7
Supersymmetric strings

The discussion of bosonic strings in the previous five chapters allowed us to uncover a great deal of the structure essential to understanding D-branes and other background solutions, in addition to the basic concepts used in discussing and working with critical string theory.

At the back of our mind was always the expectation that we would move on to include supersymmetry. Two of the main reasons are that we can remove the tachyon from the spectrum and that we will be able to use supersymmetry to endow many of our results with extra potency, since stability and non-renormalisation arguments will allow us to extrapolate beyond perturbation theory.

Let us set aside D-branes and T-duality for a while and use the ideas we discussed earlier to construct the supersymmetric string theories which we need to carry the discussion further. There are five such theories. Three of these are the *'superstrings'*, while two are the *'heterotic strings'**.

7.1 The three basic superstring theories

7.1.1 Open superstrings: type I

Let us go back to the beginning, almost. We can generalise the bosonic string action we had earlier to include fermions. In conformal gauge it is:

$$S = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma \left\{ \frac{1}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right\}, \qquad (7.1)$$

where the open string world-sheet is the strip $0 < \sigma < \pi$, $-\infty < \tau < \infty$.

* A looser and probably more sensible nomenclature is to call them all 'superstrings', but we'll choose the catch-all term to be the one we used for the title of this chapter.

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N.B. Recall that α' is the loop expansion parameter analogous to $\hbar$ on worldsheet. It is therefore natural for the fermions' kinetic terms to be normalised in this way.

We get a modification to the energy-momentum tensor from before (which we now denote as T_{B}, since it is the bosonic part):

$$T_{\mathrm{B}}(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu, \tag{7.2}$$

which is now accompanied by a fermionic energy-momentum tensor:

$$T_{\mathrm{F}}(z) = i\frac{2}{\alpha'}\psi^\mu \partial X_\mu. \tag{7.3}$$

This enlarges our theory somewhat, while much of the logic of what we did in the purely bosonic story survives intact here. Now, one extremely important feature which we encountered in section 4.7 is the fact that the equations of motion admit two possible boundary conditions on the worldsheet fermions consistent with Lorentz invariance. These are denoted the 'Ramond' (R) and the 'Neveu–Schwarz' (NS) sectors:

$$\begin{aligned} &\text{R:} \quad \psi^\mu(0,\tau) = \tilde{\psi}^\mu(0,\tau) \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau) \\ &\text{NS:} \quad \psi^\mu(0,\tau) = -\tilde{\psi}^\mu(0,\tau) \qquad \psi^\mu(\pi,\tau) = \tilde{\psi}^\mu(\pi,\tau). \end{aligned} \tag{7.4}$$

We have used the freedom to choose the boundary condition at, for example the $\sigma{=}\pi$ end, in order to have a $+$ sign, by redefinition of $\tilde{\psi}$. The boundary conditions and equations of motion are summarised by the 'doubling trick': take just left-moving (analytic) fields ψ^μ on the range 0 to 2π and define $\tilde{\psi}^\mu(\sigma,\tau)$ to be $\psi^\mu(2\pi-\sigma,\tau)$. These left-moving fields are periodic in the Ramond (R) sector and antiperiodic in the Neveu–Schwarz (NS).

On the complex z-plane, the NS sector fermions are half-integer moded while the R sector ones are integer, and we have:

$$\psi^\mu(z) = \sum_r \frac{\psi_r^\mu}{z^{r+1/2}}, \text{ where } r\in\mathbb{Z} \text{ or } r\in\mathbb{Z}+\tfrac{1}{2} \tag{7.5}$$

and canonical quantisation gives

$$\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}. \tag{7.6}$$

Similarly we have

$$\begin{aligned} T_{\mathrm{B}}(z) &= \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad \text{as before, and} \\ T_{\mathrm{F}}(z) &= \sum_r \frac{G_r}{z^{r+3/2}}, \text{ where } r\in\mathbb{Z}\text{ (R) or } \mathbb{Z}+\tfrac{1}{2}\text{ (NS)}. \end{aligned} \tag{7.7}$$

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Correspondingly, the Virasoro algebra is enlarged, with the non-zero (anti)commutators being

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s} \\
[L_m, G_r] &= \frac{1}{2}(m-2r)G_{m+r},
\end{aligned} \tag{7.8}$$

with

$$\begin{aligned}
L_m &= \frac{1}{2}\sum_m : \alpha_{m-n} \cdot \alpha_m : + \frac{1}{4}\sum_r (2r-m) : \psi_{m-r} \cdot \psi_r : + a\delta_{m,0} \\
G_r &= \sum_n \alpha_n \cdot \psi_{r-n}.
\end{aligned} \tag{7.9}$$

In the above, c is the total contribution to the conformal anomaly, which is $D + D/2$, where D is from the D bosons while $D/2$ is from the D fermions.

The values of D and a are again determined by any of the methods mentioned in the discussion of the bosonic string. For the superstring, it turns out that $D = 10$ and $a = 0$ for the R sector and $a = -1/2$ for the NS sector. This comes about because the contributions from the X^0 and X^1 directions are cancelled by the Faddeev–Popov ghosts as before, and the contributions from the ψ^0 and ψ^1 oscillators are cancelled by the superghosts. Then, the computation uses the mnemonic/formula given in equation (2.80).

$$\begin{aligned}
\text{NS sector:} \quad \text{z.p.e} &= 8\left(-\frac{1}{24}\right) + 8\left(-\frac{1}{48}\right) = -\frac{1}{2}, \\
\text{R sector:} \quad \text{z.p.e} &= 8\left(-\frac{1}{24}\right) + 8\left(\frac{1}{24}\right) = 0.
\end{aligned} \tag{7.10}$$

As before, there is a physical state condition imposed by annihilating with the positive modes of the (super) Virasoro generators:

$$G_r|\phi\rangle = 0,\ \ r > 0; \quad L_n|\phi\rangle = 0,\ \ n > 0; \quad (L_0 - a)|\phi\rangle = 0. \tag{7.11}$$

The L_0 constraint leads to a mass formula:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n,r} \alpha_{-n} \cdot \alpha_n + r\psi_{-r} \cdot \psi_r - a\right). \tag{7.12}$$

In the NS sector the ground state is a Lorentz singlet and is assigned odd fermion number, i.e. under the operator $(-1)^F$, it has eigenvalue -1.

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In order to achieve spacetime supersymmetry, the spectrum is projected on to states with even fermion number. This is called the 'GSO projection'[71], and for our purposes, it is enough to simply state that this obtains spacetime supersymmetry, as we will show at the massless level. A more complete treatment – which gets it right for all mass levels – is contained in the full superconformal field theory. The GSO projection there is a statement about locality with the gravitino vertex operator. Yet another way to think of its origin is as a requirement of modular invariance.

Since the open string tachyon clearly has $(-1)^F = -1$, it is removed from the spectrum by GSO. This is our first achievement, and justifies our earlier practice of ignoring the tachyon's appearance in the bosonic spectrum in what has gone before. From what we will do for the rest of the this book, the tachyon will largely remain in the wings, but it (and other tachyons) do have a role to play, since they are often a signal that the vacuum wants to move to a (perhaps) more interesting place.

Massless particle states in ten dimensions are classified by their $SO(8)$ representation under Lorentz rotations, that leave the momentum invariant: $SO(8)$ is the 'little group' of $SO(1,9)$. The lowest lying surviving states in the NS sector are the eight transverse polarisations of the massless open string photon, A^μ, made by exciting the ψ oscillators:

$$\psi^\mu_{-1/2}|k\rangle, \qquad M^2 = 0. \tag{7.13}$$

These states clearly form the vector of $SO(8)$. They have $(-)^F = 1$ and so survive GSO.

In the R sector the ground state energy always vanishes because the world-sheet bosons and their superconformal partners have the same moding. The Ramond vacuum has a 32-fold degeneracy, since the ψ^μ_0 take ground states into ground states. The ground states form a representation of the ten dimensional Dirac matrix algebra

$$\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}. \tag{7.14}$$

(Note the similarity with the standard Γ-matrix algebra, $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. We see that $\psi^\mu_0 \equiv \Gamma^\mu/\sqrt{2}$.)

For this representation, it is useful to choose this basis:

$$\begin{aligned} d_i^\pm &= \frac{1}{\sqrt{2}}\left(\psi_0^{2i} \pm i\psi_0^{2i+1}\right) \qquad i = 1, \dots, 4 \\ d_0^\pm &= \frac{1}{\sqrt{2}}\left(\psi_0^1 \mp \psi_0^0\right). \end{aligned} \tag{7.15}$$

In this basis, the Clifford algebra takes the form

$$\{d_i^+, d_j^-\} = \delta_{ij}. \tag{7.16}$$

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The $d_i^{\pm}$, $i = 0, \ldots, 4$ act as creation and annihilation operators, generating the $2^{10/2} = 32$ Ramond ground states. Denote these states

$$|s_0, s_1, s_2, s_3, s_4\rangle = |\mathbf{s}\rangle \tag{7.17}$$

where each of the s_i takes the values $\pm\frac{1}{2}$, and where

$$d_i^- | -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0 \tag{7.18}$$

while d_i^+ raises s_i from $-\frac{1}{2}$ to $\frac{1}{2}$. This notation has physical meaning: the fermionic part of the ten dimensional Lorentz generators is

$$S^{\mu\nu} = -\frac{i}{2} \sum_{r \in \mathbf{Z}+\kappa} [\psi^{\mu}_{-r}, \psi^{\nu}_{r}], \tag{7.19}$$

(recall equation (2.124)). The states (7.17) above are eigenstates of $S_0 = iS^{01}$, $S_i = S^{2i,2i+1}$, with s_i the corresponding eigenvalues. Since by construction the Lorentz generators (7.19) always flip an even number of s_i, the Dirac representation **32** decomposes into a **16** with an even number of $-\frac{1}{2}$s and **16′** with an odd number.

The physical state conditions (7.11), on these ground states, reduce to $G_0 = (2\alpha')^{1/2} p_\mu \psi^\mu_0$. (Note that $G_0^2 \sim L_0$.) Let us pick the (massless) frame $p^0 = p^1$. This becomes

$$G_0 = \alpha'^{1/2} p_1 \Gamma_0 \left(1 - \Gamma_0\Gamma_1\right) = 2\alpha'^{1/2} p_1 \Gamma_0 \left(\tfrac{1}{2} - S_0\right), \tag{7.20}$$

which means that $s_0 = \frac{1}{2}$, giving a 16-fold degeneracy for the *physical* Ramond vacuum. This is a representation of $SO(8)$ which decomposes into $\mathbf{8_s}$ with an even number of $-\frac{1}{2}$s and $\mathbf{8_c}$ with an odd number. One is in the **16** and the **16′**, but the two choices, **16** or **16′**, are physically equivalent, differing only by a spacetime parity redefinition, which would therefore swap the $\mathbf{8_s}$ and the $\mathbf{8_c}$.

In the R sector the GSO projection amounts to requiring

$$\sum_{i=1}^{4} s_i = 0 \pmod 2, \tag{7.21}$$

picking out the $\mathbf{8_s}$. Of course, it is just a convention that we associated an even number of $\frac{1}{2}$s with the $\mathbf{8_s}$; a physically equivalent discussion with things the other way around would have resulted in $\mathbf{8_c}$. The difference between these two is only meaningful when they are both present, and at this stage we only have one copy, so either is as good as the other.

The ground state spectrum is then $\mathbf{8}_v \oplus \mathbf{8}_s$, a vector multiplet of $D = 10$, $\mathcal{N} = 1$ spacetime supersymmetry. Including Chan–Paton factors gives again a $U(N)$ gauge theory in the oriented theory and $SO(N)$ or $USp(N)$

in the unoriented. This completes our tree-level construction of the open superstring theory.

Of course, we are not finished, since this theory is (on its own) inconsistent for many reasons. One such reason (there are many others) is that it is anomalous. Both gauge invariance and coordinate invariance have anomalies arising because it is a chiral theory: e.g. the fermion $\mathbf{8_s}$ has a specific chirality in spacetime. The gauge and gravitational anomalies are very useful probes of the consistency of any theory. These show up quantum inconsistencies of the theory resulting in the failure of gauge invariance and general coordinate invariance, and hence must be absent. See insert 7.1 for more on anomalies.

Another reason we will see that the theory is inconsistent is that, as we learned in chapter 4, the theory is equivalent to some number of space-filling D9-branes in spacetime, and it will turn out later that these are positive electric sources of a particular 10-form field in the theory. The field equation for this field asks that all of its sources must simply vanish, and so we must have a negative source of this same field in order to cancel the D9-branes' contribution. This will lead us to the closed string sector i.e. one-loop, the same level at which we see the anomaly.

Let us study some closed strings. We will find three of interest here. Two of them will stand in their own right, with two ten dimensional supersymmetries, while the third will have half of that, and will be anomalous. This latter will be the closed string sector we need to supplement the open string we made here, curing its one-loop anomalies.

7.1.2 Closed superstrings: type II

Just as we saw before, the closed string spectrum is the product of two copies of the open string spectrum, with right- and left-moving levels matched. In the open string the two choices for the GSO projection were equivalent, but in the closed string there are two inequivalent choices, since we have to pick two copies to make a closed string.

Taking the same projection on both sides gives the 'type IIB' case, while taking them opposite gives 'type IIA'. These lead to the massless sectors

$$\begin{aligned}\text{Type IIA:}\quad &(\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_c})\\ \text{Type IIB:}\quad &(\mathbf{8_v}\oplus\mathbf{8_s})\otimes(\mathbf{8_v}\oplus\mathbf{8_s}).\end{aligned}\tag{7.22}$$

Let us expand out these products to see the resulting Lorentz ($SO(8)$) content. In the NS–NS sector, this is

$$\mathbf{8_v}\otimes\mathbf{8_v}=\Phi\oplus B_{\mu\nu}\oplus G_{\mu\nu}=\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}.\tag{7.23}$$

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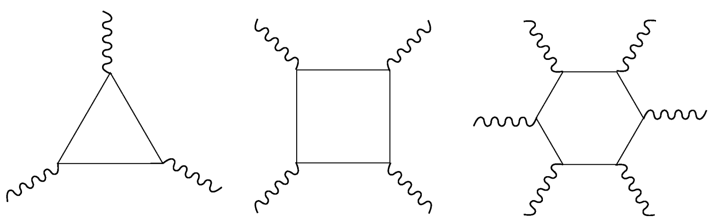
Insert 7.1. Gauge and gravitational anomalies

The beauty of the anomaly is that it is both a UV and an IR tool: UV since it represents the failure to be able to find a consistent regulator at the quantum level and IR since it cares only about the massless sector of the theory: Any potentially anomalous variations for the *effective* action $\Gamma = \ln Z$ should be written as the variation of a local term which allows it to be cancelled by adding a local counterterm. Massive fields always give effectively local terms at long distance.

An anomaly in D dimensions arises from complex representations of the Lorentz group which include chiral fermions in general but also bosonic representations if $D = 4k+2$, e.g. the rank $2k+1$ (anti)self-dual tensor. The anomalies are controlled by the so-called 'hexagon' diagram which generalises the (perhaps more familiar) triangle of four dimensional field theory or a square in six dimensions.



The external legs are either gauge bosons, gravitons, or a mixture. We shall not spend any time on the details[53], but simply state that consistency demands that the structure of the anomaly,

$$\delta \ln Z = \frac{i}{(2\pi)^{D/2}} \int \hat{I}_D,$$

is in terms of a D-form $\hat{I}_D$, polynomial in traces of even powers of the field strength two-forms $F = dA + A^2$ and $R = d\omega + \omega^2$. (Recall section 2.8.) It is naturally related to a $(D+2)$-form polynomial $\hat{I}_{D+2}$ which is gauge invariant and written as an exact form $\hat{I}_{D+2} = d\hat{I}_{D+1}$. The latter is not gauge invariant, but its variation is another exact form: $\delta\hat{I}_{D+1} = d\hat{I}_D$. A key example of this is the Chern–Simons three-form, which is discussed in insert 7.3, p. 167. See also insert 7.2 on p. 162 for explicit expressions in dimensions $D = 4k+2$. We shall see that the anomalies are a useful check of the consistency of string spectra that we construct in various dimensions.

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Insert 7.2. A list of anomaly polynomials

It is useful to list here some anomaly polynomials for later use. In ten dimensions, the contributions to the polynomial come from three sorts of field, the spinors $\mathbf{8}_{s,c}$, the gravitinos $\mathbf{56}_{c,s}$, and the fifth rank antisymmetric tensor field strength with its self-dual and anti-self-dual parts. The anomalies for each pair within each sort are equal and opposite in sign, i.e. $\hat{I}_{12}^{\mathbf{8}_s} = -\hat{I}_{12}^{\mathbf{8}_c}$, etc., and we have:

$$\begin{aligned}
\hat{I}_{12}^{\mathbf{8}_s} = & -\frac{\mathrm{Tr}(F^6)}{1440} \\
& + \frac{\mathrm{Tr}(F^4)\mathrm{tr}(R^2)}{2304} - \frac{\mathrm{Tr}(F^2)\mathrm{tr}(R^4)}{23040} - \frac{\mathrm{Tr}(F^2)[\mathrm{tr}(R^2)]^2}{18432} \\
& + \frac{n\mathrm{tr}(R^6)}{725760} + \frac{n\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} + \frac{n[\mathrm{tr}(R^2)]^3}{1327104}; \\
\hat{I}_{12}^{\mathbf{56}_c} = & -495\frac{\mathrm{tr}(R^6)}{725760} + 225\frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} - 63\frac{[\mathrm{tr}(R^2)]^3}{1327104}; \\
\hat{I}_{12}^{\mathbf{35}_+} = & +992\frac{\mathrm{tr}(R^6)}{725760} - 448\frac{\mathrm{tr}(R^4)\mathrm{tr}(R^2)}{552960} + 128\frac{[\mathrm{tr}(R^2)]^3}{1327104},
\end{aligned}$$

and n is the dimension of the gauge representation under which the spinor transforms, for which we use the trace denoted Tr. We also have suppressed the use of $\wedge$, for brevity. For $D = 6$, there are anomaly eight-forms. We denote the various fields by their transformation properties of the $D = 6$ little group $SO(4) \sim SU(2) \times SU(2)$:

$$\begin{aligned}
\hat{I}_8^{(\mathbf{1},\mathbf{2})} = & +\frac{\mathrm{Tr}(F^4)}{24} - \frac{\mathrm{Tr}(F^2)\mathrm{tr}(R^2)}{96} + \frac{n\mathrm{tr}(R^4)}{5760} + \frac{n[\mathrm{tr}(R^2)]^2}{4608}; \\
\hat{I}_8^{(\mathbf{3},\mathbf{2})} = & +245\frac{\mathrm{tr}(R^4)}{5760} - 43\frac{[\mathrm{tr}(R^2)]^2}{4608}; \\
\hat{I}_8^{(\mathbf{3},\mathbf{1})} = & +28\frac{\mathrm{tr}(R^4)}{5760} - 8\frac{[\mathrm{tr}(R^2)]^2}{4608}.
\end{aligned}$$

Note that the first two are for complex fermions. For real fermions, one must divide by two. For completeness, for $D = 2$ we list the three analogous anomaly four-forms:

$$\hat{I}_4^{1/2} = \frac{n\mathrm{tr}(R^2)}{48} - \frac{\mathrm{Tr}(F^2)}{2}, \qquad \hat{I}_4^{3/2} = -23\frac{\mathrm{tr}(R^2)}{48}, \qquad \hat{I}_4^{0} = \frac{\mathrm{tr}(R^2)}{48}.$$

It is amusing to note that the anomaly polynomials can be written in terms of geometrical characteristic classes. This should be kept at the back of the mind for a bit later, in section 9.5.

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In the R–R sector, the IIA and IIB spectra are respectively

$$\begin{aligned}\mathbf{8_s} \otimes \mathbf{8_c} &= [1] \oplus [3] = \mathbf{8_v} \oplus \mathbf{56_t} \\ \mathbf{8_s} \otimes \mathbf{8_s} &= [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+. \end{aligned} \tag{7.24}$$

Here $[n]$ denotes the n-times antisymmetrised representation of $SO(8)$, and $[4]_+$ is self-dual. Note that the representations $[n]$ and $[8-n]$ are the same, as they are related by contraction with the eight dimensional ϵ-tensor. The NS–NS and R–R spectra together form the bosonic components of $D = 10$ IIA (nonchiral) and IIB (chiral) supergravity respectively; We will write their effective actions shortly.

In the NS–R and R–NS sectors are the products

$$\begin{aligned}\mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s}. \end{aligned} \tag{7.25}$$

The $\mathbf{56_{s,c}}$ are gravitinos. Their vertex operators are made roughly by tensoring a NS field ψ^μ with a vertex operator $\mathcal{V}_\alpha = e^{-\varphi/2}\mathbf{S}_\alpha$, where the latter is a 'spin field', made by bosonising the d_is of equation (7.15) and building:

$$\mathbf{S} = \exp\left[i\sum_{i=0}^{4} s_i H^i\right]; \quad d_i = e^{\pm i H^i}. \tag{7.26}$$

(The factor $e^{-\varphi/2}$ is the bosonisation (see section 4.7) of the Faddeev–Popov ghosts (see insert 3.2), about which we will have nothing more to say here.) The resulting full gravitino vertex operators, which correctly have one vector and one spinor index, are two fields of weight $(0,1)$ and $(1,0)$, respectively, depending upon whether ψ^μ comes from the left or right. These are therefore holomorphic and anti-holomorphic world-sheet currents, and the symmetry associated to them in spacetime is the supersymmetry. In the IIA theory the two gravitinos (and supercharges) have opposite chirality, and in the IIB the same.

Consider the vertex operators for the R–R states[1]. This will involve a product of spin fields[74], one from the left and one from the right. These again decompose into antisymmetric tensors, now of $SO(9,1)$:

$$V = \mathcal{V}_\alpha \mathcal{V}_\beta (\Gamma^{[\mu_1} \ldots \Gamma^{\mu_n]} C)_{\alpha\beta} G_{[\mu_1 \cdots \mu_n]}(X) \tag{7.27}$$

with C the charge conjugation matrix. In the IIA theory the product is $\mathbf{16} \otimes \mathbf{16}'$ giving even n (with $n \cong 10 - n$) and in the IIB theory it is $\mathbf{16} \otimes \mathbf{16}$ giving odd n. As in the bosonic case, the classical equations of motion follow from the physical state conditions, which at the massless level reduce to $G_0 \cdot V = \tilde{G}_0 \cdot V = 0$. The relevant part of G_0 is just $p_\mu \psi_0^\mu$ and similarly for $\tilde{G}_0$. The p_μ act by differentiation on G, while ψ_0^μ

acts on the spin fields as it does on the corresponding ground states: as multiplication by Γ^μ. Noting the identity

$$\Gamma^\nu \Gamma^{[\mu_1} \ldots \Gamma^{\mu_n]} = \Gamma^{[\nu} \ldots \Gamma^{\mu_n]} + \left(\delta^{\nu\mu_1} \Gamma^{[\mu_2} \ldots \Gamma^{\mu_n]} + \text{perms} \right) \tag{7.28}$$

and similarly for right multiplication, the physical state conditions become

$$dG = 0 \qquad\qquad d^*G = 0. \tag{7.29}$$

These are the Bianchi identity and field equation for an antisymmetric tensor field strength. This is in accord with the representations found: in the IIA theory we have odd-rank tensors of $SO(8)$ but even-rank tensors of $SO(9,1)$ (and reversed in the IIB), the extra index being contracted with the momentum to form the field strength. It also follows that R–R amplitudes involving elementary strings vanish at zero momentum, so strings do not carry R–R charges†.

As an aside, when the dilaton background is nontrivial, the Ramond generators have a term $\Phi_{,\mu}\partial\psi^\mu$, and the Bianchi identity and field strength pick up terms proportional to $d\Phi \wedge G$ and $d\Phi \wedge {}^*G$. The Bianchi identity is non-standard, so G is not of the form dC. Defining $G' = e^{-\Phi}G$ removes the extra term from both the Bianchi identity and field strength. The field G' is thus decoupled from the dilaton. In terms of the action, the fields G in the vertex operators appear with the usual closed string $e^{-2\Phi}$ but with non-standard dilaton gradient terms. The fields we are calling G' (which in fact are the usual fields used in the literature, and so we will drop the prime symbol in the sequel) have a dilaton-independent action.

The type IIB theory is chiral since it has different numbers of left moving fermions from right-moving. Furthermore, there is a self-dual R–R tensor. These structures in principle produce gravitational anomalies, and it is one of the miracles (from the point of view of the low energy theory) of string theory that the massless spectrum is in fact anomaly free. There is a delicate cancellation between the anomalies for the $\mathbf{8}_c$ and for the $\mathbf{56}_s$ and the $\mathbf{35}_+$. The reader should check this by using the anomaly polynomials in insert 7.2, (of course, put $n = 1$ and $F = 0$) to see that

$$-2\hat{I}_{12}^{\mathbf{8}_s} + 2\hat{I}_{12}^{\mathbf{56}_c} + \hat{I}_{12}^{\mathbf{35}_+} = 0, \tag{7.30}$$

which is in fact miraculous, as previously stated[339].

† The reader might wish to think of this as analogous to the discovery that a moving electric point source generates a magnetic field, but of course is not a basic magnetic monopole source.

7.1.3 Type I from type IIB, the prototype orientifold

As we saw in the bosonic case, we can construct an unoriented theory by projecting onto states invariant under world-sheet parity, Ω. In order to get a consistent theory, we must of course project a theory which is invariant under Ω to start with. Since the left and right moving sectors have the same GSO projection for type IIB, it is invariant under Ω, so we can again form an unoriented theory by gauging. We cannot gauge Ω in type IIA to get a consistent theory, but see later.

Projecting onto $\Omega = +1$ interchanges left-moving and right-moving oscillators and so one linear combination of the R–NS and NS–R gravitinos survives, so there can be only one supersymmetry remaining. In the NS–NS sector, the dilaton and graviton are symmetric under Ω and survive, while the antisymmetric tensor is odd and is projected out. In the R–R sector, by counting we can see that the $\mathbf{1}$ and $\mathbf{35}_+$ are in the symmetric product of $\mathbf{8_s} \otimes \mathbf{8_s}$ while the $\mathbf{28}$ is in the antisymmetric. The R–R state is the product of right- and left-moving fermions, so there is an extra minus in the exchange. Therefore it is the $\mathbf{28}$ that survives. The bosonic massless sector is thus $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$, and together with the surviving gravitino, this give us the $D = 10$ $N = 1$ supergravity multiplet.

Sadly, this supergravity is in fact anomalous. The delicate balance (7.30) between the anomalies from the various chiral sectors, which we noted previously, vanishes since one each of the $\mathbf{8}_c$ and $\mathbf{56}_s$, and the $\mathbf{35}_+$, have been projected out. Nothing can save the theory unless there is an additional sector to cancel the anomaly.[107]

This sector turns out to be $N = 1$ supersymmetric Yang–Mills theory, with gauge group $SO(32)$ or $E_8 \times E_8$. Happily, we already know at least one place to find the first choice: We can use the low-energy (massless) sector of $SO(32)$ unoriented open superstring theory. This fits nicely, since as we have seen before, at one loop open strings couple to closed strings. We will not be able to get gauge group $E_8 \times E_8$ from perturbative open string theory (Chan–Paton factors can't make this sort of group), but we will see shortly that there is another way of getting this group, but from a closed string theory.

The total anomaly is that of the gravitino, dilatino and the gaugino, the latter being charged in the adjoint of the gauge group:

$$I_{12} = -\hat{I}_{12}^{\mathbf{8}_s}(R) + \hat{I}_{12}^{\mathbf{56}_c}(R) + \hat{I}_{12}^{\mathbf{8}_s}(F, R). \tag{7.31}$$

Using the polynomials given in insert 7.2, it should be easily seen that there is an irreducible term

$$(n - 496)\frac{\mathrm{tr}(R^6)}{725760}, \tag{7.32}$$

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which must simply vanish, and so n, the dimension of the group, must be 496. Since $SO(32)$ and $E_8 \times E_8$ both have this dimension, this is encouraging. That the rest of the anomaly cancels is a very delicate and important story which deserves some attention. We will do that in the next section.

Finishing the present discussion, in the language we learned in section 4.11, we put a single (space-filling) O9-plane into type IIB theory, making the type IIB theory into the unoriented $N = 1$ closed string theory. This is anomalous, but we can cancel the resulting anomalies by adding 16 D9-branes.

Another way of putting it is that (as we shall see) the O9-plane has 16 units of C_{10} charge, which cancels that of 16 D9-branes, satisfying the equations of motion for that field.

We have just constructed our first (and in fact, the simplest) example of an 'orientifolding' of a superstring theory to get another. More complicated orientifolds may be constructed by gauging combinations of Ω with other discrete symmetries of a given string theory which form an 'orientifold group' G_Ω under which the theory is invariant[28]. Generically, there will be the requirement to cancel anomalies by the addition of open string sectors (i.e. D-branes), which results in consistent new string theory with some spacetime gauge group carried by the D-branes. In fact, these projections give rise to gauge groups containing any of $U(n)$, $USp(n)$ factors, and not just $SO(n)$ sectors.

7.1.4 The Green–Schwarz mechanism

Let us finish showing that the anomalies of $\mathcal{N} = 1$, $D = 10$ supergravity coupled to Yang–Mills do vanish for the groups $SO(32)$ and $E_8 \times E_8$. We have already shown above that the dimension of the group must be $n = 496$. Some algebra shows that that the rest of the anomaly (7.31), for *this* value of n can be written suggestively as:

$$\begin{aligned} I_{12}^{(n=496)} &= \frac{1}{3\times 2^8} Y_4 X_8 \\ &\quad - \frac{1}{1440}\left(\mathrm{Tr}_{\mathrm{adj}}(F^6) - \frac{\mathrm{Tr}_{\mathrm{adj}}(F^2)\mathrm{Tr}_{\mathrm{adj}}(F^4)}{48} + \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^3}{14400}\right), \end{aligned} \tag{7.33}$$

where

$$\begin{aligned} Y_4 &= \mathrm{tr}(R^2) - \frac{1}{30}\mathrm{Tr}_{\mathrm{adj}}(F^2), \\ X_8 &= \frac{\mathrm{Tr}_{\mathrm{adj}}(F^4)}{3} - \frac{[\mathrm{Tr}_{\mathrm{adj}}(F^2)]^2}{900} - \frac{\mathrm{Tr}_{\mathrm{adj}}(F^2)\mathrm{tr}(R^2)}{30} + \\ &\qquad + \mathrm{tr}(R^4) + \frac{[\mathrm{tr}(R^2)]^2}{4}. \end{aligned} \tag{7.34}$$

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Insert 7.3. The Chern–Simons three-form

The Chern–Simons three-form is a very important structure which will appear in a number of places, and it is worth pausing a while to consider its properties. Recall from insert 2.5 that we can write the gauge potential, and the field strength as Lie Algebra–valued forms: $A = t^a A^a_\mu dx^\mu$, where the t^a are generators of the Lie algebra. We can write the Yang–Mills field strength as a matrix-valued two-form, $F = t^a F^a_{\mu\nu} dx^\mu \wedge dx^\nu$. We can define the Chern–Simons three-form as

$$\omega_{3\mathrm{Y}} = \mathrm{Tr}\left(A \wedge F - \frac{1}{3} A \wedge A \wedge A\right) = \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

One interesting thing about this object is that we can write:

$$d\omega_{3\mathrm{Y}} = \mathrm{Tr}\,(F \wedge F).$$

Furthermore, under a gauge transformation $\delta A = d\Lambda + [A, \Lambda]$:

$$\delta\omega_{3\mathrm{Y}} = \mathrm{Tr}(d\Lambda dA) = d\omega_2, \quad \omega_2 = \mathrm{Tr}(\Lambda dA).$$

So its gauge variation, while not vanishing, is an exact three-form. Note that there is a similar structure in the pure geometry sector. From section 2.8, we recall that the potential analogous to A is the spin connection one-form $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$, with a and b being Minkowski indices in the space tangent to the point x^μ in spacetime and so ω is an $SO(D-1, 1)$ matrix in the fundamental representation. The curvature is a two-form $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$, and the gauge transformation is now $\delta\omega = d\Theta + [\omega, \Theta]$. We can define:

$$\omega_{3\mathrm{L}} = \mathrm{tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right),$$

with similar properties to $\omega_{3\mathrm{Y}}$, above. Here tr means trace on the indices a, b.

On the face of it, it does not really seem possible that this can be cancelled, since the the gaugino carries gauge charge and nothing else does, and so there are a lot of gauge quantities which simply stand on their own. This seems hopeless because we have so far restricted ourselves to quantum anomalies arising from the gauge and gravitational sector. If we include the rank two R–R potential $C_{(2)}$ in a cunning way, we can generate a

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mechanism for cancelling the anomaly. Consider the interaction

$$S_{\rm GS} = \frac{1}{3 \times 2^6 (2\pi)^5 \alpha'} \int C_{(2)} \wedge X_8. \tag{7.35}$$

It is invariant under the usual gauge transformations

$$\delta A = d\Lambda + [A, \Lambda]; \quad \delta\omega = d\Theta + [\omega, \Theta], \tag{7.36}$$

since it is constructed out of the field strengths F and R. It is also invariant under the two-form potential's standard transformation $\delta C_{(2)} = d\lambda$. Let us however give $C_{(2)}$ another gauge transformation rule. While A and ω transform under (7.36), let it transform as:

$$\delta C_{(2)} = \frac{\alpha'}{4} \left(\frac{1}{30} {\rm Tr}(\Lambda F) - {\rm tr}(\Theta R) \right). \tag{7.37}$$

Then the variation of the action does not vanish, and is:

$$\delta S_{\rm GS} = \frac{1}{3 \times 2^8 (2\pi)^5} \int \left[\frac{1}{30} {\rm Tr}(\Lambda F) - {\rm tr}(\Theta R) \right] \wedge X_8.$$

However, using the properties of the Chern–Simons three-form discussed in insert 7.3, this classical variation can be written as descending *via* the consistency chain in insert 7.1 from precisely the 12-form polynomial given in the first line of equation (7.34), but with a minus sign. Therefore we cancel that offending term with this classical modification of the transformation of $C_{(2)}$. Later on, when we write the supergravity action for this field in the type I model, we will use the modified field strength:

$$\widetilde{G}^{(3)} = dC^{(2)} - \frac{\alpha'}{4} \left[\frac{1}{30} \omega_{3{\rm Y}}(A) - \omega_{3{\rm L}}(\Omega) \right], \tag{7.38}$$

where because of the transformation properties of the Chern–Simons three-form (see insert 7.3), $\widetilde{G}^{(3)}$ is gauge invariant under the new transformation rule (7.37).

N.B. It is worth noting here that this is a quite subtle mechanism. We are cancelling the anomaly generated by a one loop diagram with a tree-level graph. It is easy to see what the tree level diagram is. The kinetic term for the modified field strength will have its square appearing, and so looking at its definition (7.38), we see that there is a vertex coupling $C_{(2)}$ to two gauge bosons or to two gravitons. There is another vertex that comes from the interaction (7.35) which couples $C_{(2)}$ to four particles, pairs of gravitons and pairs of gauge bosons, or a mixture. So the tree level diagram in figure 7.1 can mix with the hexagon anomaly of insert 7.1.

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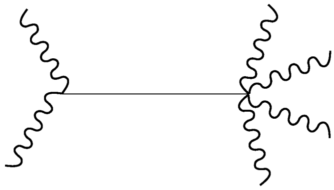

Fig. 7.1. The tree which cures the $\mathcal{N}=1$ $D=10$ anomalies. A two-form field is exchanged.

Somehow, the terms in the second line must cancel amongst themselves. Miraculously, they do for a number of groups, $SO(32)$ and $E_8 \times E_8$ included. For the first group, it follows from the fact that for the group $SO(n)$, we can write:

$$\begin{aligned}\mathrm{Tr}_{\mathrm{adj}}(t^6) &= (n-32)\mathrm{Tr}_{\mathrm{f}}(t^6) + 15\mathrm{Tr}_{\mathrm{f}}(t^2)\mathrm{Tr}_{\mathrm{f}}(t^4);\\ \mathrm{Tr}_{\mathrm{adj}}(t^4) &= (n-8)\mathrm{Tr}_{\mathrm{f}}(t^4) + 3\mathrm{Tr}_{\mathrm{f}}(t^2)\mathrm{Tr}_{\mathrm{f}}(t^2);\\ \mathrm{Tr}_{\mathrm{adj}}(t^2) &= (n-2)\mathrm{Tr}_{\mathrm{f}}(t^2),\end{aligned} \tag{7.39}$$

where the subscript ‘f’ denotes the fundamental representation. For E_8, we have that

$$\begin{aligned}\mathrm{Tr}_{\mathrm{adj}}(t^6) &= \frac{1}{7200}[\mathrm{Tr}_{\mathrm{f}}(t^2)]^3,\\ \mathrm{Tr}_{\mathrm{adj}}(t^4) &= \frac{1}{100}[\mathrm{Tr}_{\mathrm{f}}(t^2)]^2.\end{aligned} \tag{7.40}$$

In checking these (which of course the reader will do) one should combine the traces as $\mathrm{Tr}_{G_1\times G_2} = \mathrm{Tr}_{G_1} + \mathrm{Tr}_{G_2}$, etc.

Overall, the results[107] of this subsection are quite remarkable, and generated a lot of excitement which we now call the First Superstring Revolution. This excitement was of course justified, since the discovery of the mechanism revealed that there were consistent superstring theories with considerably intricate structures with promise for making contact with the physics that we see in Nature.

7.2 The two basic heterotic string theories

In addition to the three superstring theories briefly constructed above, there are actually two more supersymmetric string theories which live in ten dimensions. In addition, they have non-Abelian spacetime gauge symmetry, and they are also free of tachyons. These are the ‘Heterotic Strings’[20]. The fact that they are chiral, have fermions and non-Abelian

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gauge symmetry meant that they were considered extremely attractive as starting points for constructing 'realistic' phenomenology based on string theory. It is in fact remarkable that one can come tantalisingly close to naturally realising many of the features of the Standard Model of particle physics by starting with, say, the $E_8 \times E_8$ Heterotic String, while remaining entirely in the perturbative regime. This was the focus of much of the First Superstring Revolution. Getting many of the harder questions right led to the search for non-perturbative physics, which ultimately led us to the Second Superstring Revolution, and the realisation that all of the other string theories were just as important too, because of duality.

One of the more striking things about the heterotic strings, from the point of view of what we have done so far, is the fact that they have non-Abelian gauge symmetry and are still closed strings. The $SO(32)$ of the type I string theory comes from Chan–Paton factors at the ends of the open string, or in the language we now use, from 16 coincident D9-branes.

We saw a big hint of what is needed to get spacetime gauge symmetry in the heterotic string in chapter 4. Upon compactifying bosonic string theory on a circle, at a special radius of the circle, an enhanced $SU(2)_{\rm L} \times SU(2)_{\rm R}$ gauge symmetry arose. From the two dimensional world-sheet point of view, this was a special case of a current algebra, which we uncovered further in section 4.6. We can take two key things away from that chapter for use here. The first is that we can generalise this to a larger non-Abelian gauge group if we use more bosons, although this would seem to force us to have many compact directions. The second is that there were identical and *independent* structures coming from the left and the right to give this result. So we can take, say, the left hand side of the construction and work with it, to produce a single copy of the non-Abelian gauge group in spacetime.

This latter observation is the origin of the word 'heterotic' which comes from 'heterosis'. The theory is a hybrid of two very different constructions on the left and the right. Let us take the right hand side to be a copy of the right hand side of the superstrings we constructed previously, and so we use only the right hand side of the action given in equation (7.1) (with closed string boundary conditions). Then the usual consistency checks give that the critical dimension is of course ten, as before: the central charge (conformal anomaly) is $-26 + 11 = 15$ from the conformal and superconformal ghosts. This is cancelled by ten bosons and their superpartners since they contribute to the anomaly an amount $10 \times 1 + 10 \times \frac{1}{2} = 15$. The left hand side is in fact a purely bosonic string, and so the anomaly is cancelled to zero by the -26 from the conformal ghosts and there must be the equivalent of 26 bosonic degrees of freedom, contributing 26×1 to the anomaly.

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How can the theory make sense as a ten dimensional theory? The answer to this question is just what gives the non-Abelian gauge symmetry. Sixteen of the bosons are periodic, and so may be thought of as compactified on a torus $T^{16} \simeq (S^1)^{16}$ with very specific properties. Those properties are such that the generic $U(1)^{16}$ one might have expected from such a toroidal compactification is enhanced to one of two special rank 16 gauge groups: $SO(32)$, or $E_8 \times E_8$, via the very mechanism we saw in chapter 4: the torus is 'self-dual'. The remaining ten non-compact bosons on the left combine with the ten on the right to make the usual ten spacetime coordinates, on which the usual ten dimensional Lorentz group $SO(1,9)$ acts.

7.2.1 $SO(32)$ and $E_8 \times E_8$ from self-dual lattices

The requirements are simple to state. We are required to have a sixteen dimensional lattice, according to the above discussion, and so we can apply the results of chapter 4, but there is a crucial difference. Recalling what we learned there, we see that since we only have a left-moving component to this lattice, we do not have the Lorenzian signature which arose there, but only a *Euclidean* signature. But all of the other conditions apply: it must be even, in order to build gauge bosons as vertex operators, and it must be self-dual, to ensure modular invariance.

The answer turns out to be quite simple. There are only two choices, since even self-dual Euclidean lattices are very rare (They only exist when the dimension is a multiple of eight). For sixteen dimensions, there is either $\Gamma_8 \times \Gamma_8$ or Γ_{16}. The lattice Γ_8 is the collection of points:

$$(n_1, n_2, \ldots, n_8) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, n_2 + \tfrac{1}{2}, \ldots, n_8 + \tfrac{1}{2}), \quad \sum_i n_i \in 2\mathbb{Z},$$

with $\sum_i n_i^2 = 2$. The integer lattice points are actually the root lattice of $SO(16)$, with which the 120 dimensional adjoint representation is made. The half-integer points construct the spinor representation of $SO(16)$. A bit of thought shows that it is just like the construction we made of the spinor representations of $SO(8)$ previously; the entries are only $\pm\frac{1}{2}$ in eight different slots, with only an even number of minus signs appearing, which again gives a squared length of two. There are $2^7 = 128$ possibilities, which is the dimension of the spinor representation. The total dimension of the represetnation we can make is $120 + 128 = 248$ which is the dimension of E_8. The sixteen dimensional lattice is made as the obvious tensor product of two copies of this, giving gauge group $E_8 \times E_8$, which is 496 dimensional.

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The lattice Γ_{16} is extremely similar, in that it is:

$$(n_1, n_2, \dots, n_{16}) \quad \text{or} \quad (n_1 + \tfrac{1}{2}, n_2 + \tfrac{1}{2}, \dots, n_{16} + \tfrac{1}{2}), \quad \sum_i n_i \in 2\mathbb{Z},$$

with $\sum_i n_i^2 = 2$. Again, we see that the integer points make the root lattice of $SO(32)$, but there is more. There is a spinor representation of $SO(32)$, but it is clear that since $16 \times 1/4 = 4$, the squared length is twice as large as it need to be to make a massless vector, and so the gauge bosons remain from the adjoint of $SO(32)$, which is 496 dimensional. In fact, the full structure is more than $SO(32)$, because of this spinor representation. It is not quite the cover, which is $Spin(32)$ because the conjugate spinor and the vector representations are missing. It is instead written as $Spin(32)/\mathbb{Z}_2$. In fact, $SO(32)$ in the quotient of $Spin(32)$ by another $\mathbb{Z}_2$.

Actually, before concluding, we should note that there is an alternative construction to this one using left–moving fermions instead of bosons. This is easily arrived at from here using what we learned about fermionisation in section 4.7. From there, we learn that we can trade in each of the left-moving bosons here for *two* left–moving Majorana–Weyl fermions, giving a fermionic construction with 32 fermions $\mathbf{\Psi}^i$. The construction divides the fermions into the NS and R sectors as before, which correspond to the integer and half-integer lattice sites in the above discussion. The difference between the two heterotic strings is whether the fermions are split into two sets with independent boundary conditions (giving $E_8 \times E_8$) or if they have all the same boundary conditions ($SO(32)$). In this approach, there is a GSO projection, which in fact throws out a tachyon, etc. Notice that in the R sector, the zero modes of the 32 $\mathbf{\Psi}^i$ will generate a spinor and conjugate spinor $\mathbf{2^{31}} \oplus \overline{\mathbf{2^{31}}}$ of $SO(32)$ for much the same reasons as we saw a $\mathbf{16} \oplus \overline{\mathbf{16}}$ in the construction of the superstring. Just as there, a GSO projection arises in the construction, which throws out the conjugate spinor, leaving the sole massive spinor we saw arise in the direct lattice approach.

7.2.2 The massless spectrum

In the case we must consider here, we can borrow a lot of what we learned in section 4.5 with hardly any adornment. We have sixteen compact left-moving bosons, X^i, which, together with the allowed momenta P^i, define a lattice Γ. The difference between this lattice and the ones we considered in section 4.5 is that there is no second part coming from a family of right-moving momenta, and hence it is only half the expected dimension, and with a purely Euclidean signature. This sixteen dimensional lattice must again be self-dual and even. This amounts to the requirement of modular

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invariance, just as before. More directly, we can see what effect this has on the low-lying parts of the spectrum.

Recall that the NS and R sector of the right hand side has zero point energy equal to $-1/2$ and 0, respectively. Recall that we then make, after the GSO projection, the vector $\mathbf{8}_v$, and its superpartner the spinor $\mathbf{8}_s$ from these two sectors. On the left hand side, we have the structure of the bosonic string, with zero point energy -1. There is no GSO projection on this side, and so potentially we have the tachyon, $|0\rangle$, the familiar massless states $\alpha^{\mu}_{-1}|0\rangle$, and the current algebra elements $J^a_{-1}|0\rangle$. These must be tensored together with the right hand side's states, but we must be aware that the level–matching condition is modified. To work out what it is we must take the difference between the correctly normalised *ten dimensional* M^2 operators on each side. We must also recall that in making the ten dimensional M^2 operator, we are left with a remainder, the contribution to the internal momentum $\alpha' p_{\rm L}^2/4$. The result is:

$$\frac{\alpha' p_{\rm L}^2}{4} + N - 1 = \tilde{N} - \left\{ \begin{array}{c} -\frac{1}{2} \\ 0 \end{array} \right. ,$$

where the choice corresponds to the NS or R sectors.

Now we can see how the tachyon is projected out of the theory, even without a GSO projection on the left. The GSO on the right has thrown out the tachyon there, and so we start with $\tilde{N} = \frac{1}{2}$ there. The left tachyon is $N = 0$, but this is not allowed, and we must have the even condition $\alpha' p_{\rm L}^2/2 = 2$ which corresponds to switching on a current J^a_{-1}, making a massless state. If we do not have this state excited, then we can also make a massless state with $N = 1$, corresponding to $\alpha^{\mu}_{-1}|0\rangle$.

The massless states we can make by tensoring left and right, respecting level–matching are actually familiar. In the NS–NS sector, we have $\alpha^{\mu}_{-1}\psi^{\nu}_{-1/2}|0\rangle$, which is the graviton, $G_{\mu\nu}$ antisymmetric tensor $B_{\mu\nu}$ and dilaton Φ in the usual way. We also have $J^a_{-1}\psi^{\mu}_{-1/2}|0\rangle$, which gives an $E_8 \times E_8$ or $SO(32)$ gauge boson, $A^{\mu a}$. In the NS–R sector, we have $\alpha^{\mu}_{-1}|0\rangle_\alpha$ which is the gravitino, ψ^{μ}_{α}. Finally, we have $J^a_{-1}|0\rangle_\alpha$, which is the superpartner of the gauge boson, λ^a_α. In the language we used earlier, we can write the left hand representations under $SO(8) \times G$ (where G is $SO(32)$ or $E_8 \times E_8$) as $(\mathbf{8}_v, \mathbf{1})$ or $(\mathbf{1}, \mathbf{496})$. Then the tensoring is

$$\begin{aligned} &(\mathbf{8}_v, \mathbf{1}) \otimes (\mathbf{8}_v + \mathbf{8}_s) = (\mathbf{1}, \mathbf{1}) + (\mathbf{35}, \mathbf{1}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{56}_s, \mathbf{1}) + (\mathbf{8}_s, \mathbf{1}), \\ &(\mathbf{1}, \mathbf{496}) \otimes (\mathbf{8}_v + \mathbf{8}_s) = (\mathbf{8}_v, \mathbf{496}) + (\mathbf{8}_s, \mathbf{496}). \end{aligned}$$

So we see that we have again obtained the $\mathcal{N} = 1$ supergravity multiplet, coupled to a massless vector. The effective theory which must result at low energy must have the same gravity sector, but since the gauge fields arise

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at closed string tree level, their Lagrangian must have a dilaton coupling $e^{2\Phi}$, instead of e^{Φ} for the open string where the gauge fields arise at open string tree level.

7.3 The ten dimensional supergravities

Just as we saw in the case of the bosonic string, we can truncate consistently to focus on the massless sector of the string theories, by focusing on low energy limit $\alpha' \to 0$. Also as before, the dynamics can be summarised in terms of a low energy effective (field theory) action for these fields, commonly referred to as 'supergravity'.

The bosonic part of the low energy action for the type IIA string theory in ten dimensions may be written (cf. equation (2.106)) as (the wedge product is understood)[1, 5, 75]:

$$S_{\rm IIA} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right.$$
$$\left.-\frac{1}{4}(G^{(2)})^2-\frac{1}{48}(G^{(4)})^2\right\}-\frac{1}{4\kappa_0^2}\int B^{(2)}dC^{(3)}dC^{(3)}. \tag{7.41}$$

As before $G_{\mu\nu}$ is the metric in string frame, Φ is the dilaton, $H^{(3)}=dB^{(2)}$ is the field strength of the NS–NS two form, while the Ramond–Ramond field strengths are $G^{(2)}=dC^{(1)}$ and $G^{(4)}=dC^{(3)}+H^{(3)}\wedge C^{(1)}$‡.

For the bosonic part in the case of type IIB, we have:

$$S_{\rm IIB} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right.$$
$$\left.-\frac{1}{12}(G^{(3)}+C^{(0)}H^{(3)})^2-\frac{1}{2}(dC^{(0)})^2-\frac{1}{480}(G^{(5)})^2\right\}$$
$$+\frac{1}{4\kappa_0^2}\int\left(C^{(4)}+\frac{1}{2}B^{(2)}C^{(2)}\right)G^{(3)}H^{(3)}. \tag{7.42}$$

Now, $G^{(3)}=dC^{(2)}$ and $G^{(5)}=dC^{(4)}+H^{(3)}C^{(2)}$ are R–R field strengths, and $C^{(0)}$ is the R–R scalar. (Note that we have canonical normalisations for the kinetic terms of forms: there is a prefactor of the inverse of $-2\times p!$ for a p-form field strength.) There is a small complication due to the fact that we require the R–R four form $C^{(4)}$ to be self-dual, or we will have too many degrees of freedom. We write the action here and remind ourselves to always impose the self-duality constraint on its field strength $F^{(5)}=dC^{(4)}$ by hand in the equations of motion: $F^{(5)}={}^*F^{(5)}$.

‡ This can be derived by dimensional reduction from the structurally simpler eleven dimensional supergravity action, presented in chapter 12, but at this stage, this relation is a merely formal one. We shall see a dynamical connection later.

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Equation (2.109) tells us that, in ten dimensions, we must use

$$\widetilde{G}_{\mu\nu} = e^{(\Phi_0-\Phi)/2}G_{\mu\nu} \tag{7.43}$$

to convert these actions to the Einstein frame. As before (see discussion below equation(2.111)), Newton's constant will be set by

$$2\kappa^2 \equiv 2\kappa_0^2 g_s^2 = 16\pi G_N = (2\pi)^7\alpha'^4 g_s^2, \tag{7.44}$$

where the latter equality can be established by (for example) direct examination of the results of a graviton scattering computation. We will see that it gives a very natural normalisation for the masses and charges of the various branes in the theory. Also $g_{\rm s}$ is set by the asymptotic value of the dilaton at infinity: $g_{\rm s} \equiv e^{\Phi_0}$.

Those were the actions for the ten dimensional supergravities with thirty-two supercharges. Let us consider those with sixteen supercharges. For the bosonic part of type I, we can construct it by dropping the fields which are odd under Ω and then adding the gauge sector, plus a number of cross terms which result from cancelling anomalies, as we discussed in subsection 7.1.3:

$$S_{\rm I} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}\bigg\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2\right] \\ -\frac{1}{12}(\widetilde{G}^{(3)})^2 - \frac{\alpha'}{8}e^{-\Phi}{\rm Tr}(F^{(2)})^2\bigg\}. \tag{7.45}$$

Here, $\widetilde{G}^{(3)}$ is a modified field strength for the two-form potential, defined in equation (7.38). Recall that this modification followed from the requirement of cancellation of the anomaly via the Green–Schwarz mechanism.

We can generate the heterotic low-energy action using a curiosity which will be meaningful later. Notice that a simple redefinition of fields:

$$\begin{aligned} G_{\mu\nu}(\text{type I}) &= e^{-\Phi}G_{\mu\nu}(\text{heterotic}) \\ \Phi(\text{type I}) &= -\Phi(\text{heterotic}) \\ \widetilde{G}^{(3)}(\text{type I}) &= \widetilde{H}^{(3)}(\text{heterotic}) \\ A_\mu(\text{type I}) &= A_\mu(\text{heterotic}), \end{aligned} \tag{7.46}$$

takes one from the type I Lagrangian to:

$$S_{\rm H} = \frac{1}{2\kappa_0^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\bigg\{R+4(\nabla\Phi)^2 \\ -\frac{1}{12}(\widetilde{H}^{(3)})^2 - \frac{\alpha'}{8}{\rm Tr}(F^{(2)})^2\bigg\}, \tag{7.47}$$

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where (renaming $C^{(2)} \to B^{(2)}$)

$$\widetilde{H}^{(3)} = dB^{(2)} - \frac{\alpha'}{4}\left[\frac{1}{30}\omega_{3\mathrm{Y}}(A) - \omega_{3\mathrm{L}}(\Omega)\right]. \qquad (7.48)$$

This is the low energy effective Lagrangian for the heterotic string theories. Note that in (7.47), α' is measured in heterotic units of length.

We can immediately see two key features about these theories. The first was anticipated earlier: their Lagrangian for the gauge fields have a dilaton coupling $e^{-2\Phi}$, since they arise at closed string tree level, instead of $e^{-\Phi}$ for the open string where where the gauge fields arise at open string tree level. The second observation is that since from equation (7.46) the dilaton relations tell us that $g_{\mathrm{s}}(\text{type I}) = g_{\mathrm{s}}^{-1}(\text{heterotic})$, there is a non-perturbative connection between these two theories, although they are radically different in perturbation theory. We are indeed *forced* to consider these theories when we study the type I string in the limit of strong coupling.

7.4 Heterotic toroidal compactifications

Much later, it will be of interest to study simple compactifications of the heterotic strings, and the simplest result from placing them on tori[174, 175]. Our interest here is not in low energy particle physics phenomenology, as this would require us to compactify on more complicated spaces to break the large amount of supersymmetry and gauge symmetry. Instead, we shall see that it is quite instructive, on the one hand, and on the other hand, studying various superstring compactifications with D-brane sectors taken into account will produce vacua which are in fact strong/weak coupling dual to heterotic strings on tori. This is another remarkable consequence of duality which forces us to consider the heterotic strings even though they cannot have D-brane sectors.

Actually, there is not much to do. From our work in section 7.2 and from that in section 4.5, it is easy to see what the conditions for the consistency of a heterotic toroidal compactification must be. Placing some of the ten dimensions on a torus T^d will give us the possibility of having windings, and right-moving momenta. In addition, the gauge group can be broken by introducing Wilson lines (see insert 4.4 and section 4.9.1) on the torus for the gauge fields A^{μ}. This latter choice breaks the gauge group to the maximal Abelian subgroup, which is $U(1)^{16}$.

The compactification simply enlarges our basic sixteen dimensional Euclidean lattice from $\Gamma_8 \oplus \Gamma_8$ or Γ_{16} by two dimensions of Lorentzian signature $(1, 1)$ for each additional compact direction, for the reasons we

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already discussed in section 4.5. So we end up with a lattice with signature $(16+d, d)$, on which there must be an action of $O(d, 16+d)$ generating the lattices. Again, we will have that there is a physical equivalence between some of these lattices, because physics only depends on $p_{\rm L}^2$ and $p_{\rm R}^2$, and further, there will be the discrete equivalences corresponding to the action of a T-duality group, which is $O(d, 16+d, \mathbb{Z})$.

The required lattices are completely classified, as a mathematical exercise. In summary, the space of inequivalent toroidal compactifications turns out to be:

$$\mathcal{M}_{T^d} = [O(d) \times O(d+16)]\backslash O(d, d+16)/O(d, d+16, \mathbb{Z}). \qquad (7.49)$$

Notice, after a quick computation, that the dimension of this space is d^2+16d. So in addition to the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, we have that number of extra massless scalars in the $\mathcal{N}=2$, $D=6$ low energy theory. The first part of the result comes, as before from the available constant components, G_{mn} and B_{mn}, of the internal metric and antisymmetric tensor on T^d. The remaining part comes from the sixteen generic constant internal gauge bosons (the Wilson lines), A_m for each circle.

Let us compute what the generic gauge group of this compactified model is. There is of course the $U(1)^{16}$ from the original current algebra sector. In addition, there is a $U(1)\times U(1)$ coming from each compact dimension, since we have Kaluza–Klein reduction of the metric and antisymmetric tensor. Therefore, the generic gauge group is $U(1)^{16+2d}$.

To get something less generic, we must tune some moduli to special points. Of course, we can choose to switch off some of the Wilson lines, getting non-Abelian gauge groups from the current algebra sector, restoring an $E_8 \times E_8 \times U(1)^{2d}$ or $SO(32) \times U(1)^{2d}$ gauge symmetry. We also have the possibility of enhancing the Kaluza–Klein factor by tuning the torus to special points. We simply need to make states of the form $\exp(ik_{\rm L}\cdot X_{\rm L})\psi^{\mu}_{-1/2}|0\rangle$, where we can have left-moving momenta of $\alpha' p_{\rm L}^2/2 = 2$ (we are referring to the components of $p_{\rm L}$ which are in the torus T^d). This will give any of the A–D–E series of gauge groups up to a rank $2d$ in this sector.

The reader will have noticed that we only gave one family of lattices for each dimension d of the torus. We did not have one choice for the $E_8 \times E_8$ string and another for the $SO(32)$ string. In other words, as soon as we compactify one heterotic string on a circle, we find that we could have arrived at the same spectrum by compactifying the other heterotic string on a circle. This is of course T-duality. It is worth examining further, and we do this in section 8.1.3.

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7.5 Superstring toroidal compactification

The placement of the superstrings on tori is at face value rather less interesting than the heterotic case, and so we will not spend much time on it here, although will return to it later when we revisit T-duality, and again when we study U-duality in section 12.7.

Imagine that we compactify one of our superstring theories on the torus T^d. We simply ask that d of the directions are periodic with some chosen radius, as we did in section 4.5 for the bosonic string. This does not not affect any of our discussion of supercharges, etc., and we simply have a $(10-d)$-dimensional theory with the same amount of supersymmetry as the ten dimensional theory which we started with. As discussed in section 4.4, there is a large $O(d,d,\mathbb{Z})$ pattern of T-duality groups available to us. There are also Kaluza–Klein gauge groups $U(1)^{2d}$ coming from the internal components of the graviton and the antisymmetric tensor. In addition, there are Kaluza–Klein gauge groups coming from the possibility of some of the R–R sector antisymmetric tensors having internal indices. Note that there aren't the associated enhanced gauge symmetries present at special radii, since the appropriate objects which would have arisen in a current algebra, J^a_{-1}, do not give masses states in spacetime, and in any case level matching would have forbidden them from being properly paired with $\psi^\mu_{-1/2}$ to give a spacetime vector.

To examine the possibilities, it is probably best to study a specific example, and we do the case of placing the type IIA string theory on T^5.

Let us first count the gauge fields. This can be worked out simply by counting the number of ways of wrapping the metric and the various p-form potentials (with p odd) in the theory on the five circles of the T^5 to give a one-form in the remaining five non-compact directions. From the NS–NS sector there are five Kaluza–Klein gauge bosons and five gauge bosons from the antisymmetric tensor. There are 16 gauge bosons from the dimensional reduction of the various R–R forms: the breakdown is 10+5+1 from the forms $C^{(3)}$, $C^{(5)}$ and $C^{(1)}$, respectively, since, for example, there are ten independent ways of making two out of the three indices of $C^{(3)}$ be any two out of the five internal directions, and so on. Finally, in five dimensions, one can form a two form field strength from the Hodge dual ${}^*H$ of the three-form field strength of the NS–NS $B_{\mu\nu}$, thus defining another gauge field.

So the gauge group is generically $U(1)^{27}$. There are in fact a number of massless fields corresponding to moduli representing inequivalent sizes and shapes for the T^5. We can count them easily. We have the $5^2 = 25$ components coming from the graviton and antisymmetric tensor field. From the R–R sector there is only one way of getting a scalar from $C^{(5)}$,

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and five and ten ways from $C^{(1)}$ and $C^{(3)}$, respectively. This gives 41 moduli. Along with the dilaton, this gives a total of 42 scalars for this compactification.

By now, the reader should be able to construct the very same five dimensional spectrum but starting with the type IIB string and placing it on T^5. This is a useful exercise in preparation for later. The same phenomenon will happen with any torus, T^d. Thus we begin to uncover the fact that the type IIA and type IIB string theories are (T-dual) equivalent to each other when placed on circles. We shall examine this in more detail in section 8.1, showing that the equivalence is exact.

The full T-duality group is actually $O(5,5;\mathbb{Z})$. It acts on the different sectors independently, as it ought to. For example, for the gauge fields, it mixes the first ten NS–NS gauge fields among themselves, and the 16 R–R gauge fields among themselves, and leaves the final NS–NS field invariant. Notice that the fields fill out sensible representations of $O(5,5;\mathbb{Z})$. Thinking of the group as roughly $SO(10)$, those familiar with numerology from grand unification might recognise that the sectors are transforming as the **10**, **16**, and **1**.

A little further knowledge will lead to questions about the fact that $\mathbf{10}\oplus\mathbf{16}\oplus\mathbf{1}$ is the decomposition of the **27** (the fundamental representation) of the group E_6, but we should leave this for a later time, when we come to discuss U-duality in section 12.7.

7.6 A superstring orbifold: discovering the K3 manifold

Before we go any further, let us briefly revisit the idea of strings propagating on an orbifold, and take it a bit further. Imagine that we compactify one of our closed string theories on the four torus, T^4. Let us take the simple case where there the torus is simply the product of four circles, S^1, each with radius R. Let us choose that the four directions (say) x^6, x^7, x^8 and x^9 are periodic with period $2\pi R$. The resulting six dimensional theory has $\mathcal{N}=4$ supersymmetry.

Let us orbifold the theory by the $\mathbb{Z}_2$ group which has the action

$$\mathbf{R}: \quad x^6, x^7, x^8, x^9 \to -x^6, -x^7, -x^8, -x^9, \tag{7.50}$$

which is clearly a good symmetry to divide by. We can choose to let $\mathbf{R}$ be embedded in the $SU(2)_{\mathrm{L}}$ which acts on the $\mathbb{R}^4$ (see insert 7.4). This will leave an $SU(2)_{\mathrm{R}}$ which descends to the six dimensions as a global symmetry. It is in fact the R–symmetry of the remaining $D=6$, $\mathcal{N}=2$ model. We shall use this convention a number of times in what is to come.

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Insert 7.4. $SU(2)_{\rm L}$ versus $SU(2)_{\rm R}$

It is well worth pausing here to note a nice way of writing things, for later use. The space $\mathbb{R}^4$ with coordinates $(x_6, x_7, x_8, x_9) = (\tau, x, y, z)$ has an obvious $SO(4)$ symmetry. Note that $SO(4) \sim SU(2)_{\rm L} \times SU(2)_{\rm R}$, where the 'L' and 'R' labels denote left and right. What is the meaning of this? To see it, present two new sets of coordinates. Write $\mathbb{R}^4$ with a radial coordinate $r = (\tau^2 + x^2 + y^2 + z^2)^{1/2}$, and Euler angles on an S^3 (r, θ, ϕ, ψ), where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. The metric is:

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 = dr^2 + \frac{r^2}{4}(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\psi d\phi).$$

Further define an element $g \in SU(2)$: $g = (\tau\mathbf{1} - i\vec{\tau}\cdot\vec{x})/r$ for Pauli matrices τ_i (given, *e.g.* in equation (13.1), where they're called σ_i):

$$g = \frac{1}{r}\begin{pmatrix} \tau + iz & -y + ix \\ y + ix & \tau - iz \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\phi+\psi)}\cos\frac{\theta}{2} & -e^{\frac{i}{2}(\phi-\psi)}\sin\frac{\theta}{2} \\ e^{-\frac{i}{2}(\phi-\psi)}\sin\frac{\theta}{2} & e^{-\frac{i}{2}(\phi+\psi)}\cos\frac{\theta}{2} \end{pmatrix}.$$

There are natural independent actions of $h \in SU(2)$ on this on the left, $g \to hg$, or on the right, $g \to gh$. It is really useful to extract certain natural 'Maurer–Cartan' one-forms from this. They are $\sigma_a = -i{\rm Tr}(\tau_a g^{-1} dg)$ and are clearly invariant under the $SU(2)_{\rm L}$. The $\bar{\sigma}_a = -i{\rm Tr}(\tau_a dg g^{-1})$ are $SU(2)_{\rm R}$ invariant. Explicitly:

$$2\sigma_1 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi;$$
$$2\sigma_2 = \cos\psi d\theta + \sin\psi\sin\theta d\phi; \quad 2\sigma_3 = d\psi + \cos\theta d\phi,$$

and they satisfy $d\sigma_a = \epsilon_{abc}\sigma_b \wedge \sigma_c$. Note also that $4(\sigma_1^2 + \sigma_2^2)$ is the standard round unit radius S^2 metric, while $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$ gives the same for S^3. (The $\bar{\sigma}_i$ can be obtained by sending $\psi \leftrightarrow \phi$.) Now, our metric on $\mathbb{R}^4$ can be written as $ds^2 = dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$.

7.6.1 The orbifold spectrum

We can construct the resulting six dimensional spectrum by first working out (say) the left-moving spectrum, seeing how it transforms under $\mathbf{R}$ and then tensoring with another copy from the right in order to construct the closed string spectrum.

Let us now introduce a bit of notation which will be useful in the future. Use the label x^m, $m = 6, 7, 8, 9$ for the orbifolded directions, and use x^μ,

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$\mu = 0, \ldots, 5$, for the remaining. Let us also note that the ten dimensional Lorentz group is decomposed as

$$SO(1,9) \supset SO(1,5) \times SO(4).$$

We shall label the transformation properties of our massless states in the theory under the $SU(2) \times SU(2) = SO(4)$ little group. Just as we did before, it will be useful in the Ramond sector to choose a labelling of the states which refers to the rotations in the planes (x^0, x^1), (x^2, x^3), etc., as eigenstates $s_0, s_1, \ldots, s_4$ of the operator S^{01}, S^{23}, etc., (see equations (7.17) and (7.19) and surrounding discussion).

With this in mind, we can list the states on the left that survive the GSO projection.

sector	state	**R** charge	$SO(4)$ charge
NS	$\psi^{\mu}_{-\frac{1}{2}}\|0; k\rangle$	$+$	$(\mathbf{2}, \mathbf{2})$
	$\psi^{m}_{-\frac{1}{2}}\|0; k\rangle$	$-$	$\mathbf{4}(\mathbf{1}, \mathbf{1})$
R	$\|s_1 s_2 s_3 s_4\rangle$; $s_1 = +s_2$, $s_3 = -s_4$	$+$	$\mathbf{2}(\mathbf{2}, \mathbf{1})$
	$\|s_1 s_2 s_3 s_4\rangle$; $s_1 = -s_2$, $s_3 = +s_4$	$-$	$\mathbf{2}(\mathbf{1}, \mathbf{2})$

Crucially, we should also examine the 'twisted sectors' which will arise, in order to make sure that we get a modular invariant theory. The big difference here is that in the twisted sector, the moding of the fields in the x^m directions is shifted. For example, the bosons are now half-integer moded. We have to recompute the zero point energies in each sector in order to see how to get massless states (see (2.80)):

$$\begin{aligned} \text{NS sector:} \quad & 4\left(-\frac{1}{24}\right) + 4\left(-\frac{1}{48}\right) + 4\left(\frac{1}{48}\right) + 4\left(\frac{1}{24}\right) = 0, \\ \text{R sector:} \quad & 4\left(-\frac{1}{24}\right) + 4\left(\frac{1}{24}\right) + 4\left(\frac{1}{48}\right) + 4\left(-\frac{1}{48}\right) = 0. \end{aligned} \tag{7.51}$$

This is amusing; both the Ramond and NS sectors have zero vacuum energy, and so the integer moded sectors will give us degenerate vacua. We see that it is only states $|s_1 s_2\rangle$ which contribute from the R sector (since they are half-integer moded in the x^m directions) and the NS sector, since it is integer moded in the x^m directions, has states $|s_3 s_4\rangle$.

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N.B. It is worth seeing in equation (7.51) how we achieved this ability to make a massless field in this case. The single twisted sector ground state in the bosonic orbifold theory with energy 1/48, was multiplied by four since there are four such orbifolded directions. Combining this with the contribution from the four unorbifolded directions produced just the energy needed to cancel the contribution from the fermions.

The states and their charges are as follows (after imposing GSO).

sector	state	$\mathbf{R}$ charge	$SO(4)$ charge
NS	$\|s_3 s_4\rangle;\ s_3 = -s_4$	$+$	$\mathbf{2}(\mathbf{1},\mathbf{1})$
R	$\|s_1 s_2\rangle;\ s_1 = -s_2$	$-$	$(\mathbf{1},\mathbf{2})$

Now we are ready to tensor. Recall that we could have taken the opposite GSO choice here to get a left moving with the identical spectrum, but with the swap $(\mathbf{1},\mathbf{2}) \leftrightarrow (\mathbf{2},\mathbf{1})$. Again we have two choices: tensor together two identical GSO choices, or two opposite. In fact, since six dimensional supersymmetries are chiral, and the orbifold will keep only two of the four we started with, we can write these choices as $(0,2)$ or $(1,1)$ supersymmetry, resulting from type IIB or IIA on K3. It is useful to tabulate the result for the bosonic spectra for the untwisted sector.

sector	$SO(4)$ charge
NS–NS	$(\mathbf{3},\mathbf{3})+(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1})$ $10(\mathbf{1},\mathbf{1})+6(\mathbf{1},\mathbf{1})$
R–R (IIB)	$2(\mathbf{3},\mathbf{1})+4(\mathbf{1},\mathbf{1})$ $2(\mathbf{1},\mathbf{3})+4(\mathbf{1},\mathbf{1})$
R–R (IIA)	$4(\mathbf{2},\mathbf{2})$ $4(\mathbf{2},\mathbf{2})$

This is the result for the twisted sector.

sector	$SO(4)$ charge
NS–NS	$3(\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1})$
R–R (IIB)	$(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1})$
R–R (IIA)	$(\mathbf{2},\mathbf{2})$

Recall now that we have two twisted sectors for each orbifolded circle, and hence there are 16 twisted sectors in all, for $T^4/\mathbb{Z}_2$. Therefore, to make the complete model, we must take sixteen copies of the content of the twisted sector table above.

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Now let identify the various pieces of the spectrum. The gravity multiplet $G_{\mu\nu} + B_{\mu\nu} + \Phi$ is in fact the first line of our untwisted sector table, coming from the NS–NS sector, as expected. The field B can be seen to be broken into its self-dual and anti-self-dual parts $B^{+}_{\mu\nu}$ and $B^{-}_{\mu\nu}$, transforming as $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{3}, \mathbf{1})$. There are sixteen other scalar fields, $((\mathbf{1}, \mathbf{1}))$, from the untwisted NS–NS sector. The twisted sector NS–NS sector has 4×16 scalars. Not including the dilaton, there are 80 scalars in total from the NS–NS sector.

Turning to the R–R sectors, we must consider the cases of IIA and IIB separately. For type IIA, there are eight one-forms (vectors, $(\mathbf{2}, \mathbf{2})$) from the untwisted sector and 16 from the twisted, giving a total of 24 vectors, and have a generic gauge group $U(1)^{24}$.

For type IIB, the untwisted R–R sector contains three self-dual and three anti-self-dual tensors, while there are an additional 16 self-dual tensors $(\mathbf{1}, \mathbf{3})$. We therefore have 19 self-dual $C^{+}_{\mu\nu}$ and three anti-self-dual $C^{-}_{\mu\nu}$. There are also eight scalars from the untwisted R–R sector and 16 scalars from the twisted R–R sector. In fact, including the dilaton, there are 105 scalars in total for the type IIB case.

7.6.2 Another miraculous anomaly cancellation

This type IIB spectrum is chiral, as already mentioned, and in view of what we studied in earlier sections, the reader must be wondering whether or not it is anomaly-free. It actually is, and it is a worthwhile exercise to check this, using the polynomials in insert 7.2.

The cancellation is so splendid that we cannot resist explaining it in detail here. To do so we should be careful to understand the $\mathcal{N} = 2$ multiplet structure properly. A sensible non-gravitational multiplet has the same number of bosonic degrees of freedom as fermionic, and so it is possible to readily write out the available ones given what we have already seen. (Or we could simply finish the tensoring done in the last section, doing the NS–R and R–NS parts to get the fermions.) Either way, table 7.1 has the multiplets listed.

The 16 components of the supergravity bosonic multiplet is accompanied by two copies of the 16 components making up a gravitino and a dilatino. These two copies are the same chirality for type IIB and opposite for type IIA.

The next thing to do is to repackage the spectrum we identified earlier in terms of these multiplets. First, notice that the supergravity multiplet has one $(\mathbf{1}, \mathbf{1})$, four $(\mathbf{2}, \mathbf{1})$s and one $(\mathbf{1}, \mathbf{3})$. With four other scalars, we can make a full tensor multiplet. (The other $(\mathbf{3}, \mathbf{1})$, which is an anti-self-dual piece makes up the rest of $B_{\mu\nu}$.) That gives us 19 complete self dual

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Table 7.1. *The structure of the $\mathcal{N} = 2$ multiplets in $D = 6$*

multiplet	bosons	fermions
vector	$(\mathbf{2},\mathbf{2})+4(\mathbf{1},\mathbf{1})$	$2(\mathbf{1},\mathbf{2})+2(\mathbf{2},\mathbf{1})$
SD tensor	$(\mathbf{1},\mathbf{3})+5(\mathbf{1},\mathbf{1})$	$4(\mathbf{2},\mathbf{1})$
ASD tensor	$(\mathbf{3},\mathbf{1})+5(\mathbf{1},\mathbf{1})$	$4(\mathbf{2},\mathbf{1})$
supergravity	$(\mathbf{3},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1})$	$2(\mathbf{3},\mathbf{2})+2(\mathbf{2},\mathbf{1})$ or $2(\mathbf{2},\mathbf{3})+2(\mathbf{1},\mathbf{2})$

tensor multiplets in total and two complete anti-self-dual ones since the last one is not complete. Since there are five scalars in a tensor multiplet this accounts for the 105 scalars that we have.

So we can study the anomaly now, knowing what (anti-)self-dual tensors, and fermions we have. Consulting insert 7.2 (p. 162), we note that the polynomials listed for the fermions are for complex fermions, and so we must divide them by two to get the ones appropriate for the real components we have counted in the orbifolding. Putting it together according to what we have said above for the content of the spectrum, we have:

$$19\hat{I}_8^{(\mathbf{1},\mathbf{3})} + 19\times 4\hat{I}_8^{(\mathbf{2},\mathbf{1})} + 2\hat{I}_8^{(\mathbf{3},\mathbf{1})} + 2\times 4\hat{I}_8^{(\mathbf{2},\mathbf{1})} + 2\hat{I}_8^{(\mathbf{3},\mathbf{2})} + \hat{I}_8^{(\mathbf{3},\mathbf{1})} = 0, \quad (7.52)$$

where we have listed, respectively, the contribution of the 19 self-dual tensors, the two anti-self-dual tensors, the two gravitinos, and the remaining piece of the supergravity multiplet. That this combination of polynomials vanishes is amazing[109].

7.6.3 The K3 manifold

Quite remarkably, there is a geometrical interpretation of all of those data presented in the previous subsections in terms of compactifying type II string theory on a smooth manifold. The manifold is K3. It is a four dimensional manifold containing 22 independent two-cycles, which are topologically two-spheres more properly described as the complex surface $\mathbb{CP}^1$ (see insert 16.1), in this context. Correspondingly the space of two-forms which can be integrated over these two cycles is 22 dimensional. So we can choose a basis for this space. Nineteen of them are self-dual and three of them are anti-self-dual, in fact. The space of metrics on K3 is in fact parametrised by 58 numbers.

In compactifying the type II superstrings on K3, the ten dimensional gravity multiplet and the other R–R fields gives rise to six dimensional fields by direct dimensional reduction, while the components of the fields in the K3 give other fields. The six dimensional gravity multiplet arises by direct reduction from the NS–NS sector, while 58 scalars arise, parametrising the 58 dimensional space of K3 metrics which the internal parts of the metric, G_{mn}, can choose. Correspondingly, there are 22 scalars arising from the 19+3 ways of placing the internal components of the antisymmetric tensor, B_{mn} on the manifold. A commonly used terminology is that the form has been 'wrapped' on the 22 two-cycles to give 22 scalars.

In the R–R sector of type IIB, there is one scalar in ten dimensions, which directly reduces to a scalar in six. There is a two-form, which produces 22 scalars, in the same way as the NS–NS two-form did. The self-dual four-form can be integrated over the 22 two cycles to give 22 two forms in six dimensions, 19 of them self-dual and three anti-self-dual. Finally, there is an extra scalar from wrapping the four-form entirely on K3. This is precisely the spectrum of fields which we computed directly in the type IIB orbifold.

Alternatively, while the NS–NS sector of type IIA gives rise to the same fields as before, there is in the R–R sector a one-form, three-form and five-form. The one-form directly reduces to a one-form in six dimensions. The three-form gives rise to 22 one-forms in six dimensions while the five-form gives rise to a single one-form. We therefore have 24 one-forms (generically carrying a $U(1)$ gauge symmetry) in six dimensions. This also completes the smooth description of the type IIA on K3 spectrum, which we computed directly in the orbifold limit. See insert 7.5 for a significant comment on this spectrum.

7.6.4 Blowing up the orbifold

The connection between the orbifold and the smooth K3 manifold is as follows[78]: K3 does indeed have a geometrical limit which is $T^4/\mathbb{Z}_2$, and it can be arrived at by tuning enough parameters, which corresponds here to choosing the vev's of the various scalar fields. Starting with the $T^4/\mathbb{Z}_2$, there are 16 fixed points which look locally like $\mathbb{R}^4/\mathbb{Z}^2$, a singular point of infinite curvature. It is easy to see where the 58 geometric parameters of the K3 metric come from in this case. Ten of them are just the symmetric G_{mn} constant components, on the internal directions. This is enough to specify a torus T^4, since the hypercube of the lattice in $\mathbb{R}^4$ is specified by the ten angles between its unit vectors, $\mathbf{e}^m \cdot \mathbf{e}^n$. Meanwhile each of the 16 fixed points has three scalars associated to its metric geometry. (The remaining fixed point NS–NS scalar in the table is from the field B, about which we will have more to say later.)

The three metric scalars can be tuned to resolve or 'blow-up' the fixed point, and smooth it out into the $\mathbb{CP}^1$ which we mentioned earlier. (This accounts for 16 of the two-cycles. The other six correspond to the six $\mathbb{Z}_2$

Insert 7.5. Anticipating a string–string duality in $D = 6$

We have seen that for type IIA we have an $\mathcal{N} = 2$, $D = 6$ supergravity with 80 additional scalars and 24 gauge bosons with a generic gauge group $U(1)^{24}$. The attentive reader will have noticed an apparent coincidence between the result for the spectrum of type IIA on K3 and another six dimensional spectrum which we obtained earlier. That was the spectrum of the heterotic string compactified on T^4, obtained in section 7.4 (put $d = 4$ in the results there). The moduli space of compactifications is in fact

$$O(20, 4, \mathbb{Z})\backslash O(20, 4)/[O(20) \times O(4)]$$

on both sides. We have seen where this comes from on the heterotic side. On the type IIA side it arises too. Start with the known

$$O(19, 3, \mathbb{Z})\backslash O(19, 3)/[O(19) \times O(3)]$$

for the standard moduli space of K3s (you should check that this has 57 parameters; there is an additional one for the volume). It acts on the 19 self-dual and three anti-self-dual two-cycles. This classical geometry is supplemented by stringy geometry arising from $B_{\mu\nu}$, which can have fluxes on the 22 two-cycles, giving the missing 22 parameters. We will not prove here that the moduli space is precisely as above, and hence the same as globally and locally as the heterotic one, but it will become apparent later in chapters 12 and 16.

Perturbatively, the coincidence of the spectra must be an accident. The two string theories in $D = 10$ are extremely dissimilar. One has twice the supersymmetry of the other and is simpler, having no large gauge group, while the other is chiral. We place the simpler theory on a complicated space (K3) and the more complex theory on a simple space T^4 and result in the same spectrum. The theories cannot be T-dual since the map would have to mix things which are unrelated by properties of circles. The only duality possible would have to go beyond perturbation theory. This is what we shall see later in chapter 16. Note also that there is something missing. At special points in the heterotic moduli space we have seen that it is possible to get large enhanced non-Abelian gauge groups. There is no sign of that here in how we have described the type IIA string theory using conformal field theory. In fact, we shall see how to go beyond conformal field theory and describe these special points using D-branes in chapter 13.

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invariant forms $dX^m \wedge dX^n$ on the four-torus.) The smooth space has a known metric, the 'Eguchi–Hanson' metric[84], which is *locally* asymptotic to $\mathbb{R}^4$ (like the singular space) but with a global $\mathbb{Z}_2$ identification. Its metric is:

$$ds^2 = \left(1 - \left(\frac{a}{r}\right)^4\right)^{-1} dr^2 + r^2\left(1 - \left(\frac{a}{r}\right)^4\right)\sigma_3^2 + r^2(\sigma_1^2 + \sigma_2^2), \quad (7.53)$$

where the σ_i are defined in terms of the S^3 Euler angles (θ, ϕ, ψ) in insert 7.4. From there we learn that $4(\sigma_1^2 + \sigma_2^2) = d\theta^2 + \sin^2\theta d\phi^2$. The point $r = a$ is an example of a 'bolt' singularity. Near there, the space is topologically $\mathbb{R}^2_{r\psi} \times S^2_{\theta\phi}$, with the S^2 of radius $a/2$, and the singularity is a coordinate one provided ψ has period 2π. (See insert 7.6, (p. 188).) Since on S^3, ψ would have period 4π, the space at infinity is $S^3/\mathbb{Z}_2$, just like an $\mathbb{R}^4/\mathbb{Z}_2$ fixed point. For small enough a, the Eguchi–Hanson space can be neatly slotted into the space left after cutting out the neighbourhood of the fixed point. The bolt is in fact the $\mathbb{CP}^1$ of the blow-up mentioned earlier. The parameter a controls the size of the $\mathbb{CP}^1$, while the other two parameters correspond to how the $\mathbb{R}^2$ (say) is oriented in $\mathbb{R}^4$.

The Eguchi–Hanson space is the simplest example of an 'Asymptotically Locally Euclidean' (ALE) space, which K3 can always be tuned to resemble locally. These spaces are classified[85] according to their identification at infinity, which can be any discrete subgroup[86], Γ, of the $SU(2)$ which acts on the S^3 at infinity, to give S^3/Γ. These subgroups have been characterised by McKay[87], and have an A–D–E classification which we shall study more in chapter 13. The metrics on the A–series are known explicitly as the Gibbons–Hawking metrics[91], which we shall display later, and Eguchi–Hanson is in fact the simplest of this series, corresponding[92] to A_1. We shall later use a D-brane as a probe of string theory on a $\mathbb{R}^4/\mathbb{Z}_2$ orbifold, an example which will show that the string theory correctly recovers all of the metric data (7.53) of these fixed points, and not just the algebraic data we have seen here.

For completeness, let us compute one more thing about K3 using this description. The Euler characteristic, in this situation, can be written in two ways[82]

$$\begin{aligned}\chi(K3) &= \frac{1}{32\pi^2}\int_{K3}\sqrt{g}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right)\\ &= \frac{1}{32\pi^2}\int_{K3}\sqrt{g}\epsilon_{abcd}R^{ab}R^{cd}\\ &= -\frac{1}{16\pi^2}\int_{K3}\mathrm{Tr}R\wedge R = 24. \end{aligned}\quad (7.54)$$

Even though no explicit metric for K3 has been written, we can compute χ as follows[80, 82]. If we take a manifold M, divide by some group G, remove

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Insert 7.6. A closer look at the Eguchi–Hanson space

Let us establish some of the properties claimed in the main body of the text, while uncovering a useful technique. The S^3s in the metric (7.53) are the natural 3D 'orbits' of the $SU(2)$ action. The S^2 of (θ, ϕ) is a special 2D 'invariant submanifold'. To examine the potential singularity at $r = a$, look *near* $r = a$. Choose, if you will, $r = a + \varepsilon$ for small ε, and:

$$ds^2 = \frac{a}{4\varepsilon}\left[d\varepsilon^2 + \frac{16\varepsilon^2}{4}(d\psi + \cos\theta d\phi)^2\right] + \frac{1}{4}(a^2 + 2a\varepsilon)d\Omega_2^2,$$

which as $\varepsilon \to 0$ is obviously topologically looking locally like $\mathbb{R}^2_{\varepsilon,\psi} \times S^2_{\theta,\phi}$, where the S^2 is of radius $a/2$. (Globally, there is a fibred structure due to the $d\psi d\phi$ cross term.) Incidentally, this is perhaps the quickest way to see that the Euler number or 'Euler charachteristic' of the space has to be equal to that of an S^2, which is two. There is a potential 'bolt' singularity at $r = a$. It is a true singularity for arbitrary choices of periodicity $\Delta\psi$ of ψ, since there is a conical deficit angle in the plane. In other words, we have to ensure that as we get to the origin of the plane, $\varepsilon = 0$, the ψ-circles have circumference 2π, no more or less. Infinitesimally, we make those measures with the metric, and so the condition is:

$$2\pi = \lim_{\varepsilon\to 0}\left(\frac{d(\sqrt{a}\varepsilon^{1/2})\Delta\psi}{d\varepsilon\sqrt{(a/4)}\varepsilon^{-1/2}}\right),$$

which gives $\Delta\psi = 2\pi$. So in fact, we must spoil our S^3 which was a nice orbit of the $SU(2)$ isometry, by performing an $\mathbb{Z}_2$ identification on ψ, giving it half its usual period. In this way, the 'bolt' singularity $r = a$ is just a harmless artifact of coordinates[83, 82]. Also, we are left with an $SO(3) = SU(2)/\mathbb{Z}_2$ isometry of the metric. The space at infinity is $S^3/\mathbb{Z}_2$.

some fixed point set F, and add in some set of new manifolds N, one at each point of F, the Euler characteristic of the new manifold is

$$\chi = \frac{\chi(M) - \chi(F)}{|G|} + \chi(N). \qquad (7.55)$$

Here, $G = \mathbf{R} \equiv \mathbb{Z}_2$, and the Euler characteristic of the Eguchi–Hanson space is equal to two, from insert 7.6 (p. 188). That of a point is one, and

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of the torus is zero. We therefore get

$$\chi(K3) = -\frac{16}{2} + 16 \times 2 = 24, \tag{7.56}$$

which will be of considerable use later on.

So we have constructed the consistent, supersymmetric string propagation on the K3 manifold, using orbifold techniques. We shall use this manifold to illustrate a number of beautiful properties of D-branes and string theory in the rest of these lectures.

7.6.5 Some other K3 orbifolds

We can construct K3 in its orbifold limits using other $\mathbb{Z}_N$ group actions. We begin with the space $\mathbb{R}^4 \equiv \mathbb{C}^2$, with complex coordinates $z^1 = x^6 + ix^7$ and $z^2 = x^8 + ix^9$, upon which we make the identifications $z^i \sim z^i + 1 \sim z^i + i$, for N=2 or 4, and $z^i \sim z^i + 1 \sim z^i + \exp(\pi i/3)$ for N=3 or 6. These lattices define for us the torus T^4, upon which the discrete rotations $\mathbb{Z}_N$, acts naturally as

$$(z^1, z^2) \to (\beta z^1, \beta^{-1} z^2), \tag{7.57}$$

for $\beta = \exp(2\pi i/N)$.

We may therefore define a new space by identifying points under the action of $\mathbb{Z}_N$. This is the orbifold $T^4/\mathbb{Z}_N$, which is a smooth surface except at fixed points, which are invariant under $\mathbb{Z}_N$ or some non-trivial subgroup of it. For $N \in \{2, 3, 4, 6\}$, this procedure produces a family of compact spaces which are also orbifold limits of the K3 surface.

The smooth K3 manifold is obtained from these limits by blowing up the orbifold points, removing each of the points and replacing it by a smooth space, just as we did in the previous section. The neighbourhood of a fixed point is $\mathbb{R}^4/\mathbb{Z}_M$, where $N \geq M \in \{2, 3, 4, 6\}$, which is the asymptotic region of the A–series ALE space with which we replace the excised point. Note that the Euler characteristic of the A_n ALE space is $n + 1$.

Let us denote the generator of $\mathbb{Z}_N$ by α_N The group elements are then the powers α_N^m, for $m \in \{0, 1, \ldots, N-1\}$. In fact the number, F_M, of points fixed under the $\mathbb{Z}_M$ subgroup of $\mathbb{Z}_N$, (generated by $\alpha_N^{N/M}$) is simply $F_M = 16 \sin^4 \frac{\pi}{M}$, where M is a divisor of N.

For $T^4/\mathbb{Z}_2$, as we have already seen, we have 16 points fixed under the action of α_2, each of which are replaced by the A_1 ALE space in order to resolve to smooth K3. For $T^4/\mathbb{Z}_3$ there are nine fixed points of α_3, which are each replaced by the A_2 ALE space to make the blow-up.

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From formula (7.55), we get

$$\chi(\mathrm{K3}) = -\frac{9}{3} + 9 \times 3 = 24.$$

The case $T^4/\mathbb{Z}_4$ has 16 fixed points. Four of them are fixed under the action of α_4, while the other 12 are only fixed under α_4^2. Under α_4, these 12 $\mathbb{Z}_2$ points transform as six doublets. Consequently, the blow-up is carried out by first constructing the $\mathbb{Z}_4$-invariant region by identifying these pairs of fixed points. One can then replace each of the original four $\mathbb{Z}_4$ fixed points by an A_3 ALE space and the six pairs by an A_1. From formula (7.55), we get

$$\chi(\mathrm{K3}) = -\frac{16}{4} + 4 \times 4 + 6 \times 2 = 24.$$

For $T^4/\mathbb{Z}_6$ the situation is similar. There are 24 fixed points altogether. There is only one point fixed under α_6. It is replaced by the A_5 ALE space to make the blow-up. There are eight points fixed under the $\mathbb{Z}_3$ subgroup, generated by α_6^2, which transform as doublets under the action of α_6. They are therefore replaced by four copies of the A_2 ALE space. There are 15 points fixed under α_6^3, which transform as triplets under the action of α_6. Consequently, they are replaced by five copies of the A_1 space in performing the blow-up surgery. Once again, we get the correct value of the Euler number:

$$\chi(K3) = -\frac{24}{6} + 5 \times 2 + 4 \times 3 + 1 \times 6 = 24.$$

We can go a lot further and recover other geometric properties of the K3 in each case. For example, as we shall see later in chapter 13, the A_n ALE space is generically like $n+1$ $\mathbb{CP}^1$s (i.e. S^2s) intersecting in a particular pattern. There is in fact a self-dual cycle associated to n of these. So its contribution to the K3s count of $(19,3)$ cycles is $(n,0)$. It s

Table 7.2. *Recovering some properties of the K3 geometry in orbifold limits*

case	T^4 parameters	ALE parameters	T^4 forms	ALE forms
$\mathbb{Z}_2$	10	$16 \times 3 = 48$	(3,3)	$16 \times (1,0)$
$\mathbb{Z}_3$	4	$18 \times 3 = 54$	(1,3)	$9 \times (2,0)$
$\mathbb{Z}_4$	4	$18 \times 3 = 54$	(1,3)	$6 \times (1,0) + 4 \times (3,0)$
$\mathbb{Z}_6$	4	$18 \times 3 = 54$	(1,3)	$(5,0) + 5 \times (1,0) + 4 \times (2,0)$

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useful to combine this with the contribution from the torus to compute the result for K3, and table 7.2 has a list of the arithmetic in each case. The origin of the 58 metric parameters can similarly be computed, using the fact that some come from the torus and some from the parameters (three for each $\mathbb{CP}^1$ in fact) of the ALE spaces. This is also given in table 7.2. We've listed the $\mathbb{Z}_2$ case which we already computed in the previous subsection. Notice that it is in some sense more special than the others. In both forms and metric parameters, the bare torus contributes more than in the other cases. This is because it is more symmetric than the others. This is traceable to the fact that the T^4 is written naturally in terms of the complex parameters $z_1 = x_6 + ix_7$ and $z_2 = x_8 + ix_9$, and the form of the action on it is given by equation (7.57). It is only for $\mathbb{Z}_2$ that $\beta = 1/\beta$, and thus there is more symmetry between the x^ms.

Therefore of the 6 forms (made from $dx^m \wedge dx^n$) and 10 scalars one can make, only four survive in each non-$\mathbb{Z}_2$ case. (This can be worked out most easily by working directly with z_1 and z_2. Then the forms are $dz_1 \wedge dz_2$, $d\bar{z}_1 \wedge d\bar{z}_2$, etc., but, for example, $dz_1 \wedge d\bar{z}_2$ is clearly not invariant since it transforms as β^2.)

7.6.6 Anticipating D-manifolds

We've just made some traditional superstring compactifications by including in the internal space the pure geometry of K3, resulting in a six dimensional vacuum. Later we will see that it is possible to construct a whole new class of string 'compactification' vacua by including D-branes in the spectrum in such a way that their contribution to spacetime anomalies, etc., combines with that of the pure geometry in a way that is crucial to the consistency of the model. This gives the idea of a 'D-manifold'[116].

An analogue of the orbifold method for making these supersymmetric vacua is the generalised 'orientifold' construction already mentioned. There are constructions of 'K3 orientifolds' which follow the ideas presented in this section, combined with D-brane orbifold techniques to be developed in chapter 14[131]. We shall also encounter K3 in its orbifold limits in chapter 16, where we use our knowledge gained here to explore properties of remarkable non-perturbative type IIB vacua made using F-theory. D-branes will be present there too, but in a somewhat different way.