

PRIME NUMBERS IN INTERVALS STARTING AT A FIXED POWER OF THE INTEGERS

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(Received 27 March 2007; accepted 12 November 2008)

Communicated by I. E. Shparlinski

Abstract

The best known results about the distribution of prime numbers in short intervals imply that all intervals $[n, n + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{7/12}$, and almost all intervals $[n, n + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{1/6}$. As a natural generalization, this paper is concerned with the distribution of prime numbers in intervals of type $[n^\alpha, n^\alpha + H]$ with $\alpha > 1$.

2000 *Mathematics subject classification*: primary 11N05.

Keywords and phrases: prime numbers in short intervals.

1. Introduction

Let $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. We consider the asymptotic formula

$$\psi(x + H) - \psi(x) \sim H \quad \text{as } x \rightarrow \infty, \quad (1)$$

which is related to the number of primes in the interval $(x, x + H]$. The prime number theorem implies that (1) holds with $H \gg x$. An interval $(x, x + H]$ with $H = o(x)$ is called a short interval. The best known unconditional result about the distribution of primes in short intervals is due to Huxley [8] and asserts that (1) holds for all $H \geq x^{7/12+\varepsilon}$. This was slightly improved by Heath-Brown in [7] to $H \geq x^{7/12-o(1)}$. Under the assumption of the Riemann hypothesis, Selberg [11] proved that (1) holds for all $H \geq x^{1/2} f(x) \log x$ where $f(x) \rightarrow \infty$ arbitrarily slowly. These results imply that all intervals $[n, n + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{7/12}$ and, assuming the Riemann hypothesis, for all $H \geq N^{1/2} f(N) \log N$ where $f(N) \rightarrow \infty$ arbitrarily slowly.

We can relax our conditions and investigate if (1) holds for ‘almost all’ x . By this, we mean that the measure of $x \in [X, 2X]$ for which (1) does not hold is $o(X)$. Huxley’s zero density estimate [8], in conjunction with the method of Selberg [11], shows that (1) holds for almost all x with $H \geq x^{1/6+\varepsilon}$, slightly improved by Zaccagnini in [14] to $H \geq x^{1/6-o(1)}$. Under the assumption of the Riemann hypothesis, Selberg [11] proved that (1) holds for almost all x with $H \geq f(x) \log^2 x$, where $f(x) \rightarrow \infty$ arbitrarily slowly. These results imply that almost all intervals $[n, n + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{1/6}$ and, assuming the Riemann hypothesis, for all $H \geq f(N) \log^2 N$ with $f(N) \rightarrow \infty$ arbitrarily slowly.

As a natural generalization of the above results, this paper is concerned with the distribution of prime numbers in intervals $[n^\alpha, n^\alpha + H]$, with fixed $\alpha > 1$. Our main unconditional result is as follows.

THEOREM 1. *Let $\varepsilon > 0$ and $\alpha > 1$. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$, where*

$$c(\alpha) = \begin{cases} \frac{1}{6} & \text{if } 1 < \alpha \leq \frac{6}{5}, \\ \frac{11\alpha - 10}{16\alpha} & \text{if } \frac{6}{5} < \alpha \leq \frac{6}{5} + \Delta, \\ 1 - \sup_{(k,l)} \frac{5(1 + \alpha - l + k)}{\alpha(5k + 12)} & \text{if } \alpha \geq 4, \end{cases}$$

with Δ suitable positive constant and (k, l) running over the exponent pairs.

For the sake of simplicity, we will explicitly work out the value of the function $c(\alpha)$ only for the extreme and more interesting values of α . However, it will be clear from the proof that the same method enables one to obtain the explicit values of the function $c(\alpha)$ in the whole range $\alpha > 1$. As one might expect, we get an increasing function $c(\alpha)$ such that $c(1) = 1/6$, $c(\alpha) < 7/12$ for every α and

$$\lim_{\alpha \rightarrow +\infty} c(\alpha) = \frac{7}{12}.$$

To bound some sums which arise in our argument we employ the counting functions $N(\sigma, T)$ and $N^*(\sigma, T)$. The former is defined as the number of zeros $\rho = \beta + i\gamma$ of Riemann zeta function which satisfy $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, while $N^*(\sigma, T)$ is defined as the number of ordered sets of zeros $\rho_j = \beta_j + i\gamma_j (1 \leq j \leq 4)$, each counted by $N(\sigma, T)$, for which $|\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4| \leq 1$. If we make the heuristic assumption that

$$N^*(\sigma, T) \ll \frac{N(\sigma, T)^4}{T}, \tag{2}$$

as in Bazzanella and Perelli [2], then we can simplify and improve Theorem 1 for large values of α as follows.

THEOREM 2. *Assume (2), suppose that $\varepsilon > 0$ and $\alpha \geq 4$. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$ and*

$$c(\alpha) = \frac{7}{12} - \frac{5}{12\alpha}.$$

We conclude by presenting our results under the assumption of more standard hypotheses.

THEOREM 3. *Let $\alpha > 1$, $\varepsilon > 0$ and assume the Lindelöf hypothesis. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$ and*

$$c(\alpha) = \frac{1}{2} \left(1 - \frac{1}{\alpha} \right).$$

THEOREM 4. *Let $\alpha > 1$ and assume the Riemann hypothesis. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)} f(N) \log^2 N$ with $f(N) \rightarrow \infty$ arbitrarily slowly and*

$$c(\alpha) = \frac{1}{2} \left(1 - \frac{1}{\alpha} \right).$$

As one might expect, under the assumption of the Lindelöf hypothesis or the Riemann hypothesis, we get an increasing function $c(\alpha)$ such that $c(1) = 0$, $c(\alpha) < 1/2$ for every α and

$$\lim_{\alpha \rightarrow +\infty} c(\alpha) = \frac{1}{2}.$$

The main tools of the proofs are the Kusmin–Landau estimate for an exponential sum together with the van der Corput’s method of exponent pairs, see [4], and a result about the structure of the exceptional set for the distribution of primes in short intervals due to Bazzanella and Perelli, see [2] and [1].

2. Definitions and basic lemmas

Our starting point is the definition of the exceptional set for the number of primes in short intervals. Let $|\cdot|$ denote the modulus of a complex number or the Lebesgue measure of an infinite set of real numbers or the cardinality of a finite set. Let X be a large positive number, $\delta > 0$ and define

$$E_\delta(X, H) = \{X \leq x \leq 2X : |\psi(x + H(x)) - \psi(x) - H(x)| \geq \delta H(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, H) = \emptyset$ for all $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ , the set $E_\delta(X, H)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. We will consider increasing functions $H(x)$ of

the form $H(x) = x^{\theta+\varepsilon(x)}$, with some $0 < \theta < 1$ and a differentiable function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing, $\varepsilon(x) = o(1)$ and

$$\varepsilon(x + y) = \varepsilon(x) + O\left(\frac{|y|}{x \log x}\right). \tag{3}$$

A function satisfying these requirements will be called of type θ . It is easy to see that functions like $x^\theta \log^c x$, with c a real constant, and similar functions, are of type θ , and that for every function $H(x)$ of type θ we have $H(2x) \ll H(x)$.

REMARK. In a preceding paper, the author and Perelli [2] defined in a slightly different way the set of functions $H(x)$ of type θ , and set

$$\varepsilon(x + y) = \varepsilon(x) + O\left(\frac{|y|}{x}\right)$$

instead of (3). We remark that with this weaker condition we do not have $H(2x) \ll H(x)$ as claimed.

Our first lemma is concerned with the structure of the exceptional set above.

LEMMA 1. *Let $0 < \theta < 1$, let $H(x)$ be of type θ , let X be sufficiently large depending on the function $H(x)$ and let $0 < \delta' < \delta$ with $\delta - \delta' \geq \exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, H)$ then $E_{\delta'}(X, H)$ contains the interval $[x_0 - cH(X), x_0 + cH(X)] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$.*

PROOF. We will always assume that x and X are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence.

We first observe from the definition of a function of type θ that if $y = O(x^{\alpha+\varepsilon})$ with some $0 < \alpha < 1$, then

$$H(x + y) = H(x) + O(x^{\theta+\alpha-1+\varepsilon}) \tag{4}$$

for every $\varepsilon > 0$.

From the Brun–Titchmarsh theorem (see Montgomery and Vaughan [10]), we have that

$$\psi(x + y) - \psi(x) \leq \frac{21}{10} y \frac{\log x}{\log y} \tag{5}$$

for all $10 \leq y \leq x$. From (5) we easily obtain that

$$\psi(x + y) - \psi(x) \leq \frac{9}{4\alpha} cY \tag{6}$$

for all $X \leq x \leq 3X$ and $0 \leq y \leq cY$, where $0 < \alpha < 1$, $X^{\alpha-\varepsilon} \leq Y \leq X$ and

$$\frac{\alpha}{5} \exp(-\sqrt{\log X}) \leq c \leq 1.$$

Let $H(x)$ be of type θ , $x_0 \in E_\delta(X, H)$,

$$x \in [x_0 - cH(X), x_0 + cH(X)] \cap [X, 2X],$$

where c satisfies the above restrictions, and

$$\Delta(x, H) = \psi(x + H(x)) - \psi(x) - H(x).$$

We have

$$\begin{aligned} |\Delta(x, H)| &= |\Delta(x_0, H) + \Delta(x, H) - \Delta(x_0, H)| \\ &\geq |\Delta(x_0, H)| - |\psi(x + H(x)) - \psi(x_0 + H(x_0))| \\ &\quad - |\psi(x) - \psi(x_0)| - |H(x) - H(x_0)|. \end{aligned}$$

But from (4) with $\alpha = \theta$ we get

$$H(x_0) = H(x) + O(X^{2\theta-1+\varepsilon}),$$

hence from (6) with $\alpha = \theta$ we obtain

$$|\Delta(x, H)| \geq \delta H(x) - \frac{9}{2\theta} cH(X) + O(X^{2\theta-1+\varepsilon}) \geq \delta H(x) - \frac{5}{\theta} cH(X) \geq \delta' H(x)$$

by choosing $c = (\delta - \delta')\theta/5$, since $H(x)$ is increasing. Hence $x \in E_{\delta'}(X, H)$ and the lemma follows. \square

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [2], and essentially says that if we have a single exception in $E_\delta(X, H)$, with a fixed δ , then we necessarily have an interval of exceptions in $E_{\delta'}(X, H)$, with δ' a little smaller than δ .

We now present the necessary results about the conditional and unconditional bounds for the exceptional set for the number of primes in short intervals. With this in mind, we consider $H(x)$ of type θ and define the functions

$$\mu_\delta(\theta) = \inf\{\xi \geq 0 : |E_\delta(X, H)| \ll_\delta X^\xi\}$$

and

$$\mu(\theta) = \sup_{\delta > 0} \mu_\delta(\theta). \quad (7)$$

Our results are as follows.

LEMMA 2. *There exists a constant $\eta > 0$ such that*

$$\mu(\theta) \leq \frac{(11 - 6\theta)}{10} \quad \text{if } \frac{1}{6} < \theta \leq \frac{1}{6} + \eta.$$

PROOF. In order to prove Lemma 2 we use the classical explicit formula (see Davenport [3, Ch. 17]) to write

$$\psi(x + H(x)) - \psi(x) - H(x) = - \sum_{|\gamma| \leq T} x^\rho c_\rho(x) + O\left(\frac{X \log^2 X}{T}\right), \tag{8}$$

uniformly for all $X \leq x \leq 2X$, where $10 \leq T \leq X$, $\rho = \beta + i\gamma$ runs over the nontrivial zeros of $\zeta(s)$,

$$c_\rho(x) = \frac{(1 + H(x)/x)^\rho - 1}{\rho} \quad \text{and} \quad c_\rho(x) \ll \min\left(\frac{H(X)}{X}, \frac{1}{|\gamma|}\right). \tag{9}$$

Let $H(x)$ be of type θ . Choose

$$T = \frac{X}{H(X)} \log^3 X \tag{10}$$

and use the theorem of Montgomery (see Theorem 11.3 of Ivić [9]) which asserts that

$$N(\sigma, T) \ll T^{1600(1-\sigma)^{3/2}} \log^{15} T \tag{11}$$

for every $152/155 \leq \sigma \leq 1$. From (9)–(11) and Vinogradov’s zero-free region (see Titchmarsh [12, Ch. 6]) we deduce by a standard argument that there exists a constant $d > 0$ such that

$$\sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho c_\rho(x) \ll \frac{H(X)}{X} \log X \max_{\sigma \notin I} X^\sigma N(\sigma, T) \ll \frac{H(X)}{\log X}, \tag{12}$$

where $I = [1/2, 1 - d]$, uniformly for all $X \leq x \leq 2X$.

Again by a standard argument, from (9), (10) and the Ingham–Huxley density estimates which assert that for every $\varepsilon > 0$ we have

$$N(\sigma, T) \ll \begin{cases} T^{3(1-\sigma)/(2-\sigma)+\varepsilon}, & \frac{1}{2} \leq \sigma \leq \frac{3}{4}, \\ T^{3(1-\sigma)/(3\sigma-1)+\varepsilon}, & \frac{3}{4} \leq \sigma \leq 1, \end{cases} \tag{13}$$

we obtain

$$\int_X^{2X} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} x^\rho c_\rho(x) \right|^2 dx \ll X^{2\theta-1+\varepsilon} \max_{\sigma \in I} X^{2\sigma} N(\sigma, T) \ll X^{(11+14\theta)/10+\varepsilon},$$

for sufficiently small $\eta > 0$ and $1/6 < \theta \leq 1/6 + \eta$. Hence, for every $\varepsilon > 0$ and $\delta > 0$,

$$|E_\delta(X, H)| \ll X^{(11-6\theta)/10+\varepsilon},$$

and so the lemma is proved. □

We observe that we can take $d = 2.5 \cdot 10^{-7}$ and then $\eta = 3.125 \cdot 10^{-7}$. The value of η could be somewhat increased by using an optimized version of density estimate (11).

LEMMA 3. *Assume (2). Then*

$$\mu(\theta) \leq \frac{7}{5}(1 - \theta) \quad \text{if } \frac{23}{48} < \theta < \frac{7}{12}.$$

PROOF. Let $H(x)$ be of type θ and

$$T = \frac{X}{H(X)} \log^3 X.$$

Following the method of Heath-Brown [5], we can write

$$\int_X^{2X} |\psi(x + H(x)) - \psi(x) - H(x) + \Sigma|^4 dx \ll X^{4\theta-3+\varepsilon} \max_{1/2 \leq \sigma \leq 1} X^{4\sigma} N^*(\sigma, T),$$

with $\Sigma = o(H(X))$. Assuming (2) and using the Ingham–Huxley zero density estimates, the above estimate implies that

$$\begin{aligned} |E_\delta(X, H)| &\ll X^{-3+\varepsilon} \max_{1/2 \leq \sigma \leq 1} X^{4\sigma} N^*(\sigma, T) \ll X^{-3+\varepsilon} \max_{1/2 \leq \sigma \leq 1} X^{4\sigma} \frac{N(\sigma, T)^4}{T} \\ &\ll X^{\theta-4+\varepsilon} \left(\max_{1/2 \leq \sigma \leq 3/4} X^{4\sigma} T^{12(1-\sigma)/(2-\sigma)} \right. \\ &\quad \left. + \max_{3/4 \leq \sigma \leq 1} X^{4\sigma} T^{12(1-\sigma)/(3\sigma-1)} \right), \end{aligned}$$

for every $\delta > 0$ and $\varepsilon > 0$. With $23/48 < \theta < 7/12$ the maximum is attained at $\sigma = 3/4$, so

$$|E_\delta(X, H)| \ll X^{\frac{7}{5}(1-\theta)+\varepsilon},$$

for every $\delta > 0$ and $\varepsilon > 0$. This completes the proof of the lemma. □

LEMMA 4. *Assume the Lindelöf hypothesis, let $\varepsilon > 0$ and $\delta > 0$. For every $H \geq 1$,*

$$|E_\delta(X, H)| \ll \frac{X^{1+\varepsilon}}{H(X)}.$$

Lemma 4 may be proved along the same lines as Yu [13, Lemma B].

To deal with the problem of estimating the exceptional set for the distribution of primes in intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$, suppose that $H(x)$ is of type θ , let

$$\Delta(n, H, \alpha) = \psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha),$$

and define the set

$$A_\delta(N, H, \alpha) = \{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha} : |\Delta(n, H, \alpha)| \geq \delta H(n^\alpha)\},$$

that contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of type $[n^\alpha, n^\alpha + H(n^\alpha)] \subset [N, 2N]$. Our last lemmas allow us to link $|A_\delta(N, H, \alpha)|$ to the exceptional set for the distribution of primes in short intervals.

LEMMA 5. *Let $H(x)$ be of type θ , with $1/6 < \theta < 7/12$. Then for every $\delta > 0$ we have the following results:*

(i)

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}) \quad \text{if } 1 < \alpha \leq \frac{6}{5}.$$

(ii)

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha}) \quad \text{if } \alpha > \frac{6}{5},$$

with $f(N) \rightarrow \infty$ arbitrarily slowly.

PROOF. Recalling the explicit formula for $\psi(x)$ and putting

$$T = \frac{N}{H(N)} f(N) \log^2 N,$$

where $f(N) \rightarrow \infty$ arbitrarily slowly, we have

$$\begin{aligned} \psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) &= - \sum_{|\gamma| < T} n^{\alpha\rho} c_\rho(n) + o(H(N)) \\ &= - \sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho} c_\rho(n) + o(H(N)), \end{aligned}$$

where d and $I = [1/2, 1 - d]$ are defined as in the proof of Lemma 2,

$$c_\rho(n) = \frac{1 - (1 + H(n^\alpha)n^{-\alpha})^\rho}{\rho} \quad \text{and} \quad c_\rho(n) \ll \min\left(\frac{H(N)}{N}, \frac{1}{|\gamma|}\right).$$

Further we divide the interval I into $O(\log N)$ subintervals I_j of the form

$$I_j = \left[\frac{j-1}{\log N}, \frac{j}{\log N} \right] \cap I.$$

On applying Cauchy's inequality we find

$$\left| \sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho} c_\rho(n) \right|^2 \ll \log N \sum_j \left| \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} n^{\alpha\rho} c_\rho(n) \right|^2,$$

and so we get

$$\begin{aligned}
 H(N)^2 |A_\delta(N, H, \alpha)| &\ll \sum_{n \in A_\delta(N, H, \alpha)} |\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) + o(H(N))|^2 \\
 &\leq \sum_{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha}} \left| \sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho} c_\rho(n) \right|^2 \\
 &\ll \log N \sum_{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha}} \sum_j \left| \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} n^{\alpha\rho} c_\rho(n) \right|^2.
 \end{aligned}$$

Squaring and using partial summation we then have

$$\begin{aligned}
 |A_\delta(N, H, \alpha)| &\ll \frac{\log N}{H(N)^2} \sum_{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha}} \sum_j \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} n^{\alpha(\rho + \bar{\rho})} c_\rho(n) \overline{c_{\rho'}(n)} \\
 &\ll \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} |S|
 \end{aligned}$$

where

$$\begin{aligned}
 S &= \sum_{N^{1/\alpha} \leq n \leq (N_1)^{1/\alpha}} n^{\alpha i(\gamma - \gamma')} = \sum_{N^{1/\alpha} \leq n \leq (N_1)^{1/\alpha}} e(g(n)), \\
 e(x) &= e^{2\pi i x}, \quad g(x) = \frac{\alpha(\gamma - \gamma')}{2\pi} \log x
 \end{aligned}$$

and $N \leq N_1 \leq 2N$.

Let

$$H(N) \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N, \tag{14}$$

with $f(N) \rightarrow \infty$ arbitrarily slowly. Using the theorem of Kusmin–Landau (see Graham and Kolesnik [4, Theorem 2.1]) and the trivial bound, one finds that

$$|S| \ll \frac{N^{1/\alpha}}{|\gamma - \gamma'|} \quad \text{and} \quad |S| \ll N^{1/\alpha},$$

and hence

$$\begin{aligned}
 |A_\delta(N, H, \alpha)| &\ll \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j, |\gamma - \gamma'| \leq 1}} N^{1/\alpha} \\
 &\quad + \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j, |\gamma - \gamma'| > 1}} \frac{N^{1/\alpha}}{|\gamma - \gamma'|},
 \end{aligned}$$

which implies

$$|A_\delta(N, H, \alpha)| \ll \frac{N^{1/\alpha}}{N^2} \log^3 N \left(\sum_j \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} N^{2j/\log N} \right). \tag{15}$$

For every $1 < \alpha \leq 6/5$ and $H(x)$ of type θ with $1/6 < \theta < 7/12$, and for every $\alpha > 6/5$ and $H(x)$ satisfying (14), it follows by a standard argument and the Ingham–Huxley zero density estimates that

$$\sum_j \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} N^{2j/\log N} \ll \max_{\sigma \in I} N^{2\sigma} N(\sigma, T) \ll \frac{N^2}{\log^A N}, \tag{16}$$

for every $A > 0$. From (15) and (16), it follows that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha})$$

for every $1 < \alpha \leq 6/5$ and for every $\alpha > 6/5$ with

$$H(N) \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N.$$

Finally, let $\alpha > 6/5$ and

$$H(N) < \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N.$$

To deal with this small H we observe that if $n \in A_\delta(N, H, \alpha)$ then $N \leq n^\alpha \leq 2N$ and

$$|\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha)| \geq \delta H(n^\alpha).$$

Thus $n^\alpha \in E_\delta(N, H)$. By Lemma 1 we find a constant $c > 0$ such that

$$[n^\alpha - cH(N), n^\alpha + cH(N)] \cap [N, 2N] \subset E_{\delta/2}(N, H).$$

We now consider $m \in A_\delta(N, H, \alpha)$, with $|m - n| \geq (2/\pi) f(N) \log^2 N$ and similarly we get $m^\alpha \in E_\delta(N, H)$ and then

$$[m^\alpha - cH(N), m^\alpha + cH(N)] \cap [N, 2N] \subset E_{\delta/2}(N, H),$$

again by Lemma 1. Since

$$|m^\alpha - n^\alpha| \geq |m - n| \alpha N^{1-1/\alpha} \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N > H(N),$$

we may deduce that

$$[m^\alpha - cH(N), m^\alpha + cH(N)] \cap [n^\alpha - cH(N), n^\alpha + cH(N)] = \emptyset,$$

for c suitable small. This leads to the bound

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)},$$

for every $\delta > 0$, which proves the lemma. □

LEMMA 6. *Assume the Lindelöf hypothesis. Let $H(x)$ be of type θ , with $0 < \theta < 1/2$. Then, for every $\delta > 0$ and $\alpha > 1$,*

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)|f(N) \log^2 N}{H(N)} + o(N^{1/\alpha})$$

with $f(N) \rightarrow \infty$ arbitrarily slowly.

PROOF. We follow the proof of the Lemma 5 until the equation (15). Under the assumption of the Lindelöf hypothesis, which states that the Riemann zeta function satisfies

$$\zeta(\sigma + it) \ll t^\eta \quad (\sigma \geq \frac{1}{2}, t \geq 2),$$

for any $\eta > 0$, we have

$$N(\sigma, T) \ll \begin{cases} T^{(2+4\eta)(1-\sigma)}(\log T)^M, & 0 \leq \sigma \leq 1, \\ T^{3\eta(1-\sigma)/(\sigma-3/4)}(\log T)^M, & \frac{3}{4} < \sigma \leq 1, \end{cases} \tag{17}$$

with $T \geq 2$ and M suitable absolute constant (see Lemma 3 of Yu [13]). From (17) it follows that the bound (16) holds for every

$$H(N) \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N$$

and $\alpha > 1$. We can conclude the proof by dealing with smaller values of H in the same way as in the proof of Lemma 5. □

3. Proof of the Theorem 1

By the case (i) of Lemma 5, we can take

$$c(\alpha) = \frac{1}{6} \quad \text{if } 1 < \alpha \leq \frac{6}{5}.$$

For all $\alpha > 6/5$, by (ii) of Lemma 5,

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)|f(N) \log^2 N}{H(N)} + o(N^{1/\alpha}),$$

for every $H(x)$ of type θ , with $1/6 < \theta < 7/12$. Furthermore, by Lemma 2 there exists $\eta > 0$ such that

$$|E_{\delta/2}(N, H)| \ll N^{(11-6\theta)/10+\varepsilon},$$

for every

$$\frac{1}{6} < \theta \leq \frac{1}{6} + \eta,$$

and every $H(x)$ of type θ . These estimates together yield

$$|A_\delta(N, H, \alpha)| \ll N^{(11-16\theta)/10+\varepsilon} + o(N^{1/\alpha}),$$

and

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{11\alpha - 10}{16\alpha}$$

and sufficiently small $\alpha > 6/5$.

It follows that

$$c(\alpha) = \frac{11\alpha - 10}{16\alpha} \quad \text{if } \frac{6}{5} < \alpha \leq \frac{6}{5} + \Delta,$$

for suitable positive constant Δ . From the explicit value for η available from the Lemma 2, we can state that an admissible value is $\Delta = 7.2 \cdot 10^{-7}$.

To estimate $c(\alpha)$ for large values of α we need to follow a quite different method. In a similar way as in the proof of Lemma 5, we let

$$T = \frac{N}{H(N)} \log^3 N$$

and write

$$\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) = - \sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho} c_\rho(n) + o(H(N)),$$

where $I = [1/2, 1 - d]$, for a suitable positive constant d . Next we divide the interval I into $O(\log N)$ subintervals I_j of the form

$$I_j = \left[\frac{j-1}{\log N}, \frac{j}{\log N} \right] \cap I.$$

Using Hölder's inequality, we get

$$\left| \sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho} c_\rho(n) \right|^4 \ll \log^3 N \sum_j \left| \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} n^{\alpha\rho} c_\rho(n) \right|^4$$

and then we can deduce

$$\begin{aligned} |A_\delta(N, H, \alpha)| &\ll \frac{\log^3 N}{H(N)^4} \sum_{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha}} \sum_j \left| \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} n^{\alpha\rho} c_\rho(n) \right|^4 \\ &\ll \frac{\log^3 N}{H(N)^4} \sum_{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha}} \sum_j \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} \sum_{\substack{|\gamma''| < T \\ \beta'' \in I_j}} \sum_{\substack{|\gamma'''| < T \\ \beta''' \in I_j}} n^{\alpha(\rho + \rho' + \overline{\rho''} + \overline{\rho''''})} C_n \\ &\ll \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} \sum_{\substack{|\gamma''| < T \\ \beta'' \in I_j}} \sum_{\substack{|\gamma'''| < T \\ \beta''' \in I_j}} |S| \\ &= V_1 + V_2, \end{aligned}$$

where

$$C_n = c_\rho(n)c_{\rho'}(n)\overline{c_{\rho''}(n)c_{\rho'''}(n)},$$

$$S = \sum_{N^{1/\alpha} \leq n \leq (N_1)^{1/\alpha}} e(g(n)), \quad g(x) = \frac{\alpha(\gamma + \gamma' - \gamma'' - \gamma''')}{2\pi} \log x,$$

$$V_1 = \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} \sum_{\substack{|\gamma''| < T \\ \beta'' \in I_j}} \sum_{\substack{|\gamma'''| < T \\ \beta''' \in I_j \\ |\gamma + \gamma' - \gamma'' - \gamma'''| \leq (\pi/\alpha)N^{1/\alpha}}} |S|$$

and

$$V_2 = \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma| < T \\ \beta \in I_j}} \sum_{\substack{|\gamma'| < T \\ \beta' \in I_j}} \sum_{\substack{|\gamma''| < T \\ \beta'' \in I_j}} \sum_{\substack{|\gamma'''| < T \\ \beta''' \in I_j \\ |\gamma + \gamma' - \gamma'' - \gamma'''| > (\pi/\alpha)N^{1/\alpha}}} |S|.$$

We first proceed to estimate V_1 . For the terms in the inner sum with

$$|\gamma + \gamma' - \gamma'' - \gamma'''| < 1$$

we can estimate $|S|$ using the trivial bound. For the terms with

$$1 \leq |\gamma + \gamma' - \gamma'' - \gamma'''| \leq \frac{\pi}{\alpha} N^{1/\alpha}$$

we can use the Kusmin–Landau theorem. Hence we obtain the estimate

$$S \ll \frac{N^{1/\alpha}}{1 + |\gamma + \gamma' - \gamma'' - \gamma'''|},$$

which, by Heath-Brown’s method [5], implies

$$V_1 \ll \frac{N^{1/\alpha} \log^5 N}{N^4} \max_{\sigma \in I} N^{4\sigma} N^*(\sigma, T).$$

For $H(x)$ of type θ , with $0.342 < \theta < 7/12$, Heath-Brown’s zero-density estimates

$$N^*(\sigma, T) \ll \begin{cases} T^{(10-11\sigma)/(2-\sigma)+\varepsilon}, & \frac{1}{2} \leq \sigma \leq \frac{2}{3}, \\ T^{(18-19\sigma)/(4-2\sigma)+\varepsilon}, & \frac{2}{3} \leq \sigma \leq \frac{3}{4}, \\ T^{12(1-\sigma)/(4\sigma-1)+\varepsilon}, & \frac{3}{4} \leq \sigma \leq 1, \end{cases} \tag{18}$$

[6, Theorem 2] give upper bounds for $N^{4\sigma} N^*(\sigma, T)$ that attain their maximum at $\sigma = 1 - d$. A short calculation then shows that

$$\max_{\sigma \in I} N^{4\sigma} N^*(\sigma, T) \ll \frac{N^4}{(\log N)^4},$$

for every $A > 0$. Hence we conclude that

$$V_1 = o(N^{1/\alpha}),$$

for every $0.342 < \theta < 7/12$.

Now we turn to estimating V_2 . Let (k, l) be an exponent pair, then

$$S \ll \left(\frac{|\gamma + \gamma' - \gamma'' - \gamma'''|}{N^{1/\alpha}} \right)^k (N^{1/\alpha})^l \ll \left(\frac{T}{N^{1/\alpha}} \right)^k N^{l/\alpha} \ll N^{(k\alpha(1-\theta)-k+l)/\alpha+\varepsilon},$$

for every $\varepsilon > 0$ and $H(x)$ of type θ . This yields

$$\begin{aligned} V_2 &\ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon} \sum_j \left(\sum_{\substack{|\gamma| < T \\ \beta \in I_j}} N^{j/\log N} \right)^4 \\ &\ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon} \left(\max_{\sigma} N^{\sigma} N(\sigma, T) \right)^4. \end{aligned}$$

For H of type θ , with

$$\frac{23}{48} < \theta < \frac{7}{12}, \tag{19}$$

the density estimates of Ingham–Huxley give upper bounds for $N^{\sigma} N(\sigma, T)$ that attain their maximum at $\sigma = 3/4$. So we may deduce

$$V_2 \ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon} N^{3+12(1-\theta)/5}.$$

The above bound is $o(N^{1/\alpha})$ for every

$$\theta > 1 - \frac{5(1 + \alpha - l + k)}{\alpha(5k + 12)}, \tag{20}$$

if (k, l) is an exponent pair, H is of type θ and α is sufficiently large. Thus we can select

$$c(\alpha) = 1 - \sup_{(k,l)} \frac{5(1 + \alpha - l + k)}{\alpha(5k + 12)},$$

where (k, l) runs over the exponent pairs. Since all exponent pairs (k, l) have $0 \leq k \leq 1/2 \leq l$, we obtain

$$1 - \frac{5}{12} \frac{1 + \alpha}{\alpha} < 1 - \sup_{(k,l)} \frac{5(1 + \alpha - l + k)}{\alpha(5k + 12)} = c(\alpha),$$

which implies (19) if $\alpha \geq 4$. On the other hand from the exponent pairs

$$A^{i-1} B(0, 1) = \left(\frac{1}{2(2^i - 1)}, 1 - \frac{i}{2(2^i - 1)} \right),$$

where

$$i = \left[\frac{5\alpha}{12} \right],$$

we get

$$1 - \sup_{(k,l)} \frac{5(1 + \alpha - l + k)}{\alpha(5k + 12)} = c(\alpha) < \frac{7}{12}$$

and then, as one might expect, we conclude that

$$\lim_{\alpha \rightarrow +\infty} c(\alpha) = \frac{7}{12}.$$

This completes the proof of Theorem 1. □

NOTE. We are able to obtain the function $c(\alpha)$, in a suitable interval of α , from every estimate of the counting function $N(\sigma, T)$ in a fixed interval of σ . As an example, if we recall that

$$N(\sigma, T) \ll T^{9(1-\sigma)/(7\sigma-1)} \log^C T, \tag{21}$$

with $41/53 \leq \sigma \leq 1$ and C suitable constant (see Theorem 11.4 of Ivić [9]), we can choose H of type θ , $d = (9\theta - 3)/7 - \xi$ with $\xi > 0$, in (12) of Lemma 2, using the Ingham–Huxley density estimates and (21) we can obtain an estimate of $|E_\delta(X, H)|$. Hence, from Lemma 5, we can obtain

$$c(\alpha) = \begin{cases} \frac{5}{8} - \frac{7}{16\alpha} & \text{if } \frac{3}{2} < \alpha \leq \frac{3339}{1138}, \\ \frac{1969}{2809} - \frac{35}{53\alpha} & \text{if } \frac{3339}{1138} \leq \alpha \leq 3.447, \end{cases}$$

that cover a great part of the gap between $6/5 + \Delta$ and 4. Along the same lines we can obtain a large number of possible function $c(\alpha)$, for every $\alpha > 1$.

4. Proof of Theorems 2, 3 and 4

In order to prove Theorem 2 we assume (2) and use Lemma 5 to see that

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha}),$$

for every $H(x)$ of type θ , with $1/6 < \theta < 7/12$. So by Lemma 3

$$|E_{\delta/2}(N, H)| \ll N^{\frac{7}{5}(1-\theta)+\varepsilon},$$

with $H(x)$ of type θ and $23/48 < \theta < 7/12$. The last two estimates together yield

$$|A_\delta(N, H, \alpha)| \ll N^{\frac{7}{5}-\frac{12}{5}\theta+\varepsilon} + o(N^{1/\alpha}),$$

so that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{7}{12} - \frac{5}{12\alpha} \quad \text{and} \quad \frac{23}{48} < \theta < \frac{7}{12}.$$

Then we can define

$$c(\alpha) = \frac{7}{12} - \frac{5}{12\alpha} \quad \text{if } \alpha \geq 4.$$

This completes the proof of Theorem 2. □

Similarly we can prove Theorem 3, using Lemmas 6 and 4 instead of Lemmas 5 and 3, so obtaining

$$|A_\delta(N, H, \alpha)| \ll N^{1-2\theta+\varepsilon} + o(N^{1/\alpha}),$$

so that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{1}{2} \left(1 - \frac{1}{\alpha}\right).$$

Then we can choose

$$c(\alpha) = \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \quad \text{if } \alpha > 1.$$

This completes the proof of Theorem 3. □

To prove Theorem 4 we recall that Selberg [11] proved, under the assumption of the Riemann hypothesis, that

$$\int_X^{2X} |\psi(x + H) - \psi(x) - H|^2 dx \ll HX \log^2 X,$$

for all $H \geq 10$, which implies

$$|E_\delta(N, H)| \ll \frac{N}{H(N)} \log^2 N,$$

for every $\delta > 0$. In conjunction with Lemma 6, this gives

$$|A_\delta(N, H, \alpha)| \ll \frac{N \log^4 N}{H(N)^2} f(N) + o(N^{1/\alpha}),$$

with $f(N) \rightarrow \infty$ arbitrarily slowly, so that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

with

$$H(N) > N^{\frac{1}{2}(1-\frac{1}{\alpha})} f(N) \log^2 N,$$

for every $\alpha > 1$ and $\delta > 0$. This completes the proof of Theorem 4. □

Acknowledgement

We are particularly indebted to the referee for a very thorough reading and some helpful suggestions.

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