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THE CARATHEODORY METRIC AND ITS MAJORANT METRICS

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1. Introduction. One of the main purposes of the present paper is to provide a proof for the following statement:

THEOREM A. *Let M be a complex manifold of a complex dimension n. Let ζ be a fixed point in M such that there exists a square integrable holomorphic n-form $\alpha(z)$ on M with $\alpha(\zeta) \neq 0$. Then $C(\zeta : v) \leqq S(\zeta : v)$ for each holomorphic tangent vector $v \in \mathbf{C}^n$. Here $C = C_M$ and $S = S_M$ are the Carathéodory and the Bergman (differential) metrics on M respectively. Moreover, if for some v, $C(\zeta : v) > 0$, then $C(\zeta : v) < S(\zeta : v)$.*

When M is a bounded domain in $\mathbf{C}^n$, part of this theorem is already mentioned in the Fundamental Theorem I of Look [**7**] and was again amplified by Hahn [**4**]. Theorem A is obtained as a special case of Theorem 5. The method of proof used in this paper is that of the *method of minimum integral* (cf. Bergman [**1**, p. 26]) applied to the Bergman kernel function of M. Since this method could be well applied on any Hilbert space of holomorphic functions which possesses a reproducing kernel, we obtain a more general assertion (Theorem 1).

As in the Bergman case the reproducing kernel induces a Kählerian metric on M. We compare the sectional Riemannian curvature of this metric with the Carathéodory metric (Theorems 2,6). As a corollary of this theorem we will establish an estimate for the curvature of the Bergman metric which is a slight improvement on a result of Fuks [**3**] (see Theorem 3). We also study a function that generalizes the Bergman metric (Theorem 4).

Specifying our results for $n = 1$, we obtain relationships between the analytic capacity and curvatures of certain conformal invariant metrics. The most important relationship is the one showing that the curvature of the analytic capacity metric is always $\leqq -4$. This fact was first proved by Suita [**9**] and generalized in [**2**].

2. Kernel functions and metrics. In this section we assume that M is a domain in C^n. The case when M is a complex manifold is postponed to the next section. We assume that on M we have a Hilbert space $H_2(M)$ of all holomorphic functions f in M normed by $||f||^2 = \int |f(z)|^2 d\mu_M(z)$. Here $d\mu_M$ is a positive measure on M or on any other set determining the holomorphic

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functions in M, for example the "Šilov boundary" of M. We also assume that point evaluations are bounded linear functionals on $H_2(M)$. Therefore, $H_2(M)$ possesses a reproducing kernel $K(z, \bar{\zeta}) = K_M(z, \bar{\zeta})$ and convergence in the norm implies uniform convergence on compacta of M. $K(z, \bar{z})$ is real analytic and non negative. Let $\mathscr{M} = \{\zeta \in M\colon$ there exists $f \in H_2(M)$ with $f(\zeta) \neq 0\}$. Then $\mathscr{M}$ is open in M and clearly $\mathscr{M} = \{\zeta \in M\colon K(\zeta, \bar{\zeta}) > 0\}$.

Let $v \in \mathbf{C}^n - \{0\}$ and $z \in \mathbf{C}^n$. The following vector fields (in the direction of v) are defined for any C^∞-function at z;

$$\partial_v^{\,m} = \partial_v \partial_v^{\,m-1}, \quad \partial_v = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j}, \quad m = 0, 1, \ldots .$$

Thus $\partial_v^{\,0} f(z) = f(z)$ for each f. Likewise we can define $\bar{\partial}_v^{\,m}$ where $\bar{\partial}_v = \sum_{j=1}^n \bar{v}_j \,\partial/\partial \bar{z}_j$.

Let $H(M : \Delta)$ be the family of holomorphic functions from M into the unit disc Δ in $\mathbf{C}$. For fixed $\zeta \in M$ we write $H_\zeta(M : \Delta) = \{f \in H(M : \Delta)\colon f(\zeta) = 0\}$. The Carathéodory metric (cf. Reiffen [**8**]) is given by

$$C(\zeta : v) = C_M(\zeta : v) = \sup \{|\partial_v f(\zeta)|\colon f \in H(M : \Delta)\}.$$

Evidently

$$C(\zeta : v) = \sup \{|\partial_v f(\zeta)|\colon f \in H_\zeta(M : \Delta)\}.$$

By a normal family argument there exists an $F \in H_\zeta(M : \Delta)$ such that $\partial_v F(\zeta) = C(\zeta : v)$. Here $F(z) = F(z; \zeta, v)$ and $C(\zeta : v) \geqq 0$. The Carathéodory metric is a function defined on the complex tangent space of M at ζ. In fact, if $\phi\colon M \to M^*$ is a holomorphic mapping then

$$C_{M^*}(\phi(\zeta)\colon \phi_*(v)) \leqq C_M(\zeta : v),$$

where $[\phi_*(v)]_j = \partial_v \phi_j(\zeta), j = 1, \ldots, n$, and $\phi(\zeta) = (\phi_1(\zeta), \ldots, \phi_n(\zeta))$. Consequently, if $\phi\colon M \to M^*$ is a biholomorphic mapping of M onto M^* then $C_{M^*}(\phi(\zeta)\colon \phi_*(v)) = C_M(\zeta : v)$. Let $\mathscr{A} = \{(\zeta, v) \in M \times (\mathbf{C}^n - \{0\})\colon$ there exists $f \in H(M : \Delta)$ with $\partial_v f(\zeta) \neq 0\}$. Again, $\mathscr{A}$ is open in $M \times (\mathbf{C}^n - \{0\})$ and for $(\zeta, v) \in \mathscr{A}$, $\partial_v F(\zeta) = C(\zeta : v) > 0$ where F is as before. Clearly, $\mathscr{A} = M \times (\mathbf{C}^n - \{0\})$ whenever M is bounded.

Let $\zeta \in \mathscr{M}$ and $(\zeta, v) \in \mathscr{A}$. Consider the sets

$$A_m(\zeta : v) = \{f \in H_2(M)\colon \partial_v^{\,k} f(\zeta) = \delta_{km}, k = 0, 1, \ldots, m\}; \qquad m = 0, 1, \ldots .$$

These sets are convex and closed subsets of $H_2(M)$ and they are not empty; for, the function

$$\text{(2.1)} \qquad \varphi_m(z) = \frac{F(z)^m K(z, \bar{\zeta})}{m! C^m K}; \quad C = C(\zeta : v), K = K(\zeta, \bar{\zeta}),$$

belongs to $A_m(\zeta : v)$ for every $m = 0, 1, \ldots$. Let ψ_m be the unique solution of

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the minimum problem

(2.2) $\lambda_m = \lambda_m(\zeta : v) = \min \{||f||^2 : f \in A_m(\zeta : v)\}.$

Then (cf. [**1**, p. 26])

(2.3) $\lambda_m = J_{m-1}/J_m,$

where

$$J_m = J_m(\zeta : v) = W_\zeta[K, \bar{\partial}_v K, \ldots, \bar{\partial}_v{}^m K], \quad K = K(\zeta, \bar{\zeta}),$$

is the $(m + 1)$ order Wronskian with respect to the vector field ∂_v evaluated at ζ. Here $J_{-1} \equiv 1$ and $J_0 = K(\zeta, \bar{\zeta})$. Thus

$$J_1(\zeta : v) = \begin{vmatrix} K & \bar{\partial}_v K \\ \partial_v K & \partial_v \bar{\partial}_v K \end{vmatrix} = K^2 \partial_v \bar{\partial}_v \log K.$$

Therefore, $K^{-2}J_1(\zeta : v)$ is exactly the Bergman metric $S^2(\zeta : v)$ when $d\mu_M$ is the usual volume element of M.

Using (2.3) we find that $J_m = (\prod_{k=0}^m \lambda_k)^{-1} > 0$. Let

(2.4) $$R_m(\zeta : v) = \left(\prod_{k=1}^{m} k!\right)^{-2} K^{-(m+1)} J_m(\zeta : v), \quad m \geqq 1,$$

and we retain the alternative symbol $S^2(\zeta : v)$ for $R_1(\zeta : v)$.

THEOREM 1. *Let* $\zeta \in \mathscr{M} \subset M$. *Then*

(i) $[C_M(\zeta : v)]^{m(m+1)} \leqq R_m(\zeta : v)$ *for each* $v \in \mathbf{C}^n - \{0\}$.

(ii) *If also* $(\zeta, v) \in \mathscr{A}$ *and* $d\mu_M$ *acts on* M, *then* $[C_M(\zeta : v)]^{m(m+1)} < R_m(\zeta : v)$.

Proof. (i) We can assume that $(\zeta, v) \in \mathscr{A}$. By (2.1) and (2.2) we have $||\varphi_k||^2 \geqq \lambda_k$. But

$$||\varphi_k||^2 = \frac{||F^k K(\ \ , \bar{\zeta})||^2}{(k!)^2 C^{2k} K^2} \leqq \frac{1}{(k!)^2 C^{2k} K}$$

and so

$$[C(\zeta : v)]^{2k} \leqq \frac{1}{(k!)^2 K \lambda_k}, \quad k \geqq 1.$$

Upon multiplying these inequalities by running from $k = 1$ through $k = m$ the assertion follows.

(ii) If $d\mu_M$ acts on M, then $||F^k K(\ \ , \bar{\zeta})||^2 < K(\zeta, \bar{\zeta})$ for $k \geqq 1$. Indeed, if equality holds then $\int |K(z, \bar{\zeta})|^2 (1 - |F(z)|^{2k}) d\mu_M(z) = 0$. Thus $|F(z)| = 1$ on M which is a contradiction.

Remarks. (i) For $m = 1$ we have $C_M{}^2(\zeta : v) \leqq R_1(\zeta : v) = S_M{}^2(\zeta : v)$. A special case of this result when M is bounded was proved by Look [**7**], using a different method. See also [**4**].

(ii) We actually proved a little more. Let

$$\nu_m = \nu_m(\zeta : v) = \frac{\lambda_0}{(m!)^2 C^{2m} \lambda_m}$$

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and

$$\mu_m = \mu_m(\zeta : v) = \frac{R_m(\zeta : v)}{C^{m(m+1)}}.$$

Then $\mu_1 = \nu_1 \geqq 1$ and $\mu_m \geqq \nu_m \geqq 1$, $m > 1$, where we have strict inequalities in the inequalities when $d\mu_m$ acts on M. This again implies Theorem 1.

Assume now that $S_M^2(\zeta : v) > 0$ (this, of course, occurs, according to Theorem 1, if $\zeta \in \mathscr{M}$ and $(\zeta, v) \in \mathscr{A}$). Then, using the summation convention,

$$S_M^{\,2}(\zeta : v) = T_{j\bar{k}} v_j \bar{v}_k; \quad T_{j\bar{k}} = \frac{\partial^2}{\partial \zeta_j \partial \bar{\zeta}_k} \log K,$$

and so $S_M^2(\zeta : v)$ is a Kähler metric. Let $T^{j\bar{k}}$ be the elements of the inverse matrix of the matrix $(T_{j\bar{k}})$. The sectional Riemannian curvature at $\zeta \in M$ in a two-dimensional holomorphic direction is given by

$$R = R_M(\zeta : v) = S_M^{-4}(\zeta : v) R_{\bar{q} l p \bar{m}} \bar{v}_q v_l v_p \bar{v}_m,$$

where

$$R_{\bar{q} l p \bar{m}} = -\frac{\partial^2 T_{p\bar{q}}}{\partial \zeta_l \partial \bar{\zeta}_m} + T^{j\bar{k}} \frac{\partial T_{p\bar{k}}}{\partial \zeta_l} \frac{\partial T_{j\bar{q}}}{\partial \bar{\zeta}_m}.$$

Here, again, we used the summation convention. Then (cf. Fuks [**3**])

(2.5) $\quad \mu^{-1} = (2 - R) K S^4,$

where

$$\mu = \mu(\zeta : v) = \min \{ ||f||^2 : f \in A(\zeta : v) \},$$
$$A(\zeta : v) = \{ f \in H_2(M) : f(\zeta) = \partial f/\partial \zeta_1 = \ldots = \partial f/\partial \zeta_n = 0, \quad \partial_v^2 f(\zeta) = 1 \}.$$

Clearly, if M is bounded and $\mu_M(M) < \infty$ then $R < 2$ at each point and every direction. We have

THEOREM 2. *Let $\zeta \in \mathscr{M} \subset M$ and $(\zeta, v) \in \mathscr{A}$. Then*
(i) $4C^4 \leqq (2 - R) S^4$, $C = C_M(\zeta : v)$, *and hence* $R < 2$.
(ii) *If $d\mu_M$ acts on M then* $4C^4 < (2 - R) S^4$.

Proof. Exactly as in Theorem 1. φ_2 of (2.1) belongs to $A(\zeta : v)$. Thus $||\varphi_2||^2 \geqq \mu$. But $||\varphi_2||^2 \leqq 1/4C^4 K$ and the assertion follows from (2.5).

If M is a bounded domain and S is the Bergman metric (i.e., $d\mu_M$ is the usual volume element of M) we have the following improvement on Fuks' result [**3**].

THEOREM 3. *Let M be a bounded domain in $\mathbf{C}^n$. The Riemannian curvature of the Bergman metric on M satisfies*

$$R_M(\zeta : v) < 2 - \frac{4}{(n+1)^2} \left(\frac{d_M(\zeta)}{\rho_M} \right)^{4(n+1)}; \quad \zeta \in M, v \in \mathbf{C}^n - \{0\},$$

where $d_M(\zeta) = \operatorname{dist}(\zeta, \partial M)$ *and ρ_M is the radius of the smallest ball containing M.*

Proof. Let ζ be fixed in M and let $t \in \mathbf{C}^n$ be the center of the smallest ball containing M. Then $B \subset M \subset A$ where $B = B(\zeta; d_M(\zeta))$ and $A = B(t; \rho_M)$. Here $B(z_0; r) = \{z \in \mathbf{C}^n : ||z - z_0|| < r\}$, $r > 0$, with $||z|| = (z, z)^{1/2}$ and $(z, w) = \sum_{j=1}^n z_j \bar{w}_j$ for $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$ in $\mathbf{C}^n$. In the Bergman metric case we have $\lambda_j^{(B)} \leqq \lambda_j^{(M)} \leqq \lambda_l^{(A)}$, $j = 0, 1, \ldots$. Here $\lambda_j = \lambda_j(\zeta : v)$. Therefore,

$$(2.6) \qquad S_M^2(\zeta : v) = \frac{\lambda_0^{(M)}}{\lambda_1^{(M)}} \leqq S_B^2(\zeta : v) \frac{\lambda_0^{(A)}}{\lambda_0^{(B)}}.$$

Clearly, $C_M(\zeta : v) \geqq C_A(\zeta : v)$. For the unit ball $\Omega = B(0; 1)$ we have [**8**]

$$C_\Omega^2(z : v) = \frac{(1 - ||z||^2)||v||^2 + |(z, v)|^2}{(1 - ||z||^2)^2}.$$

Also, the Bergman kernel for Ω is

$$K_\Omega(z, \bar{z}) = \frac{n!}{\pi^n}(1 - ||z||^2)^{-(n+1)}$$

and therefore

$$(2.7) \qquad S_\Omega^2(z : v) = (n + 1) C_\Omega^2(z : v).$$

$w = r^{-1}(z - z_0)$ maps $B(z_0; r)$ biholomorphically onto Ω and so

$$C^2_{B(z_0;r)}(z : v) = C_\Omega^2(w; v^*) = r^{-2} C_\Omega^2(w; v).$$

Consequently

$$(2.8) \qquad r^{-2}||v||^2 \leqq \frac{||v||^2}{r^2 - ||z - z_0||^2} \leqq C^2_{B(z_0:r)}(z_0: v) \leqq r^2 \frac{||v||^2}{(r^2 - ||z - z_0||^2)^2}.$$

Similarly

$$(2.9) \qquad K_{B(z_0:r)}(z : \bar{z}) = \frac{n!}{\pi^n} r^2 (r^2 - ||z - z_0||^2)^{-(n+1)}.$$

By (2.5) - (2.9) and using the fact that $K = 1/\lambda_0$ we have

$$(2.10) \qquad S_M^4(\zeta : v) \leqq (n + 1)^2 \left(\frac{\rho_M}{d_M(\zeta)}\right)^{4(n+1)} \frac{||v||^4}{\rho_M^4}.$$

Likewise

$$(2.11) \qquad C_M^4(\zeta : v) \geqq \frac{||v||^4}{\rho_M^4}.$$

The theorem now follows from Theorem 2 (ii) and (2.10) - (2.11).

Before we turn to the case where M is a complex manifold we make the following observation. Let $\phi: M \to M^*$ be a biholomorphic mapping of M on M^* with the non-vanishing Jacobian $J_\phi = \partial w/\partial z$, $w = \phi(z)$. Assume that there

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is an $\alpha > 0$ such that $[J_\phi(z)]^\alpha$ is holomorphic in M and that $\mu_{M^*}(\phi(N)) = \int_N |J_\phi(z)|^{2\alpha} d\mu_M(z)$ for each Borel subset N of M. Then, $f \to (f_0\phi)J_\phi^\alpha$ is an isometry of $H_2(M^*)$ onto $H_2(M)$. Therefore

$$(2.12) \quad K_M(z, \bar{z}) = K_{M^*}(\phi(z), \phi(z))|J_\phi(z)|^{2\alpha}, \quad z \in M.$$

THEOREM 4. *Let the assumptions of formula* (2.12) *prevail. Then* (*see formula* (2.4))

$$R_m^{(M)}(z; v) = R_m^{(M^*)}(\phi(z); \phi_*(v)), \quad m = 1, 2, \ldots.$$

That is, the non-negative function $R_m(z; v)$ is biholomorphic invariant.

Proof. We have $J_m^{(M)}(z; v) = W_z[K_M, \bar{\partial}_v K_M, \ldots, \bar{\partial}_v^m K_M]$, $K_M = K_M(z, \bar{z})$. Using (2.12) and standard properties of Wronskians we obtain, writing $K_{M^*} = K_{M^*}(w, \bar{w})$, $w = \phi(z)$ and $J = J_\phi(z)$,

$$\begin{aligned} J_m^{(M)}(z; v) &= W_z[J^\alpha \bar{J}^\alpha K_{M^*}, J^\alpha \bar{\partial}_v(\bar{J}^\alpha K_{M^*}), \ldots, J^\alpha \bar{\partial}_v^m(\bar{J}^\alpha K_{M^*})] \\ &= (J^\alpha)^{m+1} W_z[\bar{J}^\alpha K_{M^*}, \bar{\partial}_v(\bar{J}^\alpha K_{M^*}), \ldots, \bar{\partial}_v^m(\bar{J}^\alpha K_{M^*})] \\ &= (J^\alpha)^{m+1} W_z\Big[\bar{J}^\alpha K_{M^*}, \sum_{j=0}^{1} \binom{1}{j} \bar{\partial}_v^j \bar{J}^\alpha \bar{\partial}_v^{1-j} K_{M^*}, \ldots, \\ &\qquad \sum_{j=0}^{m} \binom{m}{j} \bar{\partial}_v^j \bar{J}^\alpha \bar{\partial}_v^{m-j} K_{M^*}\Big] \\ &= (J^\alpha)^{m+1} (\bar{J}^\alpha)^{m+1} W_z[K_{M^*}, \bar{\partial}_v K_{M^*}, \ldots, \bar{\partial}_v^m K_{M^*}] \\ &= |J|^{2\alpha(m+1)} W_w[K_{M^*}, \bar{\partial}_{v^*} K_{M^*}, \ldots, \bar{\partial}_{v^*}^m K_{M^*}], \end{aligned}$$

where $w = \phi(z)$ and $v^* = \phi_*(v)$. Consequently,

$$J_m^{(M)}(z : v) = J_m^{(M^*)}(\phi(z); \phi_*(v))|J_\phi(z)|^{2\alpha(m+1)}.$$

The theorem now follows from this, (2.12) and (2.4).

Remarks. Some comments about the nature of the function $R_m(z, v)$ are in order. Clearly, $R_0(z; v) \equiv 1$. For $m \geqq 1$, however, let the assumptions of formula (2.12) prevail. Then, a close examination of the definition of $R_m(z : v)$ via (2.2) - (2.4) and Theorem 4 shows that, for each $z \in M$, $R_m(z; \quad)$ is a function which is defined on the complex tangent space of M at z. For $m = 1$, $R_1(z : v) = S^2(z : v)$ is a complex tensor field of M which is covariant of degree 2. It behaves exactly as the Bergman metric and induces a Riemannian structure on M which is Kählerian.

3. Extension to manifolds. We now assume that M is a complex n-dimensional manifold. For sake of simplicity we consider only the Bergman kernel case. The extension of the Bergman kernel function theory to complex manifolds is by now standard (cf. [**5** and **6**]). However, here we take a slightly different course.

Let $F_2(M)$ be the space of holomorphic n-forms α on M normed by $||\alpha||^2 = i^{n2}\int_M \alpha \wedge \bar{\alpha}$. Here, $\alpha = a dz_1 \wedge \ldots \wedge dz_n$ in a local coordinate neighborhood U of $z \in M$. Since $|a(z)|^2$ is plurisubharmonic in U, we have, for each compact subset A of U

(3.1) $\quad |a(\zeta)| \leqq N_A ||\alpha||, \quad \zeta \in A \subset U,$

where N_A is a positive constant depending only on A. Therefore, $F_2(M)$ is a separable Hilbert space with the scalar product

(3.2) $$(\alpha, \beta) = i^{n2} \int_M \alpha \wedge \bar{\beta}.$$

Let $v \in \mathbf{C}^n - \{0\}$. For a fixed $\zeta \in U \subset M$ we let

(3.3) $\quad D^m_{v,\zeta}\, \alpha = \partial_v{}^m a(\zeta), \quad m = 0, 1, \ldots.$

We write $l_\zeta = D^0_{v,\zeta}$ and thus $l_\zeta \alpha = a(\zeta)$. According to (3.1) each $D^m_{v,\zeta}$ is a bounded linear functional on $F_2(M)$ and therefore there exists a unique $\mathscr{K}_\zeta$ in $F_2(M)$ such that

(3.4) $\quad l_\zeta \alpha = a(\zeta) = (\alpha, \mathscr{K}_\zeta)$

for every $\alpha \in F_2(M)$. We write $\mathscr{K}(\ \ , \bar{\zeta}) = \mathscr{K}_\zeta$ and so

(3.5) $\quad \mathscr{K}(z, \bar{\zeta}) = k(z, \bar{\zeta}) dz_1 \wedge \ldots \wedge dz_n$

in a local neighborhood V of $z \in M$. Let $\eta \in V \subset M$. According to (3.4)

(3.6) $\quad k(\zeta, \bar{\eta}) = (\mathscr{K}_\eta, \mathscr{K}_\zeta) = \overline{(\mathscr{K}_\zeta, \mathscr{K}_\eta)} = \overline{k(\eta, \bar{\zeta})}$

and especially $k(\zeta, \bar{\zeta}) \geqq 0$. $\mathscr{K}_\zeta = \mathscr{K}(\ \ , \bar{\zeta})$ is called the *Bergman n-form* at $\zeta \in M$ and in terms of a local coordinate system z of M it is given by (3.5) with a locally defined Bergman kernel function $k(z, \bar{\zeta})$, $(z, \zeta) \in V \times U$. Clearly, $k(z, \bar{\zeta})$ is holomorphic in $V \times \bar{U}$, where $\bar{U}$ is the complex conjugate neighborhood of U.

We should emphasize again that the Bergman n-form is not a globally defined form. On the other hand one can also define, although not essential for our present purposes, the Bergman $2n$-form (cf. Kobayashi [**5**])

$$K(z, \bar{\zeta}) = \mathscr{K}(z, \bar{\zeta}) d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_n$$

which is invariant under the group of holomorphic transformations of M. $K(\ \ , \bar{\zeta})$ does not belong to $F_2(M)$. However, it reproduces $F_2(M)$ in the following sense; for fixed $\zeta \in M$ and each $\alpha \in F_2(M)$

$$\alpha(\zeta) = i^{n^2} \int_M \alpha \wedge \overline{K(\ \ , \bar{\zeta})}.$$

This immediately follows from (3.2) and (3.4) - (3.5). Of course, $K(z, \bar{\zeta})$ can be expressed in the usual way in terms of an orthonormal basis for $F_2(M)$ and it is a holomorphic $2n$-form on $M \times \bar{M}$. We also note that the latter $2n$ form differs from our n-form by only a constant in a given coordinate neighborhood.

LEMMA 1. *The Riesz representer of* $D^m_{v,\zeta}$ *in* $F_2(M)$ *is* $\bar{\partial}_v{}^m\mathscr{K}(\ ,\bar{\zeta})$, *where, in a local neighborhood* V *of* $z \in M$, $\bar{\partial}_v{}^m\mathscr{K}(z,\bar{\zeta}) = \bar{\partial}_{v,\zeta}{}^m k(z,\bar{\zeta})dz_1 \wedge \ldots \wedge dz_n$. *Moreover*, $(D^m_{v,\zeta}, D^k_{v,\zeta}) = \partial_v{}^k\bar{\partial}_v{}^m k(\zeta,\bar{\zeta})$ *and so* $\|D^m_{v,\zeta}\|^2 = \partial_v{}^m\bar{\partial}_v{}^m k(\zeta,\bar{\zeta})$.

Proof. This follows from (3.1) - (3.6).

The Carathéodory metric on the manifold M is defined exactly as before. The same applies for the set $\mathscr{A}$ namely, $\mathscr{A} = \{(\zeta, v) \in M \times (\mathbf{C}^n - \{0\})$: there exists $f \in H(M:\Delta)$ with $\partial_v f(\zeta) \neq 0\}$. It is open in $M \times (\mathbf{C}^n - \{0\})$ and for $(\zeta, v) \in \mathscr{A}$, $\partial_v F(\zeta) = C(\zeta : v) > 0$ with $F \in H_\zeta(M:\Delta)$. Let $\mathscr{M}_1 = \{\zeta \in M$: there exists $\alpha \in F_2(M)$ with $\alpha(\zeta) \neq 0\}$. Again $\mathscr{M}_1$ is open in M and $\zeta \in \mathscr{M}_1$ implies, using (3.6), that $k(\zeta, \bar{\zeta}) > 0$.

We are now in a position to formulate minimum problems which are similar to the ones stated previously. Let

$$A_m(\zeta : v) = \{\alpha \in F_2(M): D^j_{v,\zeta}\alpha = \delta_{jm}, j = 0, 1, \ldots, m\}, \quad m = 0, 1, \ldots.$$

As before, $A_m(\zeta : v)$ is a closed convex subset of $F_2(M)$. Moreover, $A_0(\zeta : v) \neq \emptyset$ if $\zeta \in \mathscr{M}_1$ and $A_m(\zeta : v) \neq \emptyset$, $m \geqq 1$, if also $(\zeta, v) \in \mathscr{A}$. Indeed, the n-form

$$\varphi_m = \frac{F^m\mathscr{K}_\zeta}{m!C^m k(\zeta,\bar{\zeta})}, \quad C = C_M(\zeta : v),$$

is in $A_m(\zeta : v)$ for each $m = 0, 1, \ldots$. Let ψ_m be the unique solution to the minimum problem

$$\lambda_m = \lambda_m(\zeta : v) = \min\{\|\alpha\|^2 : \alpha \in A_m(\zeta : v)\}.$$

By Lemma 1,

$$\lambda_m = J_{m-1}/J_m,$$

where

$$J_m = J_m(\zeta : v) = W_\zeta[k, \bar{\partial}_v k, \ldots, \bar{\partial}_v{}^m k], \quad k = k(\zeta, \bar{\zeta}).$$

Here again, $J_{-1} \equiv 1$, $J_0 = k(\zeta, \bar{\zeta})$ and $J_m = (\prod_{j=0}^m \lambda_j)^{-1}$.

The function

$$R_m(\zeta : v) = \left(\prod_{j=1}^m j!\right)^{-2} k^{-(m+1)} J_m(\zeta : v), \quad k = k(\zeta, \bar{\zeta}),$$

is independent of a choice of a coordinate system and exactly as in Theorem 4 is invariant under the holomorphic transformations of M. Concerning the nature of $R_m(\zeta : v)$ see the remarks at the end of the previous section. Especially, $R_1(\zeta : v) = S^2(\zeta : v)$ is the Bergman metric given by

$$(3.7) \qquad S^2(\zeta : v) = T_{i\bar{j}} v_i \bar{v}_j, \quad T_{i\bar{j}} = \frac{\partial^2}{\partial\zeta_i \partial\bar{\zeta}_j} \log k(\zeta, \bar{\zeta}).$$

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THEOREM 5. *Let* $\zeta \in \mathcal{M}_1 \subset M$. *Then*
(i) $[C_M(\zeta : v)]^{m(m+1)} \leqq R_m(\zeta : v)$ *for each* $v \in \mathbf{C}^n - \{0\}$.
(ii) *If also* $(\zeta, v) \in \mathcal{A}$, *then* $[C_M(\zeta : v)]^{m(m+1)} < R_m(\zeta : v)$.

Proof. Exactly as in Theorem 1. We only have to prove (ii). We have $||\varphi_j||^2 \geqq \lambda_j$. But

$$||\varphi_j||^2 = \frac{||F^j\mathcal{K}_\zeta||^2}{(j!)^2C^{2j}k(\zeta, \bar{\zeta})^2} < \frac{1}{(j!)^2C^{2j}k(\zeta, \bar{\zeta})}$$

because F is not constant on M. The proof now proceeds as in Theorem 1.

The remarks following Theorem 1 also apply here. Moreover, for $\zeta \in \mathcal{M}_1$ and $(\zeta, v) \in \mathcal{A}$ the Bergman metric is strictly positive by Theorem 5 (ii). Therefore, we can define the sectional Riemannian curvature $R = R_M(\zeta : v)$ as before where now the $T_{i\bar{j}}$ are given in (3.7). Here

$$\mu^{-1} = (2 - R)kS^4; \quad k = k(\zeta, \bar{\zeta}), S = S(\zeta : v),$$

with $\mu = \mu(\zeta : v) = \min \{||\alpha||^2 : \alpha \in A(\zeta : v)\}$ and

$$A(\zeta : v) = \{\alpha \in F_2(M) : l_\zeta\alpha = D_{e_j,\zeta}\alpha = 0, \quad j = 1, \ldots, n, \quad D^2_{v,\zeta}\, \alpha = 1\}.$$

Here, e_j stands for the unit vector $(0, \ldots, 0, 1_j, 0, \ldots, 0) \in C^n$. Exactly as in Theorem 2 we have:

THEOREM 6. *Let* $\zeta \in \mathcal{M}_1 \subset M$ *and* $(\zeta, v) \in \mathcal{A}$. *Then* $4C^4 < (2 - R)S^4$, $C = C_M(\zeta : v)$, $S = S(\zeta : v)$, *and in particular* $R(\zeta : v) < 2$.

We conclude this section by remarking that the same results, in view of Theorem 4, could be obtained if we introduce the "volume element" $v_M^{(\alpha)} = i^{n^2}k^{1-\alpha}dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$, $k = k(z, \bar{z})$, α an integer $\geqq 2$. One then speaks about automorphic forms of weight α instead of holomorphic forms. We shall not pursue this topic further.

4. Plane domains. If M is a plane domain having non constant bounded analytic functions then the Carathéodory metric becomes $C(\zeta)|d\zeta|$. Here $C(\zeta) = C_M(\zeta) = \sup \{|f'(\zeta)| : f \in H_\zeta(M : \Delta)\}$ is the analytic capacity of M at ζ. There exists a unique $F \in H_\zeta(M : \Delta)$, called the *Ahlfors function*, with $F'(\zeta) = C(\zeta) > 0$. According to Theorem 1, for $\zeta \in \mathcal{M}$,

$$(4.1) \qquad C(\zeta)^{m(m+1)} \leqq \left(\prod_{k=1}^{m} k!\right)^{-2} K^{-(m+1)} J_m(\zeta), \quad m \geqq 1,$$

with $J_m(\zeta) = \det ||K_{j\bar{k}}||_{j,k=0}{}^m$ and $K = K(\zeta, \bar{\zeta})$. Here $K_{j\bar{k}} = \partial^{j+k}K/(\partial\zeta^j\partial\bar{\zeta}^k)$. If $d\mu_M$ acts on M we have a strict inequality in (4.1). This is the case if for example K is the Bergman kernel.

If one assumes that M is bounded by a finite number of analytic curves then

$C(\zeta) = 2\pi\hat{K}(\zeta, \bar{\zeta})$ (cf. [**1**, p. 118] where $\hat{K}(\zeta, \bar{\zeta})$ is the Szegö kenrel for M. Hence

$$(4.2) \qquad C^{(m+1)^2} \leqq \left(\prod_{k=1}^{m} k!\right)^{-2} \det ||C_{j\bar{k}}||_{j,k=0}^{m}, \quad m \geqq 1,$$

where $C_{j\bar{k}} = \partial^{j+k}C/(\partial\zeta^j\partial\bar{\zeta}^k)$, $C = C(\zeta)$. Equality holds, at one point $\zeta \in M$ and any $m \geqq 1$, if and only if M is simply connected [**2**].

In the general case one uses a canonical exhaustion process (cf. [**9**]) to show that $C(\zeta)$ is real analytic and that (4.2) still holds. Especially, for $m = 1$, we have that $C^2 \leqq \frac{1}{4} \Delta \log C$, ($\Delta$ is the Laplacian), or that the curvature of $C(\zeta)|d\zeta|$ is always $\leqq -4$. This last result is due to Suita [**9**] and is generalized in [**2**].

References

1. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys *5* (Amer. Math. Soc., Providence, 1970).

2. J. Burbea, *The curvatures of the analytic capacity*, submitted for publication.

3. B. A. Fuks, *On the Ricci curvature of a Bergman metric invariant under biholomorphic mappings*, Dokl. Akad. Nauk SSSR *167* (1966), 996–999.

4. K. T. Hahn, *On completeness of the Bergman metric and its subordinate metric*, to appear, Proc. Nat. Acad. Sc. U.S.A.

5. S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. *92* (1959), 267–290.

6. A. Lichnerowicz, *Variétés complexes et tenseur de Bergmann*, Ann. Inst. Fourier (Grenoble) *15* (1965), 345–408.

7. K. H. Look, *Schwarz Lemma and analytic invariants*, Sci. Sinica *7* (1958), 453–504.

8. H. J. Reiffen, *Die differentialgeometrischen Eigenschaften der invarianten Distanzfunktion von Carathéodry*, Schrift Math. Inst. Univ. Münster *26* (1963).

9. N. Suita, *On a metric induced by analytic capacity*, Kodai Math. Sem. Rep. *25* (1973), 215–218.

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