$ is unit-regular if and only if it is regular and morphic.*

EXAMPLE 30. Let ${}_D V$ be vector space over a division ring D on basis $\{v_0, v_1, v_2, \dots\}$. Then ${}_D V$ is unit-regular but not morphic, and $\text{end}({}_D V)$ is not a unit-regular ring.

Proof. Let $v \in V = {}_D V$. As V is regular (by Proposition 2), let $v = (v\lambda)v$ where $\lambda \in V^*$. If $v \neq 0$, then $\lambda \neq 0$ so $V\lambda = D$, that is λ is epic. Hence, ${}_D V$ is a unit-regular module. If $\alpha : V \rightarrow V$ is the shift operator ($v_i\alpha = v_{i+1}$), we saw in Example 13 that α is not unit-regular. Moreover, $V/V\alpha \cong Dv_0$ and $\ker(\alpha) = 0$, so V is not a morphic module. \square

A ring R is called left morphic (respectively regular, unit-regular) if every element has the corresponding property, regarded as an element of $\text{end}_R R$ via right multiplication. Camillo and Khurana [3] have given the following characterization of unit-regular rings.

THEOREM. *A ring R is unit-regular if and only if every element $a \in R$ can be written as $e + u$ where $e^2 = e$, $u^{-1} \in R$ and $aR \cap eR = 0$.*

Our final result is to extend the Camillo–Khurana theorem:

THEOREM 31. *Let $M = {}_R M$ be a module and write $E = \text{end } M$. Assume that M is morphic and quasi-projective. Let $\theta \in E$ be such that $M\theta + \ker(\theta)$ and $M\theta \cap \ker(\theta)$ are direct summands of M . Then θ is unit-regular if and only if $\theta = \pi + \sigma$ where $\pi^2 = \pi \in E$, σ is a unit in E , and $M\theta \cap M\pi = 0$.*

Proof. If the condition holds, then

$$\pi\sigma^{-1}\theta = \pi\sigma^{-1}(\pi + \sigma) = \pi\sigma^{-1}\pi + \pi = (\pi\sigma^{-1} + 1)\pi$$

so $m\pi\sigma^{-1}\theta \in M\theta \cap M\pi = 0$ for every $m \in M$. It follows that $\pi\sigma^{-1}\theta = 0$, and hence that $\theta\sigma^{-1}\theta = (\pi + \sigma)\sigma^{-1}\theta = \pi\sigma^{-1}\theta + \theta = \theta$. Thus, θ is unit-regular.

For the converse, write $P = M\theta + \ker(\theta)$ and $K = M\theta \cap \ker(\theta)$ for convenience, and let $P \oplus Y = M$ and $K \oplus Z = M$. We have $M\theta \subseteq^{\oplus} P$ because $M\theta \subseteq^{\oplus} M$ (as θ is unit-regular). Since P is quasi-projective (being a direct summand of M), it follows from [9, Lemma 2.8] that there exists ${}_R X \subseteq \ker(\theta)$ such that $P = M\theta \oplus X$. In particular, $M = M\theta \oplus X \oplus Y$. Moreover, since $X \subseteq \ker(\theta) \subseteq X \oplus M\theta$ the modular law gives $\ker(\theta) = X \oplus K$. Since M is morphic, we obtain

$$X \oplus K = \ker(\theta) \cong M/M\theta \cong X \oplus Y,$$

so $K \cong Y$ because M has IC by [10, Corollary 47]. So let

$$\phi : K \rightarrow Y \quad \text{and} \quad \eta : X \oplus K \rightarrow X \oplus Y$$

be isomorphisms. Since $K \subseteq M\theta$ and $K \subseteq^{\oplus} M$, we have $M\theta = K \oplus L$ where ${}_R L \subseteq M\theta$. Thus, we obtain

$$M = K \oplus L \oplus X \oplus Y,$$

and so we can define ν and ω in $\text{end}_R M$ as follows:

$$\begin{array}{ccccccccc} M & = & X \oplus Y & \oplus & K & \oplus & L & & & M & = & X \oplus K \oplus Y & \oplus & L \\ \nu \downarrow & & \eta^{-1} \downarrow & & \phi \downarrow & & 0 \downarrow & & & \text{and} & \omega \downarrow & & \eta \downarrow & & 1_Y \downarrow & & 0 \downarrow & & . \\ M & = & X \oplus K \oplus Y & \oplus & L & & & & & M & \supseteq & X \oplus Y & + & Y & + & L \end{array}$$

Then we have $\omega\nu\omega = \omega$. Indeed:

$$\begin{aligned} (y)\omega\nu\omega &= (y)\nu\omega = (y\eta^{-1})\omega = (y\eta^{-1})\eta = y = (y)\omega \quad \text{for all } y \in Y, \\ (x)\omega\nu\omega &= (x\eta)\nu\omega = (x\eta\eta^{-1})\omega = (x)\omega \quad \text{for all } x \in X, \\ (k+z)\omega\nu\omega &= (k)\omega\nu\omega = (k\eta)\nu\omega = (k\eta\eta^{-1})\omega = (k)\omega = (z+k)\omega \quad \text{for all } z \in L \text{ and } k \in K. \end{aligned}$$

Hence, $\pi = \nu\omega$ is an idempotent in E . Also, $M\pi \cap M\theta = 0$ because $M\pi \subseteq X \oplus Y$:

$$M\pi = M\nu\omega \subseteq (X \oplus K \oplus Y)\omega \subseteq (X \oplus Y) + Y = X \oplus Y.$$

Thus, it remains to show that $\theta - \pi$ is invertible in E . Since M is morphic, it suffices to show that $\theta - \pi$ is monic [10, Corollary 2]. So suppose that $m \in \ker(\theta - \pi)$. Write $m = x + y + z + k$ with the obvious notation, so that

$$m\pi = m\nu\omega = [(x + y)\eta^{-1} + k\phi]\omega = (x + y)\eta^{-1}\eta + k\phi = x + y + k\phi.$$

Also, $m\theta = (y + z)\theta$ because $\ker(\theta) = K \oplus X$. Hence, $m(\theta - \pi) = 0$ becomes

$$(y + z)\theta = m\theta = m\pi = x + y + k\phi.$$

Since $k\phi \in Y$, this means that $(y + z)\theta \in M\theta \cap (X \oplus Y) = 0$, so that

$$y + z \in \ker(\theta) \cap (Y \oplus L) = 0.$$

This means that $y = 0 = z$, and hence that $x + k\phi = 0$. Thus, $k\phi = -x \in Y \cap X = 0$, whence $x = 0 = k$ because ϕ is monic. □

REMARK 1. In the above proof, we showed $X \oplus Y \subseteq (X \oplus K)(\theta - \pi)$. In fact, this is equality: Given x and y write $y = k\phi$ and compute

$$(-x - k)(\theta - \pi) = -(x + k)(\theta - \pi) = -[0 - (x + k)\nu\omega] = x + k\phi = x + y.$$

QUESTION 3. If $M\theta + \ker(\theta)$ and $M\theta \cap \ker(\theta)$ are direct summands of M , is θ regular?

ACKNOWLEDGEMENTS. The research of the first author was supported by the Natural Science Foundation of Zhejiang Province (LY13A010019), and the second author is grateful for the support of the International Cooperation and Exchange Division of Hangzhou Normal University during his visits there. The third author was supported by NSERC Grant OGP0194196.

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