ON COMMUTATIVE V^*-ALGEBRAS II

by P. G. SPAIN

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1. Introduction. We show that the commutative V^*-algebras with relatively weakly compact unit spheres are those that are representable by means of hermitian spectral measures. This provides a more unified approach to the results of [**15**], and allows us to generalise some of them.

Let X be a complex Banach space with dual space X' and semi-inner-product $[\ ,\ ]$ compatible with the norm. Let $\langle x, x'\rangle$ be the value of the functional x' in X' at the point x in X. When Y is a subset of X, we write Y^w for the weak closure of Y. Let $\mathscr{L}(X)$ be the algebra of bounded linear operators on X. When $\mathscr{T}$ is a subset of $\mathscr{L}(X)$, we write $\mathscr{T}^w$ for the closure of $\mathscr{T}$ in the weak (operator) topology and $\mathscr{T}^s$ for the closure of $\mathscr{T}$ in the strong (operator) topology. We write $\mathscr{T}_k$ for $\{T \in \mathscr{T} : \|T\| \leqq k\}$.

For x in X, we define the point state $\omega_x : \mathscr{L}(X) \to \mathbf{C} : T \mapsto [Tx, x]$. T is said to be hermitian if $W(T) = \{\omega_x(T) : \|x\| = 1\}$, the numerical range of T, is a set of real numbers. (These topics are discussed in [**10**].)

Let $\mathscr{A}$ be a closed subalgebra of $\mathscr{L}(X)$ and let $\mathscr{H}$ be the set of hermitian operators in $\mathscr{A}$. $\mathscr{A}$ is called a V^*-algebra if $I \in \mathscr{A}$ and $\mathscr{A} = \mathscr{H} + i\mathscr{H}$. Then $\mathscr{A}$ is a V^*-algebra if and only if $I \in \mathscr{A}$ and $\mathscr{A}$ is a C^*-algebra under the operator norm and the (Vidav) involution $^* : R + iJ \mapsto R - iJ$ $(R, J \in \mathscr{H})$ [**11**].

We refer the reader to [**15**] for the definitions of (possibly unbounded) normal, self-conjugate and strongly self-conjugate operators. A bounded operator is normal if and only if it is contained in a commutative V^*-algebra.

Let Λ be a compact Hausdorff space and let $C(\Lambda)$ be the space of continuous complex functions on Λ, with the supremum norm. Let $S(\Lambda)$ $(S_0(\Lambda))$ be the family of Borel (Baire) sets of Λ; let $B(\Lambda)$ $(B_0(\Lambda))$ be the space of bounded Borel (Baire) measurable functions on Λ, with the supremum norm.

We refer to [**5**] for the definition and properties of spectral measures.

Throughout this paper, $\mathscr{A}$ will be a commutative V^*-algebra on X with maximal ideal space Λ and inverse Gelfand map $\psi : C(\Lambda) \to \mathscr{A}$. $\mathscr{H}$ will be the set of hermitian operators in $\mathscr{A}$.

2. Commutative V^*-algebras with weakly compact unit spheres.

LEMMA 1. *If $T \in \mathscr{H}$, then $(I+T^2)^{-1} \in \mathscr{H}_1$ and $2T(I+T^2)^{-1} \in \mathscr{H}_1$. Conversely, if $S \in \mathscr{H}_1$, there is a T in $\mathscr{H}$ such that $S = 2T(I+T^2)^{-1}$.*

Proof. This lemma holds in any C^*-algebra with identity. (See Lemma 6 of [**14**, p. 24].)

LEMMA 2. *There is a unique regular spectral measure $F(\cdot)$ of class $(S(\Lambda), X)$ with values in $\mathscr{L}(X')$ such that*

$$\langle \psi(f)x, x'\rangle = \int_\Lambda f(\lambda)\langle x, F(d\lambda)x'\rangle \qquad (x \in X,\ x' \in X',\ f \in C(\Lambda)).$$

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Proof. For x in X and x' in X', the map $\psi_{x,x'}: C(\Lambda) \to \mathbf{C}: f \mapsto \langle \psi(f)x, x' \rangle$ is a functional on $C(\Lambda)$ bounded by $\|x\| \ \|x'\|$. By the Riesz representation theorem, there is a unique regular Borel measure $\mu(\cdot\,; x, x')$ on Λ such that $\|\mu(\cdot\,; x, x')\| \leqq \|x\| \ \|x'\|$ and

$$\langle \psi(f)x, x' \rangle = \int_\Lambda f(\lambda)\mu(d\lambda; x, x') \qquad (x \in X,\ x' \in X',\ f \in C(\Lambda)).$$

For τ in $S(\Lambda)$ and x' in X', the map $x \mapsto \mu(\tau; x, x')$ is a bounded functional on X. Therefore there is an $F_{x'}(\tau)$ in X' such that $\mu(\tau; x, x') = \langle x, F_{x'}(\tau) \rangle$. Clearly $\|F_{x'}(\tau)\| \leqq \|x'\|$. By linearity and the uniqueness of $\mu(\cdot\,; x, x')$, there is an operator $F(\tau)$ in $\mathscr{L}(X')$ such that $F_{x'}(\tau) = F(\tau)x'$. Also $\|F(\tau)\| \leqq 1$. Since each $\mu(\cdot\,; x, x')$ is a regular measure and ψ is an algebra isomorphism, it is routine to check that $F(\cdot)$ is a regular spectral measure of class $(S(\Lambda), X)$. Then $\|F(\tau)\| = 1$ $(\tau \in S(\Lambda))$, since each $F(\tau)$ is a projection.

LEMMA 3. $\|Sx\| = \|S^*x\|$ $(x \in X, S \in \mathscr{A})$.

Proof. Let $S = \psi(f)$; then $S^* = \psi(\bar{f})$. We define g on Λ by $g(\lambda) = \bar{f}(\lambda)/f(\lambda)$ if $f(\lambda) \neq 0$, $g(\lambda) = 0$ if $f(\lambda) = 0$. Then $g \in B(\Lambda)$ and $\|g\| = 1$. We define U in $\mathscr{L}(X')$ by $U = \int_\Lambda g(\lambda)F(d\lambda)$. For x in X and x' in X', we have

$$|\langle x, Ux' \rangle| = \left| \int_\Lambda g(\lambda)\mu(d\lambda; x, x') \right| \leqq \|g\| \ \|\mu(\cdot\,; x, x')\| \leqq \|x\| \ \|x'\|.$$

Therefore $\|U\| \leqq 1$. Also, $(S^*)' = S'U$. Hence

$$\|S^*x\| = \sup_{\|x'\| \leqq 1} |\langle S^*x, x' \rangle| = \sup_{\|x'\| \leqq 1} |\langle x, S'Ux' \rangle| = \sup_{\|x'\| \leqq 1} |\langle Sx, Ux' \rangle| \leqq \|Sx\|.$$

By symmetry, $\|S^*x\| = \|Sx\|$.

Lemmas 2 and 3 are similar to Theorem 2.5(ii) and Lemma 2.7 of [**12**]. The first part of the next theorem is the same as Theorem 2.8 of [**12**].

THEOREM 1. *$\mathscr{A}^w$ is a commutative V^*-algebra and* $(\mathscr{A}^w)_1 = (\mathscr{A}_1)^w$.

Proof. If $S \in \mathscr{A}^w$, there is a net $\{S_s = R_s + iJ_s : s \in \sigma\}$ in $\mathscr{A}$ with strong limit S. Lemma 3 shows that $\{S_s^*\}$ converges strongly. Hence $\{R_s : s \in \sigma\}$ and $\{J_s : s \in \sigma\}$ converge strongly to R and J in $\mathscr{H}^w$ and $S = R + iJ$. Hence $\mathscr{A}^w = \mathscr{H}^w + i\mathscr{H}^w$. Therefore $\mathscr{A}^w$ is a commutative V^*-algebra and $\mathscr{H}^w$ is the set of hermitian operators in $\mathscr{A}^w$.

Let $S \in (\mathscr{H}^w)_1$ and let T in $\mathscr{H}^w$ be such that $S = 2T(I+T^2)^{-1}$. Let $\{T_s : s \in \sigma\}$ be a net in $\mathscr{H}$ converging strongly to T; put $S_s = 2T_s(I+T_s^2)^{-1}$. Then, as in [**14**, p. 25, Theorem 2] or [**6**, p. 47],

$$S_s - S = 2(I+T_s^2)^{-1}(T_s - T)(I+T^2)^{-1} + \tfrac{1}{2}S_s(T - T_s)S.$$

Therefore S is the strong limit of $\{S_s\}$ in $\mathscr{H}_1$; so $(\mathscr{H}^w)_1 \subset (\mathscr{H}_1)^w$.

By the Russo and Dye theorem [**11**, p. 538], $(\mathscr{A}^w)_1 \subset (\mathscr{A}_1)^w$. Hence $(\mathscr{A}^w)_1 = (\mathscr{A}_1)^w$.

DEFINITION. We say that $\mathscr{A}$ is representable by a spectral measure if there is a regular hermitian spectral measure $E(\cdot)$ of class $(S(\Lambda), X')$ with values in $\mathscr{L}(X)$ such that $\psi(f) = \int_\Lambda f(\lambda)E(d\lambda)$ $(f \in C(\Lambda))$. Such a spectral measure is unique.

THEOREM 2. *$\mathscr{A}$ is representable by a spectral measure if and only if $\mathscr{A}_1$ is relatively weakly compact. If this is so, then $\mathscr{A}^w$ is a commutative W^*-algebra and any faithful representation of $\mathscr{A}^w$ as a von Neumann algebra is weakly and strongly bicontinuous on bounded spheres.*

Proof. Let $\mathscr{A}$ be represented by the spectral measure $E(\cdot)$. For each x in X the map $\psi_x : C(\Lambda) \to X : f \mapsto \psi(f)x$ is weakly compact [**1**, Theorem 3.2]; hence $\mathscr{A}_1 x$ is relatively weakly compact in X. The argument suggested in [**7**, p. 511, Exercise 2] shows that $\mathscr{A}_1$ is relatively weakly compact in $\mathscr{L}(X)$.

Let $\mathscr{A}_1$ be relatively weakly compact. Theorem 3 of [**16**] shows that any von Neumann representation $\phi : \mathscr{A}^w \to \mathscr{B} \subset \mathscr{L}(H)$ is weakly bicontinuous on bounded spheres; also, $(\mathscr{A}_1)^w = (\mathscr{A}^w)_1$. By [**7**, X.2.1] the map $\phi\psi : C(\Lambda) \to \phi\mathscr{A}$ has the form $f \mapsto \int_\Lambda f(\lambda) E^h(d\lambda)$, where $E^h(\cdot)$ is a unique regular hermitian spectral measure of class $(S(\Lambda), H)$ with values in $\mathscr{L}(H)$. Also $E^h(\tau) \in \mathscr{B}$ $(\tau \in S(\Lambda))$. We define $E(\cdot)$ by $E(\cdot) = \phi^{-1} E^h(\cdot)$. Then $E(\cdot)$ is a regular hermitian spectral measure of class $(S(\Lambda), X')$, since ϕ is weakly bicontinuous on bounded spheres; and $\psi(f) = \int_\Lambda f(\lambda) E(d\lambda)$ $(f \in C(\Lambda))$ because ϕ is an isometry.

Suppose that $\mathscr{A}_1$ is relatively weakly compact. We have still to prove that any von Neumann representation ϕ of $\mathscr{A}^w$ is strongly bicontinuous on bounded spheres.

Let $\{T_s : s \in \sigma\}$ be a bounded net in $\mathscr{A}^w$ and let $\lim_\sigma T_s = 0$ in the strong topology. Then $\lim_\sigma \omega_x(T_s^* T_s) = 0$ $(x \in X)$. We next show that the converse holds.

Let $\tilde{\Lambda}$ be the maximal ideal space of $\mathscr{A}^w$, let $\tilde{E}(\cdot)$ be its representing spectral measure and $\tilde{\psi}$ the inverse Gelfand map defined by $\tilde{\psi} : C(\tilde{\Lambda}) \to \mathscr{A}^w : f \mapsto \int_{\tilde{\Lambda}} f(\lambda) \tilde{E}(d\lambda)$. Let $f_s = \tilde{\psi}^{-1} T_s$ $(s \in \sigma)$. Since $\{T_s\}$ is a bounded net and the weak topology on a bounded sphere is the weak topology induced by the point states [**16**, Lemma 1], it follows that

$$\lim_\sigma \langle T_s^* T_s x, x' \rangle = \lim_\sigma \int_{\tilde{\Lambda}} |f_s(\lambda)|^2 \langle \tilde{E}(d\lambda)x, x' \rangle = 0 \qquad (x \in X,\ x' \in X').$$

Therefore $\lim_\sigma f_s = 0$ in var$(\langle \tilde{E}(\cdot)x, x' \rangle)$-measure, and $\lim_\sigma \int_{\tilde{\Lambda}} f_s(\lambda) \langle \tilde{E}(d\lambda)x, x' \rangle = 0$. For fixed x in X, the set $\{\langle \tilde{E}(\cdot)x, x' \rangle : \|x'\| \leqq 1\}$ is a relatively weakly compact set of measures [**7**, IV.10.2]; hence, by [**9**, Théorème 2], $\lim_\sigma \int_{\tilde{\Lambda}} f_s(\lambda) \langle \tilde{E}(d\lambda)x, x' \rangle = 0$ uniformly for $\|x'\| \leqq 1$. Therefore $\lim_\sigma \int_{\tilde{\Lambda}} f_s(\lambda) \tilde{E}(d\lambda)x = 0$; that is, $\lim_\sigma T_s = 0$ in the strong topology.

Thus, if $\{T_s : s \in \sigma\}$ is a bounded net in $\mathscr{A}^w$, $\lim_\sigma T_s = 0$ in the strong topology if and only if $\lim_\sigma T_s^* T_s = 0$ in the weak topology. It follows that ϕ is strongly bicontinuous on bounded spheres.

REMARK 1. The hypotheses of the theorem hold if $E(\cdot)$ is a hermitian spectral measure of class $(S_0(\Lambda), X')$ and $\psi(f) = \int_\Lambda f(\lambda) E(d\lambda)$ $(f \in C(\Lambda))$ [**1**, Theorem 3.2].

COROLLARY 1. *If $\mathscr{A}_1$ is relatively weakly compact and $\{R_s : s \in \sigma\}$ is a bounded monotone increasing net in $\mathscr{H}$, then $\bigvee_\sigma R_s = \lim_\sigma R_s$ in the strong topology.*

Proof. We already know that $\bigvee_\sigma R$ exists and is the weak limit of $\{R_s\}$ [**16**, Lemma 2]. Let ϕ be a von Neumann representation of $\mathscr{A}^w$. Then $\bigvee_\sigma \phi R_s = \lim_\sigma \phi R_s$ in the strong topology [**6**, Appendice II]; whence the result (cf. [**4**, Theorem 4.2] and [**13**, Lemma 3]).

COROLLARY 2. *Let X be weakly (sequentially) complete. Then $\mathscr{A}$ is representable by a spectral measure.*

Proof. For each x in X, the map $\psi_x : C(\Lambda) \to X : f \mapsto \psi(f)x$ is weakly compact [**7**, VI. 7.6]. It follows as in the theorem that $\mathscr{A}_1$ is relatively weakly compact. (This corollary is Theorem 2.5(i) of [**12**].)

REMARK 2. Since any bounded Boolean algebra of projections can be made hermitian by a suitable equivalent renorming of X [**4**, §3, Remark 2], the theorem includes Corollary 2 to Theorem 3 of [**8**].

3. Applications of Theorem 2. The results of [**15**] are based on the use of Theorem 2.5(i) of [**12**] and Corollary 2 of Theorem 3 of [**8**]. These are both corollaries of Theorem 2 above.

Theorem 2 and the proof of Theorem 1 of [**15**] give our next result.

THEOREM 3. *Let T be a normal operator on X and let $\mathscr{A}$ be the commutative V^*-algebra generated by T. Let $\mathscr{A}_1$ be relatively weakly compact. Then $\sigma(T)$ (the spectrum of T) is the maximal ideal space of $\mathscr{A}$. If $E(\cdot)$ is the representing spectral measure for $\mathscr{A}$, then λ in $\sigma(T)$ is an eigenvalue of T if and only if $E(\{\lambda\}) \neq 0$.*

Theorem 2 and the proof of Theorem 2 of [**15**] give our next result.

THEOREM 4. *Let S be a strongly self-conjugate operator on X and let its generated group of isometries $\{U(t, S) : t \in \mathbf{R}\}$ be contained in a commutative V^*-algebra with relatively weakly compact unit sphere. Then there is a regular hermitian spectral measure $E(\cdot)$ of class $(S(\mathbf{R}), X')$ such that*

$$U(t, S) = \lim_n \int_{-n}^{n} e^{it\lambda} E(d\lambda) \qquad (t \in \mathbf{R}),$$

$$Sx = \lim_n \int_{-n}^{n} \lambda E(d\lambda)x \qquad (x \in \mathscr{D}(S)).$$

Theorem 5 of [**15**] may be further generalised.

THEOREM 5. *Let $\mathscr{B}'$ be a bounded Boolean algebra of projections on a Banach space X, and let the closed algebra generated by $\mathscr{B}'$ have relatively weakly compact unit sphere. Then $\mathscr{B}'$ has a σ-complete extension contained in $(\mathscr{B}')^s$.*

Proof. By Remark 2, there is no loss of generality in assuming that each projection in $\mathscr{B}'$ is hermitian. Let Λ be the Stone space of $\mathscr{B}'$, $K'(\Lambda)$ the set of characteristic functions of open-and-closed subsets of Λ, and let $\psi' : K'(\ ) \to \mathscr{B}' : \chi_\tau \mapsto B(\tau)$ be the representation isomorphism. We extend ψ' to an algebra isomorphism $\psi : K(\Lambda) \to \mathscr{B} : \sum c_j \chi_{\tau_j} \mapsto \sum c_j B(\tau_j)$, where $K(\Lambda)$ is the algebra generated by $K'(\ )$, $\mathscr{B}$ that generated by $\mathscr{B}'$. Then ψ is an isometry [3, Theorem 2.1].

Since Λ is totally disconnected, $K(\Lambda)$ is norm dense in $C(\Lambda)$. We extend ψ to an isometric isomorphism (also denoted by) $\psi : C(\Lambda) \to \bar{\mathscr{B}}$ (norm closure of $\mathscr{B}$). Then $\bar{\mathscr{B}} = K + iK$, where K is the set of hermitian operators in $\bar{\mathscr{B}}$. Thus $\bar{\mathscr{B}}$ is a commutative V^*-algebra.

Let $E(\cdot)$ be the representing spectral measure for $\bar{\mathscr{B}}$. Let $\bar{\bar{\mathscr{B}}} = \{E(\tau) : \tau \in S_0(\Lambda)\}$. Then $\bar{\bar{\mathscr{B}}}$ is a Boolean algebra of hermitian projections containing $\mathscr{B}'$.

Let $\{E(\tau_n) : n = 1, 2, \ldots\}$ be a sequence in $\bar{\bar{\mathscr{B}}}$ and let

$$\tau = \bigcup_1^\infty \tau_n.$$

Put

$$\tau_1' = \tau_1, \quad \tau_{n+1}' = \tau_{n+1} \backslash \bigcup_1^n \tau_k \qquad (n = 1, 2, \ldots).$$

Then

$$E(\tau) = \lim_n E(\bigcup_1^n \tau_k') = \lim_n \bigvee_1^n E(\tau_k') = \lim_n \bigvee_1^n E(\tau_k),$$

where the limits exist in the strong topology (by the Banach–Orlicz–Pettis theorem). It is clear that $E(\tau)X = \operatorname{clm}\{E(\tau_n)X\}$.

The proof of the existence of $\bigwedge E(\tau_n)$ and that $(\bigwedge E(\tau_n))X = \bigcap (E(\tau_n)X)$ is similar. Thus $\bar{\bar{\mathscr{B}}}$ is σ-complete.

Since Λ is totally disconnected, $S_0(\Lambda)$ is contained in the σ-algebra generated by the open-and-closed sets. Hence $\bar{\bar{\mathscr{B}}} \subset (\mathscr{B}')^s$.

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UNIVERSITY OF GLASGOW
GLASGOW, G12 8QQ