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EVEN WHITEHEAD SQUARES ARE NOT PROJECTIVE

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The projectivity of the Whitehead square $w_n = [i_n, i_n]$ in $\pi_{2n-1}(S^n)$ has been studied by Randall [**6**] who proved that if w_n is projective then n must be a power of 2 or one less than a power of 2. Here we solve the question in the even case, proving by means of b_0 homology:

THEOREM. $w_{2n} \in \pi_{4n-1}{}^{\mathrm{Proj}}(S^{2n})$ *if and only if* $n = 1, 2, 4$.

We should also note that all attempts by either author to prove this result using ordinary K-cohomology theory have failed. Hence, the techniques developed here may be of independent interest. Since writing this note it has come to our attention that Randall has also proved the "only if" part of the above theorem, by rather different methods [**7**].

1. Factorization of w_{2n}**.** For completeness we first give a short proof due to I. M. James, which simplifies a previous proof of ours, that w_2, w_4, and w_8 are projective.

1.1 PROPOSITION. $w_{2n} \in \pi_{4n-1}{}^{\mathrm{Proj}}(S^{2n})$ *if* $n = 1, 2, 4$.

Proof. Let ξ_2, ξ_4, ξ_8 be the respective Hopf maps, and let i_{2n} generate $\pi_{2n}(S^{2n})$. Since ξ_{2n} is projective, so is $(2i_{2n}) \circ \xi_{2n}$ by naturality. From the left distributive law (cf. [**2**, p. 93])

$$(2i_{2n}) \circ \xi_{2n} = \xi_{2n} + \xi_{2n} + H(\xi_{2n}) \cdot w_{2n} = 2\xi_{2n} + w_{2n}.$$

Hence, by [**4**, Lemma 1.1], w_{2n} is projective.

From now on we suppose that

$$w_{2n} : S^{4n-1} \to S^{2n}$$

is projective, which gives a factorization

$$\begin{tikzcd} S^{4n-1} \arrow[rr, "w_{2n}"] \arrow[dr, "\gamma"'] & & S^{2n} \\ & P_{2n}{}^{4n-1} \arrow[ur, "f"'] & \end{tikzcd}$$

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where $P_{2n}^{4n-1} = RP^{4n-1}/RP^{2n-1}$, and γ is the composition of the collapsing map with the standard double covering.

We now sharpen this somewhat by proving:

1.2 PROPOSITION. *For some odd number t there is a factorization*

$$\begin{tikzcd} S^{4n-1} \arrow[rr, "w_{2n}\cdot t"] \arrow[dr, "\gamma"'] & & S^{2n} \\ & P_{2n+1}^{4n-1} \arrow[ur, "h"'] & \end{tikzcd}$$

if $n \neq 1, 2, 4$.

Proof. We need two lemmas.

1.3 LEMMA. *There exists a map* $\delta : P_{2n}^{4n-1} \to P_{2n}^{4n-1} \vee S^{2n}$ *such that*
(a) $\delta_*(e_{2n}) = e_{2n} + 2\epsilon_{2n}$ *in* $H_{2n}(\ \ ; Z)$, *and*
(b) $[\delta\gamma] = ([\gamma], 0, s[i_{2n}', i_{2n}]) \in \pi_{4n-1}(P_{2n}^{4n-1} \vee S^{2n})$ *for some integer s.*

Here e_{2n}, ϵ_{2n} are the respective homology generators of $H_{2n}(P_{2n}^{4n-1})$, $H_{2n}(S^{2n})$, and i_{2n}' generates $\pi_{2n}(P_{2n}^{4n-1}) = Z$.

Proof. Let $p : P_{2n}^{4n-1} \to S^{2n}$ be defined by the cofibration sequence:

$$\begin{tikzcd} S^{2n-1} \arrow[r, hook, "J"] & P_{2n-1}^{4n-1} \arrow[r] & P_{2n}^{4n-1} \arrow[r, "p"] & S^{2n} \\ & & S^{4n-1} \arrow[ul, "\gamma'"] \arrow[u, "\gamma"'] & \end{tikzcd}$$

The diagram also shows that $p\gamma$ is null homotopic, and one readily finds that $p_*(e_{2n}) = 2\epsilon_{2n}$. We now define δ by means of the following diagram and cellular approximation:

$$\begin{tikzcd} P_{2n}^{4n-1} \arrow[r, dashed, "\delta"] \arrow[d, "\Delta"] & P_{2n}^{4n-1} \vee S^{2n} = (P_{2n}^{4n-1} \times S^{2n})_{(4n-1)} \arrow[d, hook, "i"] \\ P_{2n}^{4n-1} \times P_{2n}^{4n-1} \arrow[r, "1 \times p"] & P_{2n}^{4n-1} \times S^{2n}. \end{tikzcd}$$

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Property (a) is immediate, and

$$i_*[\delta\gamma] = [(1 \times p]\Delta\gamma] = ([\gamma], [p\gamma]) = ([\gamma], 0)$$

so

$$[\delta\gamma] = ([\gamma], 0, s \cdot [i', i])$$

as desired.

1.4 LEMMA. *$f_* : H_{2n}(P_{2n}{}^{4n-1}) \to H_{2n}(S^{2n})$ has even degree provided $n \neq 1, 2, 4$.*

Proof. If $\deg f_* = 2k + 1$ is odd, then the composition

$$P_{2n}{}^{4n-1} \xrightarrow{\quad\delta\quad} P_{2n}{}^{4n-1} \vee S^{2n} \xrightarrow{f \vee (-k)} S^{2n} \vee S^{2n} \xrightarrow{\quad F\quad} S^{2n}$$

has degree 1 in $H_{2n}(\quad; Z)$, contradicting [**1**, Theorem 1.2] unless $n = 1, 2, 4$.

Now we can complete the proof of 1.2. By 1.4, $\deg f_* = 2k$ in H_{2n} for some k. The composition $F \circ (f \vee (-k)) \circ \delta$ then has degree 0 in H_{2n}, hence factors through $P_{2n+1}{}^{4n-1}$. Furthermore, by 1.3(b)

$$\begin{aligned}[F \circ (f \vee (-k)) \circ \delta \circ \gamma] &= F_*(f \vee (-k))_*([\gamma], 0, s[i', i]) \\ &= F_*(w_{2n}, 0, s[2ki, -ki]) = w_{2n}(1 - 2sk^2).\end{aligned}$$

Using 1.2 we now produce a further factorization of $w_{2n} \cdot t$. Consider the fibration

$$F \xrightarrow{u} S^{2n} \xrightarrow{i} K(Z, 2n).$$

Since $P_{2n+1}{}^{4n-1}$ is $2n$ connected, $h : P_{2n+1}{}^{4n-1} \to S^{2n}$ can be factored through F, giving a commutative diagram

$$\begin{tikzcd}
P_{2n+1}{}^{4n-1} \arrow[r, "g"] \arrow[dr, "h"] & F \arrow[d, "u"] \\
S^{4n-1} \arrow[u, "\gamma"] \arrow[r, "wt"] & S^{2n}
\end{tikzcd}$$

According to [**5**, p. 153], the natural map

$$\psi : \Sigma F \to K(Z, 2n)/S^{2n}$$

induces isomorphisms in homology through dimension $4n$. We write $x_i \in H^i(P_{2n+1}{}^{4n-1}; Z/2)$, $\iota \in H^{2n}(K(Z, 2n); Z/2)$ for the respective generators, and set $y = \Sigma^{-1}\psi^*(\iota^2) \in H^{4n-1}(F; Z/2)$, where we identify $H^q(K(Z, 2n)) \approx H^q(K(Z, 2n)/S^{2n})$, $q > 2n$.

1.5 LEMMA. $g^*(y) = x_{4n-1}$.

Proof. Clearly, $\mathrm{Sq}_u^{2n}(\iota) = y$ with zero indeterminacy. Now consider the cofibration sequence

$$P_{2n+1}{}^{4n-1} \xrightarrow{h} S^{2n} \xrightarrow{j} C_h \xrightarrow{l} \Sigma P_{2n+1}{}^{4n-1},$$

and let v generate $H^{2n}(C_h; Z) \approx Z$, $\bar{x}_{4n-1}$ generate $H^{4n-1}(P_{2n+1}{}^{4n-1}; Z) \approx Z$, so $l^*(\Sigma\bar{x}_{4n-1})$ generates $H^{4n}\ (C_h; Z) \approx Z$. Following the proof of Theorem A in [**6**] we obtain

$$v \cup v = t^2 l^*(\Sigma \bar{x}_{4n-1}).$$

In $Z/2$-cohomology this implies $\mathrm{Sq}_h^{2n}(\iota) = x_{4n-1}$ with zero indeterminacy, so by naturality

$$x_{4n-1} = \mathrm{Sq}_{ug}{}^{2n}(\iota) = g^*\mathrm{Sq}_u{}^{2n}(\iota) = g^*(y).$$

1.6 COROLLARY. *Writing τ for the composition*

$$\Sigma P_{2n+1}{}^{4n-1} \xrightarrow{\Sigma g} \Sigma F \xrightarrow{\psi} K(Z, 2n)/S^{2n}$$

then $\tau^*(\iota^2) = \Sigma(x_{4n-1})$ *in* $H^{4n}(\Sigma P_{2n+1}{}^{4n-1}; Z/2)$.

2. Application of b_0-homology. Now we show that there can be no map τ satisfying 1.6 ($n \neq 1, 2, 4$ as before). If $n \equiv 1, 3\ (4)$ then $\iota^2 = \mathrm{Sq}^2(\mathrm{Sq}^{2n-2}\iota)$ in $H^*(K/S^{2n})$ but x_{4n-1} is not in im (Sq^2). Similarly, using the relation

$$\mathrm{Sq}^{8s+4}\iota = \mathrm{Sq}^2(\mathrm{Sq}^{8s}\mathrm{Sq}^2\iota) + \mathrm{Sq}^1(\mathrm{Sq}^{8s}\mathrm{Sq}^3\iota) + \mathrm{Sq}^4(\mathrm{Sq}^{8s}\iota)$$

we see that 1.6 is impossible for $n \equiv 2(4)$. In the remainder of this section we use b_0-theory localized at 2 as in [**3**] (especially § 4) to prove the result for $n \equiv 0(4)$.

Let $\nu_2(m)$ be the greatest power of 2 occurring in m, so $\nu_2(8) = 3$ etc. Then we have

2.1 LEMMA. a) $b_{0(2n-1)}(P^{2n-1}) = Z/2^{n-1} \oplus Z_{(2)}$, *generated by* a, b *respectively, and for any operation* φ *detecting* Sq^4 (cf. [**4**]),

$$\varphi(b) = 2^{\nu_2(n)+1} \cdot (\mathrm{odd}) \cdot a.$$

b) $b_{0(4n-1)}(P_{2n+1}{}^{4n-1}) = Z/2^{n-1} \oplus Z_{(2)}$ *generated by* $\bar{a}, \bar{b}$ *respectively, and*

$$\varphi(\bar{b}) = 2^{\nu_2(n)+2} \cdot (\mathrm{odd}) \cdot \bar{a}.$$

Proof. (a) Apply $b_{0(2n)}$ to the cofibration sequence $S^0 \to M_{2n-1} \to \Sigma(P^{2n-1})$ (see e.g. [**4**, Lemma 4.1]) which shows $b_{0(2n)}(\Sigma P^{2n-1}) = Z/2^{n-1} \oplus Z_{(2)}$.

Next, consider the map $i : P^{2n-1} \to P^\infty$. From [**4**; 4.6] $b_{0(2n-1)}(P^\infty) = Z/2^n$ with generator λ and it is easily checked that $i_*(a) = 2\lambda$, $i_*(b) = \lambda$. Now [**4**; 4.6] shows $\varphi(\lambda) = 2^{\nu_2(n)+2} \cdot (\mathrm{odd}) \cdot \lambda$. On the other hand, if we project

$p : P^{2n-1} \to P^{2n-1}/P^{2n-2} = S^{2n+1}$ then $p_*(a) = 0$ and $p_*(b)$ is the generator θ in $b_{0(2n-1)}(S^{2n-1}) = Z_{(2)}$. But $\varphi(\theta) = 0$, hence $\varphi(b) = w \cdot a$ and (a) follows. The proof of (b) is similar.

2.2 PROPOSITION. *There is a map*

$$\kappa : \Sigma^{2n+1}(P^{2n-1}) \to K(Z, 2n)/S^{2n}$$

inducing $\kappa_* : b_{0(4n)}(\Sigma^{2n+1} P^{2n-1}) \to b_{0(4n)}(K(Z, 2n)/S^{2n})$ *and* κ_* *is injective onto a direct summand with complementary summand a direct sum of* $Z/2$'s.

Proof. Consider the diagram

$$\begin{tikzcd}
S^{\circ} \arrow[r] \arrow[d, "="] & M_{2n-1} \arrow[r] \arrow[d, "U_{2n-1}"] \arrow[ddd, bend right] & \Sigma^1 P^{2n-1} \arrow[d, "\kappa_{2n-1}"] \arrow[ddd, bend right] & \\
S^{\circ} \arrow[r] \arrow[d, "="] & \Sigma^{\circ} K(Z, 2n) \arrow[r] \arrow[d] & \Sigma^{\circ}(K(Z, 2n)/S^{2n}) \arrow[d] & \\
S^{\circ} \arrow[r] & K(Z, 0) \arrow[r] & K(Z, 0)/S^{\circ} \arrow[r] & X = K(Z, 0)/\Sigma^{\circ} K(Z, 2n) \\
S^{\circ} \arrow[r] \arrow[u, "="] & M_{\infty} \arrow[r] \arrow[u, "U_{\infty}"] & \Sigma^1 P^{\infty} \arrow[r] \arrow[u, "\kappa_{\infty}"] & \Sigma^1(P^{\infty}/P^{2n-1}) \arrow[u]
\end{tikzcd}$$

where U_{2n-1}, U_∞ represent the respective fundamental cohomology classes and κ_{2n-1}, κ_∞ are induced by the respective cofibration sequences. The result is verified for κ_∞ using the change of rings isomorphism $\mathrm{Ext}_{\mathscr{A}(2)}(H^*(X) \otimes \mathscr{A}(2)/\mathscr{A}(2)\overline{\mathscr{A}}_1, Z/2) = \mathrm{Ext}_{\mathscr{A}_1}(H^*(X), Z/2)$ and direct calculation to show $\mathrm{Ker}(\kappa_\infty{}^*)$ is a free sum of copies of $\mathscr{A}_1$, and then applying the Adams spectral sequence. Next, we consider the cofibration in the diagram

$$\Sigma^{\circ} K(Z, 2n) \to K(Z, 0) \to X = K(Z, 0)/\Sigma^{\circ} K(Z, 2n).$$

We claim

$$b_{0(i)}(X) = \begin{cases} 0 & i \leqq 2n \\ Z_{(2)} & i = 2n + 1. \end{cases}$$

To see this, note that the $Z/2$ cohomology of X through $2n + 3$ is

$$\ldots, 0, 0, \mathrm{Sq}^{2n+1}\iota, \mathrm{Sq}^{2n+2}\iota, \begin{cases} \mathrm{Sq}^{2n+3}\iota \\ \mathrm{Sq}^2\iota \cup \iota \end{cases}.$$

From this $\pi_{2n+1}(X \wedge b_0) = H_{2n+1}(X \wedge b_0) = Z_{(2)}$ and the claim follows.

Now the proof of 2.2 is easily completed when we check against our diagram and see that $b_{0(2n+1)}\Sigma^1(P^\infty/P^{2n-1}) = Z_{(2)}$ also and maps isomorphically to the $Z_{(2)}$ in $b_{0(2n)}(X)$, after we have checked $Z/2$ generators.

We now have our main result.

2.3 THEOREM. *For $n = 4m$, $m > 1$, there cannot exist a map $\tau : \Sigma P_{2n+1}{}^{4n-1} \to K(Z, 2n)/S^{2n}$ such that $\tau^*(\iota^2) = \Sigma x_{4n-1}$ in $H^{4n}(\Sigma P_{2n+1}{}^{4n-1}; Z/2)$.*

Proof. Assuming τ exists, application of $b_{0(4n)}$ gives

$$\tau_*(\Sigma\bar{b}) = r(\kappa_*(a)) + s(\kappa_*(b)) + \lambda$$

where λ is of order 2. The hypothesis on $\tau^*(\iota^2)$ guarantees that s is odd. Applying φ and using naturality gives

$$\begin{aligned} 2^{\nu_2(n)+2}\tau_*(\Sigma\bar{a}) &= r\, 2^{\nu_2(n)+2}\kappa_*(a) + s\, 2^{\nu_2(n)+1}\kappa_*(a) + \epsilon 2^{n-2}\kappa_*(a) \\ &= 2^{\nu_2(n)+1}\cdot(\text{odd})\cdot\kappa_*(a). \end{aligned}$$

But 2.1 and 2.2 show $\kappa_*(a)$ generates a direct summand in $b_{0(4n)}$ $(K(Z, 2n)/S^{2n})$ of order 2^{n-1}, and the above equation has no solution for $m > 1$.

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