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AUTOMATIC CONTINUITY OF SEPARATING LINEAR ISOMORPHISMS

BY

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ABSTRACT. A linear map $A : C(T) \longrightarrow C(S)$ is called separating if $f \cdot g \equiv 0$ implies $Af \cdot Ag \equiv 0$. We describe the general form of such maps and prove that any separating isomorphism is continuous.

Let T, S be compact Hausdorff spaces and let A be a linear map from the Banach space $C(T)$ into $C(S)$. The map A is said to be *separating* or *disjointness preserving* if $f \cdot g \equiv 0$ implies $Af \cdot Ag \equiv 0$ for all f, g in $C(T)$. For f in $C(T)$ or $C(S)$ we define the cozero set of f by $\text{coz}(f) = \{t : f(t) \neq 0\}$. Hence A is separating if and only if it maps functions with disjoint cozero sets into functions with disjoint cozero sets.

The concept of separating maps in this context was introduced by E. Beckenstein and L. Narici [5–7]. However, disjointness preserving maps between general vector lattices and similar automatic-continuity problems were considered earlier by other authors; see e.g., [1, 2, 8] and [3, 4]. In [7] the authors prove that if A is separating and satisfies a number of additional conditions then it is automatically continuous.

In this note we describe the general form of a separating linear map $A : C(T) \longrightarrow C(S)$. Roughly speaking we can always divide S into three subsets. On the first part A is just the zero map, on the second part A is given by a composition of a continuous map from a subset of S into T and a multiplication by a continuous scalar function. The third part of S is finite, possibly empty, and A is discontinuous at every point of this part. As a consequence we prove that any separating isomorphism is automatically continuous but we also show that there is always a discontinuous separating linear map A from $C(T)$ into $C(S)$, provided T is infinite.

Our results hold both in the real and in the complex case.

THEOREM. *Let A be a linear separating map from $C(T)$ into $C(S)$. Then S is a sum of three disjoint sets S_1, S_2, S_3 where S_2 is open and S_3 is closed, there is a continuous map $\varphi : S_1 \cup S_2 \longrightarrow T$ and a continuous, bounded, non-vanishing scalar-valued function χ on S_1 such that for any $f \in C(T)$*

$$(*) \qquad \begin{aligned} A(f)(s) &= \chi(s) f \circ \varphi(s) && \forall s \in S_1 \\ A(f)(s) &= 0 && \forall s \in S_3. \end{aligned}$$

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Furthermore the set $F = \varphi(S_2)$ *is finite, all functionals of the form*

$$C(T) \ni f \longmapsto A(f)(s) \quad \text{for } s \in S_2$$

are discontinuous and

$$A(f)|_{S_2} \equiv 0 \quad \text{if } \operatorname{supp} f \cap F = \emptyset.$$

PROOF. For any $s \in S$ we denote by δ_s the functional "*evaluation at the point s*". We define $S_3 = \{s \in S : \delta_s \circ A = 0\}$, $S_2 = \{s \in S : \delta_s \circ A$ is discontinuous$\}$ and $S_1 = S \setminus (S_2 \cup S_3)$. For any $s \in S$ we define $\operatorname{supp}(\delta_s \circ A)$ to be the set of all $t \in T$ such that for any open neighborhood U of t there is an f in $C(T)$ with $A(f)(s) \neq 0$ and $\operatorname{coz}(f) \subseteq U$. We contend that $\operatorname{supp}(\delta_s \circ A)$ contains at most one point. Assuming the contrary we get two open, disjoint sets U_1 and U_2, both having non-empty intersection with $\operatorname{supp}(\delta_s \circ A)$ and then $f_1, f_2 \in C(T)$ with $\operatorname{coz}(f_j) \subseteq U_j$, $A(f_j)(s) \neq 0$, $j = 1, 2$ which contradicts the assumption that A is separating. Assume now $\operatorname{supp}(\delta_s \circ A) = \emptyset$. Then there is an open finite cover of T, $T = U_1 \cup U_2 \cup \ldots \cup U_n$ such that $Af(s) = 0$ if $\operatorname{coz}(f) \subset U_j$, for some $j = 1, \ldots, n$. Let $\mathbf{1} = \sum_{j=1}^{n} f_j$ be a continuous decomposition of the identity subordinate to $\{U_j\}_{j=1}^{n}$. For any $f \in C(T)$ we have $Af(s) = A(\sum_{j=1}^{n} f_j f)(s) = \sum_{j=1}^{n} A(f_j f)(s) = 0$, and this means $\delta_s \circ A = 0$, so $s \in S_3$. Hence we can define a function $\varphi : S_1 \cup S_2 \longrightarrow T$ by $\{\varphi(s)\} = \operatorname{supp}(\delta_s \circ A)$.

Note that by exactly very similar arguments as above, we also get $Af(s) = 0$ for any $f \in C(T)$ such that $\varphi(s) \notin \overline{\operatorname{coz}(f)} =: \operatorname{supp} f$.

LEMMA 1. φ *is continuous.*

PROOF OF THE LEMMA. Assuming the contrary, by the compactness of T, there is a net $(s_\alpha)_{\alpha \in \Gamma}$ in $S_1 \cup S_2$ convergent to $s_0 \in S_1 \cup S_2$ such that $\varphi(s_\alpha) = t_\alpha$ converges to $t_1 \neq t_0 = \varphi(s_0)$. Let U_0, U_1 be open, disjoint neighborhoods respectively of t_0 and t_1, and let $f_0 \in C(T)$ be such that $\operatorname{coz}(f_0) \subset U_0$ and $Af_0(s_0) \neq 0$. Fix an $\alpha \in \Gamma$ such that $Af_0(s_\alpha) \neq 0$ and $t_\alpha \in U_1$. Let $f_1 \in C(T)$ be such that $\operatorname{coz}(f_1) \subset U_1$ and $Af_1(s_\alpha) \neq 0$. We get $f_0 \cdot f_1 \equiv 0$ but $Af_0 . Af_1(s_\alpha) \not\equiv 0$, which contradicts the assumption that A is separating.

The definition of φ and Lemma 1 are taken from [7]; we present the above proof here for the sake of completeness.

LEMMA 2. *Let* $(s_n)_{n=1}^{\infty}$ *be a sequence in* $S_1 \cup S_2$ *such that* $t_n = \varphi(s_n)$, $n \in \mathbf{N}$ *are distinct points of* T. *Then*

$$\lim\ \sup \|\delta_{s_n} \circ A\| < \infty.$$

Note that the above says, in particular, that $\|\delta_{s_n} \circ A\| < \infty$ for all, but finitely many $n \in \mathbf{N}$.

PROOF OF THE LEMMA. Assume the contrary. Taking an appropriate subsequence, we can assume without loss of generality that

$$\|\delta_{s_n} \circ A\| > n^2, \quad \forall n \in \mathbf{N}, \tag{1}$$

and that there is a sequence (U_n) of pairwise disjoint open subsets of T with $t_n \in U_n$. By the definition of φ and (1), there is a sequence (f_n) in $C(T)$ such that

$$\operatorname{supp} f_n \subset U_n, \|f_n\| \leqq 1/n, \quad \text{and } |Af_n(s_n)| \geqq n.$$

Put

$$f = \sum_{n=1}^{\infty} f_n.$$

By the comment preceding Lemma 1 we have

$$|Af(s_{n_0})| = |A(f_{n_0})(s_{n_0}) + A\left(\sum_{n \neq n_0} f_n\right)(s_{n_0})| = |A(f_{n_0})(s_{n_0})| \geqq n_0.$$

Hence Af is unbounded, which is not possible. This proves the lemma.

Put

$$F = \{t \in T : \sup\{\|\delta_s \circ A\| : s \in \varphi^{-1}(t)\} = \infty\}.$$

By Lemma 2, F is a finite set. We want to show that $F = \varphi(S_2)$. The inclusion $\varphi(S_2) \subseteq F$ is obvious by the definition of S_2; to show the converse one fix a $t \in F$ and define

$$\Phi : C(T) \longrightarrow C(\varphi^{-1}(t)) \quad \text{by } \Phi(f) = Af|_{\varphi^{-1}(t)}.$$

Since $t \in F$, the map Φ is discontinuous, and by the closed graph theorem there is a sequence $(f_n)_{n=1}^{\infty}$ in $C(T)$ convergent to 0 and such that $(\Phi(f_n))_{n=1}^{\infty}$ is convergent to a non-zero function $g_0 \in C(\varphi^{-1}(t))$. Let $s \in \varphi^{-1}(t)$ be such that $g_0(s) \neq 0$. We have $f_n \underset{n=\infty}{\longrightarrow} 0$ and $\delta_s \circ A(f_n) \longrightarrow g_0(s) \neq 0$ so $s \in S_2$ and hence $F \subseteq \varphi(S_2)$.

Fix now an $s \in S_1$ and put

$$J_s = \{f \in C(T) : \varphi(s) \notin \operatorname{supp} f\}$$
$$K_s = \{f \in C(T) : f(\varphi(s)) = 0\}.$$

Fix $g \in K_s$ and $\epsilon > 0$. Put $T_1 = \{t \in T : |g(t)| \geqq \epsilon\}$, $T_2 = \{t \in T : |g(t)| \leqq (1/2)\epsilon\}$ and let $g' \in C(T)$ be such that $\|g'\| = 1$, $g'|_{T_1} \equiv \mathbf{1}$, $g'|_{T_2} \equiv 0$. We have $g \cdot g' \in J_s$ and $\|g \cdot g' - g\| \leqq \epsilon$, so J_s is a dense subspace of K_s. Moreover, since $s \in S_1$, $\delta_s \circ A$ is a non-zero continuous functional and by the remark before Lemma 1 $J_s \subseteq \ker(\delta_s \circ A)$. Hence $K_s \subseteq \ker(\delta_s \circ A)$ and since the codimensions of these spaces are both equal to one we have $\ker(\delta_s \circ A) = K_s$ and so $\delta_s \circ A$ is of the form

$$\delta_s \circ A(f) = \chi(s) f(\varphi(s)), \quad \forall f \in C(T),$$

for some scalar $\chi(s) \neq 0$. Let $f \in C(T)$ be such that $f(\varphi(s)) \neq 0$. In some neighborhood of s, namely on $\{s \in S_1 : f(\varphi(s)) \neq 0\}$ we have $\chi = A(f)/f \circ \varphi$. Since s is an arbitrary point of S_1, by Lemma 1, χ is locally a well-defined quotient of two continuous functions and so is continuous itself on S_1. It is also a bounded function, since otherwise $A(\mathbf{1})$ would be unbounded.

It remains to prove that S_2 is open. For any $f \in C(T)$ we have

$$\sup\{|Af(s)| : s \in \overline{S_1 \cup S_3}\} = \sup\{|Af(s)| : s \in S_1 \cup S_3\} \leqq \|\chi\| \, \|f\|.$$

Hence $S_1 \cup S_3 = \{s \in S : \delta_s \circ A \text{ is continuous}\}$ is closed, and we are done.

From the theorem and the definition of φ, we can immediately deduce the following observations:

(2) A is surjective $\Rightarrow S_3 = \emptyset$ and $\varphi|_{S_1}$ is injective.
(3) $S_3 = \emptyset \Rightarrow S_1$ is a compact subset of S.
(4) F consists of non-isolated points only.
(5) A is injective $\Leftrightarrow \overline{\varphi(S_1)} = \overline{\varphi(S_1 \cup S_2)} = T$.

Statements (2) and (3) are obvious. To prove (4), assume $\varphi(s_0) = t_0$ is an isolated point of T. By the definition of φ, $A(f)(s_0) = 0$ if $f(t_0) = 0$, hence $\delta_{s_0} \circ A = \alpha\delta_{t_0}$ for some scalar α, so $\varphi(s_0) \notin F$. Implication "$\Leftarrow$" of (5) is obvious; to get "$\Rightarrow$" assume $\overline{\varphi(S_1)} \subsetneq T$. By (4) and since F is finite, we get $\overline{\varphi(S_1)} \cup F \subsetneq T$, so there is an $f \in C(T)$ such that $f \neq 0$ and $\operatorname{supp} f \cap (\overline{\varphi(S_1)} \cup F) = \emptyset$. By Theorem, $Af = 0$ and A is not injective.

Corollary. *Assume A is a linear, separating isomorphism from $C(T)$ onto $C(S)$. Then A is continuous and S and T are homeomorphic.*

Proof. By (2), (3), and (5), since φ is continuous we get $\varphi(S_1) = T$. For any $f \in C(T)$ we have

$$Af|_{S_1} \equiv 0 \Rightarrow f \equiv 0 \Rightarrow Af|_{S_2} \equiv 0.$$

Hence, since A is surjective, we get $S_2 = \emptyset$, and by (2) φ is a homeomorphism from S onto T.

Example. Let T be an infinite compact set, S a compact set, and let E be a linear subspace of $C(S)$ with $\dim E \leqq \mathfrak{c} :=$ *continuum*. We show that there is a discontinuous, linear separating map A from $C(T)$ onto E. Observe that the cardinality of any separable metric space is at most $\mathfrak{c}$, so E may be any separable linear subspace of $C(S)$. There are also many non-separable Banach spaces E with $\dim E \leqq \mathfrak{c}$, e.g., $E = l^\infty$. Hence, in particular we have an example of a discontinuous, linear, separating map from $c =$ *Banach space of all convergent sequences* onto l^∞.

Let $(U_n)_{n=1}^\infty$ be a sequence of pairwise disjoint, non-empty, open subsets of T, and let $t_n \in U_n$, for $n \in \mathbf{N}$. Fix an $x_0 \in \beta\mathbf{N} - \mathbf{N}$, where $\beta\mathbf{N}$ is the Stone-Čech compactification of the set of positive integers. Any sequence $(a_n)_{n=1}^\infty$ of non-negative real numbers can be extended to a continuous function from $\beta\mathbf{N}$ into $\mathbf{R}^+ \cup \{+\infty\}$, which we denote by $[(a_n)_{n=1}^\infty]$. We define two vector spaces

$$V = \{(f(t_n))_{n=1}^\infty \in l^\infty : f \in C(T)\}$$
$$V_0 = \{(a_n)_{n=1}^\infty \in V : x_0 \notin \operatorname{supp}[(a_n)_{n=1}^\infty]\}.$$

LEMMA. $\dim(V/V_0) = \mathfrak{c}$.

The equation $\dim(V/V_0) \leqq \mathfrak{c}$ is obvious since $\dim(V/V_0) \leqq \dim V \leqq \dim l^\infty = \mathfrak{c}$. The converse equation can be proven in several ways. Probably the shortest one is to observe that V/V_0 can be seen as a subset of the non-standard model $^*\mathbf{C}(^*\mathbf{R})$ of the set of all complex (real) numbers, that V/V_0 contains the monad M_0 of 0, and that M_0 is a $\mathfrak{c}$-dimensional linear space over $\mathbf{C}(\mathbf{R})$ [9, 10]. To get a more elementary and self-contained proof let $f_n \in C(T)$ be such that $\operatorname{supp} f_n \subset U_n$ and $\|f_n\| = 1 = f(t_n)$. Let $\mathcal{A}$ be the set of all infinite subsets of $\mathbf{N}$. Clearly $\operatorname{card}(\mathcal{A}) = \mathfrak{c}$. Let $(\alpha_n)_{n=1}^\infty$ be any decreasing sequence of positive numbers tending to zero and such that

$$\lim_n \frac{\alpha_{n+1}}{\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n} = 0.$$

For any $A \in \mathcal{A}$ we define a sequence $(a_n^A)_{n=1}^\infty$ by

$$a_n^A = \prod_{k \in A(n)} \alpha_k \quad \text{for } n \in \mathbf{N},$$

where $A(n) = \{k \in \mathbf{N} : n - k \in A\}$; if $A(n) = \emptyset$ then we understand that $a_n^A = 1$. Let A, B be distinct subsets of $\mathbf{N}$ and let k_0 be the smallest integer which is contained in exactly one of these sets, say $k_0 \in A$. Then

$$0 < \frac{a_n^A}{a_n^B} \leqq \frac{\alpha_{n-k_0}}{\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{n-k_0-1}} \xrightarrow[n \to \infty]{} 0. \tag{6}$$

For any $A \in \mathcal{A}$ we now define $f_A \in C(T)$ by

$$f_A = \sum_{n=1}^\infty a_n^A f_n.$$

Let $A_1, \ldots, A_2$ be distinct subsets of $\mathbf{N}$. By (6) sequences $(f_{A_j}(t_n))_{n=1}^\infty$ tend to zero with quite "different speed", this means in particular that there is one set among $A_1, \ldots, A_k$, say A_1, such that

$$\lim_n \frac{f_{A_j}(t_n)}{f_{A_1}(t_n)} = +\infty \quad \text{for } j = 2, \ldots, k.$$

Hence a non-trivial linear combination of $(f_{A_1}(t_n))_{n=1}^\infty, \ldots, (f_{A_k}(t_n))_{n=1}^\infty$ is distinct from zero for all, except possibly finitely many, indices; hence the set

$$\{(f_A(t_n))_{n=1}^\infty + V_0 \in (V/V_0) : A \in \mathcal{A}\}$$

is linearly independent, so $\dim(V/V_0) = \mathfrak{c}$.

Let Φ be any linear map from V onto E such that $V_0 \subseteq \ker\Phi$ and

$$\Phi\left(\left(\frac{1}{n}\right)_{n=1}^\infty\right) \neq 0.$$

We define $A : C(T) \longrightarrow C(S)$ by

$$Af(s) = \Phi((f(t_n))_{n=1}^{\infty}) \quad \text{for } s \in S.$$

The map A is evidently discontinuous. We prove that it is separating.

Let $h_1, h_2 \in C(T)$ be such that

$$\{t \in T : h_1(t) \neq 0\} \cap \{t \in T : h_2(t) \neq 0\} = \emptyset.$$

Put

$$N_i = \{n \in \mathbf{N} : h_i(t_n) \neq 0\}, \quad \text{for } i = 1, 2.$$

We have $N_1 \cap N_2 = \emptyset$ and hence the closures of these sets in $\beta\mathbf{N}$ are also disjoint. This means that at most one of the sets $\overline{N_1}$ or $\overline{N_2}$ contains x_0. Assume that $x_0 \notin \overline{N_1}$. Then $(h_1(t_n))_{n=1}^{\infty} \in V_0$ and $Ah_1 \equiv 0$.

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