# GENERALISATIONS OF SOME HYPERGEOMETRIC FUNCTION TRANSFORMATIONS

*by* T. M. MACROBERT

(Received 3rd January, 1949)

§ 1. *Introductory.* The formulae

$$F\begin{pmatrix}\alpha,\ \beta\ ;\ x\\ \gamma\end{pmatrix}=F\left\{\begin{matrix}\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\beta\ ;\ 4x(1-x)\\ \gamma\end{matrix}\right\}, \qquad \text{.........(1)}$$

where $\gamma=\frac{1}{2}(\alpha+\beta+1)$, and

$$F\begin{pmatrix}\alpha,\ \beta\ ;\ x\\ \gamma\end{pmatrix}=(1-x)^{-\alpha}F\left\{\begin{matrix}\tfrac{1}{2}\alpha,\ \tfrac{1}{2}+\tfrac{1}{2}\alpha-\beta\ ;\ \dfrac{-4x}{(1-x)^2}\\ \gamma\end{matrix}\right\}, \qquad \text{.........(2)}$$

where $\gamma=1+\alpha-\beta$, were given by Gauss (*Ges. Werke*, iii, pp. 225, 226). It is here proposed to find the corresponding expressions for the hypergeometric function when $\gamma$ has general values (not zero or negative integral). These will be derived in section 2 by applying Lagrange's expansion

$$F(x)=F(\lambda)+\sum_{n=1}^{\infty}\frac{w^n}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}[F'(\lambda)\{\phi(\lambda)\}^n], \qquad \text{.........(3)}$$

where

$$x=\lambda+w\phi(x), \qquad \text{.........(4)}$$

and that root of equation (4) in $x$ is taken which is equal to $\lambda$ when $w=0$. Two generalisations of Whipple's Transformation will be obtained in section 3.

§ 2. *Expressions for the Hypergeometric Function.* Let $x=\frac{1}{2}\{1-\surd(1-\xi)\}$, so that

$$x=\tfrac{1}{4}\xi(1-x)^{-1}\ ;$$

and in Lagrange's Expansion put $\lambda=0$, $w=\frac{1}{4}\xi=x(1-x)$ and $\phi(x)=(1-x)^{-1}$. Then

$$F\begin{pmatrix}\alpha,\ \beta\ ;\ x\\ \gamma\end{pmatrix}=1+\sum_{n=1}^{\infty}\frac{w^n}{n!}\left[\frac{d^{n-1}}{d\lambda^{n-1}}\left\{F'\begin{pmatrix}\alpha,\ \beta\ ;\ \lambda\\ \gamma\end{pmatrix}(1-\lambda)^{-n}\right\}\right]_{\lambda=0}$$

$$=1+\sum_{n=1}^{\infty}\frac{w^n}{n!}\left\{\frac{(\alpha\ ;\ n)(\beta\ ;\ n)}{(\gamma\ ;\ n)}+{}^{n-1}C_1\frac{(\alpha\ ;\ n-1)(\beta\ ;\ n-1)}{(\gamma\ ;\ n-1)}(n\ ;\ 1)+\ldots\right\},$$

where $(\alpha\ ;\ 0)=1$ and $(\alpha\ ;\ n)=\alpha(\alpha+1)\ldots(\alpha+n-1)$.

Thus

$$F\begin{pmatrix}\alpha,\ \beta\ ;\ x\\ \gamma\end{pmatrix}=\sum_{n=0}^{\infty}\{x(1-x)\}^n\frac{(\alpha\ ;\ n)(\beta\ ;\ n)}{n!(\gamma\ ;\ n)}F\begin{pmatrix}1-n,\ n,\ 1-\gamma-n\ ;\ 1\\ 1-\alpha-n,\ 1-\beta-n\end{pmatrix}. \qquad \text{.........(5)}$$

This formula is the generalisation of (1). If the order of the terms in the generalised hypergeometric series on the right is reversed the expansion may be written

$$F\begin{pmatrix}\alpha,\ \beta\ ;\ x\\ \gamma\end{pmatrix}=1+\frac{\alpha.\beta}{\gamma}\sum_{n=1}^{\infty}\{x(1-x)\}^n\frac{(2n-2)!}{n!(n-1)!}\psi(n\ ;\ \alpha,\beta\ ;\ \gamma), \qquad \text{.........(6)}$$

where

$$\psi(n\ ;\ \alpha,\beta\ ;\ \gamma)=\textit{first } n \textit{ terms of } F\begin{pmatrix}\alpha+1,\ \beta+1,\ 1-n\ ;\ 1\\ \gamma+1,\ 2-2n\end{pmatrix}.$$

https://doi.org/10.1017/S2040618500032925 Published online by Cambridge University Press

In considering the convergence of the series on the right, it is convenient to use form (6). Then

$$|\,\psi(n\,;\,\alpha, \beta\,;\,\gamma)\,| \leqq \sum_{r=0}^{n-1} \left| \frac{(\alpha+1\,;\,r)(\beta+1\,;\,r)}{r!\,(\gamma+1\,;\,r)} \right| \frac{1}{2^r}$$
$$< \sum_{r=0}^{\infty} \left| \frac{(\alpha+1\,;\,r)(\beta+1\,;\,r)}{r!\,(\gamma+1\,;\,r)} \right| \frac{1}{2^r} = K,$$

where $K$ is a positive number independent of $n$. Hence the modulus of the $n$th term in the series on the right of (6) is less than

$$|\,4x(1-x)\,|^n \frac{(\frac{1}{2}\,;\,n-1)}{n!} K\,;$$

and, consequently, by the comparison and ratio tests, the series converges absolutely if

$$|\,4x(1-x)\,| < 1.$$

Conversely, let $\xi = \frac{1}{2}\{1-\surd(1-x)\}$, so that $x = 4\xi \,.\, \frac{1}{2}\{1+\surd(1-x)\} = 4\xi(1-\xi)$; and, in Lagrange's Expansion, put $\lambda=0$, $w=4\xi$, $\phi(x)=\frac{1}{2}\{1+\surd(1-x)\}$; then

$$F\begin{pmatrix}\alpha, \beta\,;\, x\\ \gamma\end{pmatrix} = 1 + \sum_{n=1}^{\infty} \frac{(4\xi)^n}{n!} \left[ \frac{d^{n-1}}{d\lambda^{n-1}} \left\{ F'\begin{pmatrix}\alpha, \beta\,;\, \lambda\\ \gamma\end{pmatrix} \left( \frac{1+\surd(1-\lambda)}{2} \right)^n \right\} \right]_{\lambda=0}.$$

But [Phil. Mag., Ser. 7, XXVI, p. 86], if $\alpha$ is not a positive integer,

$$\left\{ \frac{1+\surd(1-\lambda)}{2} \right\}^{\alpha} = F\begin{pmatrix}-\frac{1}{2}\alpha, \frac{1}{2}-\frac{1}{2}\alpha\,;\, \lambda\\ 1-\alpha\end{pmatrix}.$$

Hence, when $\alpha \to n$, a positive integer,

$$\left\{ \frac{1+\surd(1-\lambda)}{2} \right\}^{n} = \textit{the first } \tfrac{1}{2}n+\tfrac{1}{2} \textit{ or } \tfrac{1}{2}n+1 \textit{ terms of } F\begin{pmatrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n\,;\, \lambda\\ 1-n\end{pmatrix}$$
$$+ \textit{terms of degree not less than } n \textit{ in } \lambda.$$

Therefore,

$$F\begin{pmatrix}\alpha, \beta\,;\, x\\ \gamma\end{pmatrix} = \sum_{n=0}^{\infty} [2\{1-\surd(1-x)\}]^n \frac{(\alpha\,;\,n)(\beta\,;\,n)}{n!\,(\gamma\,;\,n)} F\begin{pmatrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, 1-\gamma-n\,;\, 1\\ 1-\alpha-n, 1-\beta-n\end{pmatrix}. \quad \ldots\ldots(7)$$

On interchanging $x$ and $\xi$, it is seen that

$$F\begin{Bmatrix}\alpha, \beta\,;\, 4x(1-x)\\ \gamma\end{Bmatrix} = \sum_{n=0}^{\infty} (4x)^n \frac{(\alpha\,;\,n)(\beta\,;\,n)}{n!\,(\gamma\,;\,n)} F\begin{pmatrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, 1-\gamma-n\,;\, 1\\ 1-\alpha-n, 1-\beta-n\end{pmatrix}. \quad \ldots\ldots(8)$$

The series on the right is convergent if $|\,x\,| < \frac{1}{2}$ [Cf. previous paper].

Again, in the formula

$$F\begin{pmatrix}\alpha, \beta\,;\, x\\ \gamma\end{pmatrix} = (1-x)^{-\alpha} F\begin{pmatrix}\alpha, \gamma-\beta\,;\, \frac{x}{x-1}\\ \gamma\end{pmatrix},$$

apply (5) to the hypergeometric function on the right and get

$$F\begin{pmatrix}\alpha, \beta\,;\, x\\ \gamma\end{pmatrix} = (1-x)^{-\alpha} \sum_{n=0}^{\infty} \left\{ \frac{-x}{(1-x)^2} \right\}^n \frac{(\alpha\,;\,n)(\gamma-\beta\,;\,n)}{n!\,(\gamma\,;\,n)} F\begin{pmatrix}1-n, n, 1-\gamma-n\,;\, 1\\ 1-\alpha-n, 1-\gamma+\beta-n\end{pmatrix}. \quad \ldots(9)$$

This series, which is convergent for $x$ small and $|\,4x/(1-x)^2\,| < 1$, is the generalisation of (2).

*Note.*—Formulae (1) and (2) may be deduced from (5), (8) and (9) by using Saalschutz's theorem or Whipple's theorem (W. N. Bailey, *Generalized Hypergeometric Series*, p. 16).

https://doi.org/10.1017/S2040618500032925 Published online by Cambridge University Press

§ 3. *Generalisations of Whipple's Transformation.* Whipple's Transformation may be written

$$F\left(\begin{matrix}-\nu, \nu+1 ; & \frac{1-x}{2} \\ \mu+1 & \end{matrix}\right)=\left(\frac{1+x}{2x}\right)^{\mu} x^{\nu} F\left(\begin{matrix}\frac{\mu-\nu}{2}, \frac{\mu-\nu+1}{2} ; & 1-\frac{1}{x^2} \\ \mu+1 & \end{matrix}\right) . \quad \text{............(10)}$$

Now, if $x$ is small,

$$(1-x)^{-\frac{1}{2}\alpha} \sum_{n=0}^{\infty}\left(\frac{x}{x-1}\right)^n \frac{(\alpha ; n)(\beta ; n)}{n!(\gamma ; n) 4^n} F\left(\begin{matrix}1-n, n, 1-\gamma-n ; & 1 \\ 1-\alpha-n, 1-\beta-n & \end{matrix}\right)=\sum_{n=0}^{\infty} \phi(n ; \alpha, \beta ; \gamma) x^n, \quad \text{...(11)}$$

where

$$\phi(n ; \alpha, \beta ; \gamma) = \sum_{r=0}^{n} \frac{(\frac{1}{2}\alpha+n-r ; r)(\alpha ; n-r)(\beta ; n-r)}{r!(n-r)!(\gamma ; n-r)(-4)^{n-r}} F\left(\begin{matrix}r+1-n, n-r, r+1-\gamma-n ; & 1 \\ r+1-\alpha-n, r+1-\beta-n & \end{matrix}\right) . \quad \text{......(11a)}$$

This formula can be established by expanding the powers of $(1-x)$ in (11) in powers of $x$ and picking out the coefficient of $x^n$. Hence, on replacing $x$ by $x/(x-1)$, we have

$$\sum^{\infty} \frac{(\alpha ; n)(\beta ; n)}{n!(\gamma ; n) 4^n} F\left(\begin{matrix}1-n, n, 1-\gamma-n ; & 1 \\ 1-\alpha-n, 1-\beta-n & \end{matrix}\right) x^n=(1-x)^{-\frac{1}{2}\alpha} \sum_{n=0}^{\infty} \phi(n ; \alpha, \beta ; \gamma)\left(\frac{x}{x-1}\right)^n . \quad \text{...(12)}$$

Now, apply (5) to the hypergeometric function on the right of the identity

$$F\left(\begin{matrix}\alpha, \beta ; & \frac{1-x}{2} \\ \gamma & \end{matrix}\right)=\left(\frac{1+x}{2}\right)^{\gamma-\alpha-\beta} F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta ; & \frac{1+x}{2} \\ \gamma & \end{matrix}\right)$$

and it becomes

$$F\left(\begin{matrix}\alpha, \beta ; & \frac{1-x}{2} \\ \gamma & \end{matrix}\right)=\left(\frac{1+x}{2}\right)^{\gamma-\alpha-\beta} \times \sum_{=0}^{\infty} \frac{(\gamma-\alpha ; n)(\gamma-\beta ; n)}{n!(\gamma ; n) 4^n} F\left(\begin{matrix}1-n, n, 1-\gamma-n ; & 1 \\ 1-\gamma+\alpha-n, 1-\gamma+\beta-n & \end{matrix}\right)(1-x^2)^n . \quad \text{.....(13)}$$

Next, apply (12) to the R.H.S. of (13), and so obtain

$$F\left(\begin{matrix}\alpha, \beta ; & \frac{1-x}{2} \\ \gamma & \end{matrix}\right)=\left(\frac{1+x}{2}\right)^{\gamma-\alpha-\beta} x^{\alpha-\gamma} \sum_{n=0}^{\infty} \phi(n ; \gamma-\alpha, \gamma-\beta ; \gamma)\left(1-\frac{1}{x^2}\right)^n . \quad \text{........(14)}$$

This is the first generalisation of (10), to which form it can be reduced when $\alpha+\beta=1$ by applying Whipple's formula.

The second generalisation can be derived as follows. In formula (7) replace $x$ by $1-x^2$ and replace $\beta$ by $\gamma-\beta$; then

$$F\left(\begin{matrix}\alpha, \beta ; & 1-\frac{1}{x^2} \\ \gamma & \end{matrix}\right)=x^{2\alpha} \sum_{n=0}^{\infty}\{2(1-x)\}^n \frac{(\alpha ; n)(\gamma-\beta ; n)}{n!(\gamma ; n)} F\left(\begin{matrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, 1-\gamma-n ; & 1 \\ 1-\alpha-n, 1-\gamma+\beta-n & \end{matrix}\right) . \quad \text{...(15)}$$

Now it can easily be verified that

$$(1-x)^{2\alpha+2\beta-\gamma} \sum_{n=0}^{\infty}(4x)^n \frac{(\alpha ; n)(\beta ; n)}{n!(\gamma ; n)} F\left(\begin{matrix}-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, 1-\gamma-n ; & 1 \\ 1-\alpha-n, 1-\beta-n & \end{matrix}\right) = \sum_{n=0}^{\infty} \chi(n ; \alpha, \beta ; \gamma)(4x)^n, \quad \text{......(16)}$$

where

$$\chi(n ; \alpha, \beta ; \gamma)=\frac{(\alpha ; n)(\beta ; n)}{n!(\gamma ; n)} \sum_{r=0}^{\infty} \frac{(\gamma-2\alpha-2\beta ; r)(-n ; r)(1-\gamma-n ; r)}{r!(1-\alpha-n ; r)(1-\beta-n ; r)} \cdot \frac{1}{4^r} \times F\left(\begin{matrix}\frac{r-n}{2}, \frac{1+r-n}{2}, 1-\gamma+r-n ; & 1 \\ 1-\alpha+r-n, 1-\beta+r-n & \end{matrix}\right) . \quad \text{..............(16a)}$$

https://doi.org/10.1017/S2040618500032925 Published online by Cambridge University Press

Here replace $x$ by $\frac{1}{2}(1-x)$ and $\beta$ by $\gamma-\beta$ and substitute on the right of (15); then

$$\sum_{n=0}^{\infty} \chi(n\,;\ \alpha, \gamma-\beta\,;\ \gamma)\left(4\,\frac{1-x}{2}\right)^n = \left(\frac{1+x}{2x}\right)^{2\alpha-2\beta+\gamma} x^{\gamma-2\beta} F\begin{pmatrix}\alpha, \beta\,; & 1-\dfrac{1}{x^2}\\ \gamma & \end{pmatrix}. \qquad \text{.........(17)}$$

When $\gamma=\alpha+\beta+\frac{1}{2}$, formula (16$a$) can be simplified by applying Saalschutz's theorem. Formula (10) is thus obtained as a particular case.

UNIVERSITY OF GLASGOW