CORRESPONDENCE.

APPROXIMATE DIFFERENTIATION.

To the Editors of the Journal of the Institute of Actuaries.

SIRS—In the *Journal* for July 1923, I wrote concerning approximate integration obtained by appropriately weighting successive values of a function. The converse process of determining, from the definite integrals of a function between successive series of limits, the numerical values of the function corresponding to certain values of the variable, is also of some practical interest.

2. For example, since $d_x = \int_0^1 l_{x+t}\mu_{x+t}dt$, the process in question would enable the numerical values of $l_x\mu_x$, the curve of deaths, to be obtained from an appropriate series of values of d_x, given, say, as

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portion of the data of a mortality investigation. Similarly, since $L_x = \int_0^1 l_{x+t}dt$, the numerical values of l_x could be obtained from a series of values of L_x.

3. If we put $F(x) = \int_0^1 f(x+t)dt$, and assume that

$$f(x) = a + bx + cx^2 + dx^3$$

we have

$$\int f(x)dx = K + ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4$$

Hence

$$F(-2) = \int_0^1 f(-2+t)dt = a - \frac{3b}{2} + \frac{7c}{3} - \frac{15d}{4}$$

$$F(-1) = \int_0^1 f(-1+t)dt = a - \frac{b}{2} + \frac{c}{3} - \frac{d}{4}$$

$$F(0) = \int_0^1 f(t)dt = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$$

$$F(1) = \int_0^1 f(1+t)dt = a + \frac{3b}{2} + \frac{7c}{3} + \frac{15d}{4}$$

4. If we assume $F(x)$ to be given for four successive values -2, -1, 0, and 1 of x, the value of $f(0)$ may be obtained by suitably weighting and adding these values of $F(x)$. If such weights be denoted respectively by w_{-2}, w_{-1}, w_0, and w_1, then since $f(0) = a$ we have

$$w_{-2} + w_{-1} + w_0 + w_1 = 1$$

$$-\frac{3}{2}w_{-2} - \frac{1}{2}w_{-1} + \frac{1}{2}w_0 + \frac{3}{2}w_1 = 0$$

$$\frac{7}{3}w_{-2} + \frac{1}{3}w_{-1} + \frac{1}{3}w_0 + \frac{7}{3}w_1 = 0$$

$$-\frac{15}{4}w_{-2} - \frac{1}{4}w_{-1} + \frac{1}{4}w_0 + \frac{15}{4}w_1 = 0$$

In solving this it may be assumed that $w_{-2} = w_1$ and $w_{-1} = w_0$.

Hence

$$2w_{-2} + 2w_{-1} = 1$$

$$\frac{14}{3}w_{-2} + \frac{2}{3}w_{-1} = 0$$

and

$$w_{-2} = -\frac{1}{12} = w_1$$

$$w_{-1} = \frac{7}{12} = w_0$$

5. The weighting is thus $\frac{1}{12}(-1,\ 7,\ 7,\ -1)$ which will be recognized as that usually employed for determining μ_x from given successive values of d_x.

The usual formula follows from the fact that, as shown above,

$$l_x\mu_x = \frac{1}{12}\{-d_{x-2}+7d_{x-1}+7d_x-d_{x+1}\}$$

and consequently

$$\mu_x = \frac{1}{12l_x}\{-d_{x-2}+7d_{x-1}+7d_x-d_{x+1}\}$$

or, as usually given,

$$\mu_x = \frac{7(d_{x-1}+d_x)-(d_{x-2}+d_{x+1})}{12l_x}$$

6. Similarly it follows that

$$l_x = \frac{7(L_{x-1}+L_x)-(L_{x-2}+L_{x+1})}{12}$$

A combination of these results gives

$$\mu_x = \frac{7(d_{x-1}+d_x)-(d_{x-2}+d_{x+1})}{7(L_{x-1}+L_x)-(L_{x-2}+L_{x+1})}$$

Hence also

$$q_x = \frac{d_x}{l_x} = \frac{12d_x}{7(L_{x-1}+L_x)-(L_{x-2}+L_{x+1})}$$

These results are of some interest in the construction of mortality tables from census and registration returns where the crude data are usually in the form L_x and d_x.

Again, since

$$\text{col}_e p_x = \int_0^1 \mu_{x+t}dt$$

we have

$$\mu_x = \frac{7(\text{col}_e p_{x-1}+\text{col}_e p_x)-(\text{col}_e p_{x-2}+\text{col}_e p_{x+1})}{12}$$

and since $m_x = \text{col}_e p_x$ approx.

We have also

$$\mu_x = \frac{7(m_{x-1}+m_x)-(m_{x-2}+m_{x+1})}{12} \text{ approx.}$$

7. If desired, a wider range of values of $F(x)$ could be used. For example, to determine $f(0)$ from the six values of $F(x)$ for $x=-3, -2, -1, 0, 1,$ and 2, an infinite series of weightings can readily be obtained by the process indicated above. The most interesting members of this series are the following:

$$\frac{1}{24}\{-2, 4, 10, 10, 4, -2\}$$

$$\frac{1}{24}\{-1, 1, 12, 12, 1, -1\}$$

$$\frac{1}{24}\{0, -2, 14, 14, -2, 0\}$$

The last of these is, of course, $\frac{1}{12}\{-1, 7, 7, -1\}$ in another form; while the first reduces to $\frac{1}{12}\{-1, 2, 5, 5, 2, -1\}$. If a range of six terms were to be used in practice, this latter would probably be the most suitable. From it, for example, we have for the curve of deaths

$$l_x\mu_x = \frac{5(d_{x-1}+d_x)+2(d_{x-2}+d_{x+1})-(d_{x-3}+d_{x+2})}{12}$$

and for the life curve

$$l_x = \frac{5(L_{x-1}+L_x)+2(L_{x-2}+L_{x+1})-(L_{x-3}+L_{x+2})}{12}$$

8 It is of interest to note that the weightings

$$\frac{1}{12}\{-1, 7, 7, -1\} \text{ and } \frac{1}{12}\{-1, 2, 5, 5, 2, -1\}$$

may be conveniently applied to a series of values of $F(x)$ by means of the summation method, the former being equivalent to $\frac{1}{12}[2]\{9-[3]\}F$ and the latter to $\frac{1}{12}[2][3]\{5-[3]\}F$. If in summing in twos the sum in each case is placed opposite the space between the two items summed, the step is analogous to indicating the length of the ordinate corresponding to the central position in a series of group values in a graph. If applied to crude data, for example, L_x and d_x, these processes have a graduating effect which is by no means inconsiderable.

Yours faithfully,

CHAS. H. WICKENS.

Melbourne,
28 *March* 1924.

[It will be observed that Mr. Wickens uses the symbol L_x to denote $\int_0^1 l_{x+t}dt$ instead of $\frac{1}{2}(l_x+l_{x+1})$. The formulas involving L do not therefore hold good for the Life Table. In theory they would appear to be applicable only if d and L were the deaths and numbers living in a stationary population, although in practice the formulas for the ratios μ and q might no doubt be applied more generally.—EDS. *J.I.A.*]

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