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CONSTRUCTION OF SOME NEW HADAMARD MATRICES

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We prove that there exist skew type Hadamard matrices of order $4n$ for $n = 67, 113$, 127, 157, 163, 181 and 241 which have not been constructed so far. In particular there exists a Hadamard matrix of order $4 \cdot 163$, which was unknown until now. We mention that very recently we have constructed skew type Hadamard matrices of orders $4n$ for $n = 37$ and 43.

1.

A *Hadamard matrix* of order m is a $(1,-1)$-matrix H of order m satisfying $HH^T = mI_m$. (X^T denotes the transpose of a matrix X, and I_m the identity matrix of order m.) The order m of a Hadamard matrix H must be 1, 2 or a multiple of 4, $m = 4n$. A $(1,-1)$-matrix A of order m is said to be of *skew type* if $A + A^T = 2I_m$. A *skew Hadamard* matrix is a Hadamard matrix of skew type.

It has been conjectured that Hadamard matrices as well as skew Hadamard matrices exist for all orders m which are multiples of 4. According to Appendix K of the book [**2**, p.416] published in 1979, Hadamard matrices of order $m = 4n$ with $n < 200$ were not known only for the following nine values of n :

$$67, 103, 107, 127, 151, 163, 167, 179, 191. \tag{1}$$

For orders of skew Hadamard matrices $m = 4n$ with n odd and $n < 250$ only the following 53 values were dubious (see [**5**]) :

$$\begin{gathered}29,37,39,43,47,49,59,65,67,69,81,89,93,97,101,103,107,109,113,119,121,\\127,129,133,145,149,151,153,157,163,167,169,177,179,181,191,193,\\201,205,209,213,217,219,223,225,229,233,235,239,241,245,247,249.\end{gathered} \tag{2}$$

The number 67 has been removed from the list (1) by Sawade [**3**], and the numbers 103, 127 and 151 by Yamada [**7**, Theorem 4]. From the list (2), the number 29 has been removed by Szekeres [**6**] and the numbers 37 and 43 were removed recently by the author [**1**]. Here we announce the existence of skew Hadamard matrices of orders $m = 4n$ for $n = 67$, 113, 127, 157, 163, 181 and 241. Hence these numbers should be removed from the list (2) and consequently the number 163 should be removed from the list (1).

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2.

We shall identify the integer $i \in \{0, 1, 2, \ldots, n-1\}$ with the corresponding residue class modulo n. We say that subsets S_i, $1 \leqslant i \leqslant k$, of $\{0,1,2,\ldots,n-1\}$ are

$$k - (n; n_1, \ldots, n_k; \lambda)$$

supplementary difference sets modulo n (abbreviated as *sds*) if $|S_i| = n_i$ for $1 \leqslant i \leqslant k$ and for each non-zero residue r modulo n the congruences

$$x_i - y_i \equiv r (\operatorname{mod} n)$$

have in total exactly λ solutions with $x_i, y_i \in S_i$ and $1 \leqslant i \leqslant k$ (i is not fixed). Our construction of skew Hadamard matrices is based on some new $4 - (n; n_1, \ldots, n_4; \lambda)$ sds with $\lambda = n_1 + \cdots + n_4 - n$.

We say that a subset S of $\{1,2,\ldots,n-1\}$ is of *skew type* if

$$i \in S \iff -i \notin S.$$

They exist if and only if n is odd. Given any subset S of $\{1,2,\ldots,n-1\}$ we denote by $A_S = (a_{ij})$ the circulant $(1,-1)$-matrix of order n, $0 \leqslant i,j \leqslant n-1$, whose first row is given by

$$a_{0,j} = \begin{cases} -1 & \text{if } j \in S, \\ 1 & \text{if } j \notin S. \end{cases}$$

Since $0 \notin S$, all diagonal entries of A_S are 1's. A_S is of skew type if and only if S is of skew type. For later use we introduce the permutation matrix $R = (r_{ij})$ of order n, $0 \leqslant i,j \leqslant n-1$, such that

$$r_{ij} = \begin{cases} 1 & \text{if } i + j \equiv -1 \ (\operatorname{mod} n), \\ 0 & \text{otherwise.} \end{cases}$$

3.

We say that a $(1,-1)$-matrix H of order $m = 4n$ is of *Goethals–Seidel type* if

$$H = \begin{pmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & -A_4^TR & A_3^TR \\ -A_3R & A_4^TR & A_1 & -A_2^TR \\ -A_4R & -A_3^TR & A_2^TR & A_1 \end{pmatrix}$$

where A_i are circulant matrices of order n and R is the matrix defined in the previous section. Such H is a Hadamard matrix if and only if

$$\sum_{i=1}^{4} A_i A_i^T = 4nI_n,$$

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see [4]. Furthermore H is of skew type if and only if A_1 is of skew type. All Hadamard matrices constructed in this paper are of Goethals–Seidel type.

Let $S = (S_1, S_2, S_3, S_4)$ be a 4-tuple of subsets of $\{1, 2, \dots, n-1\}$ and let H_S be the matrix H above with $A_i = A_{S_i}$, $1 \leqslant i \leqslant 4$. We write n_i for the cardinality $|S_i|$ of S_i. The following proposition is well known.

Proposition 1. *(See [4]) The matrix H_S is a Hadamard matrix if and only if S_1, S_2, S_3, S_4 are $4-\left(n; n_1, n_2, n_3, n_4; \sum_{i=1}^{4} n_i - n\right)$ supplementary difference sets modulo n. Furthermore H_S is of skew type if and only if S_1 is of skew type.*

4.

In all cases that we consider below, n is a prime number and so $G = \{1, 2, \dots, n-1\}$ is a group under multiplication modulo n. We choose a subgroup H of order k for some odd integer $k > 1$ and set $b = [G : H] = (n-1)/k$. Then we enumerate the cosets of H in G :

$$\alpha_0, \alpha_1, \dots, \alpha_{b-1}.$$

In all cases our enumeration is such that

$$\alpha_{2i+1} = -1 \cdot \alpha_{2i}, \quad 0 \leqslant i < b/2,$$

and so it suffices to list the cosets α_{2i} only. The sets S_i are constructed as unions of cosets α_i. By the above proposition, in order to construct a skew Hadamard matrix of order $m = 4n$ it suffices to produce four such sets S_i with S_1 of skew type which are $4-(n; n_1, \dots, n_4; \lambda)$ sds where $\lambda = \sum n_i - n$ and $n_i = |S_i|$.

Theorem 2. *There exist skew Hadamard matrices of order $4n$ for n =67, 113, 127, 157, 163, 181 and 241.*

Proof: It suffices to list the required sds's. Although in some cases we have constructed several non-equivalent sds's, for the sake of brevity, we shall give here only one sds for each n listed in the theorem.

Case. n =67. Let $H = \{1, 29, 37\}$ be the subgroup of G of order 3. Enumerate the cosets α_{2i} as follows :

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 4H,\ \alpha_8 = 5H,\ \alpha_{10} = 6H,$$
$$\alpha_{12} = 8H,\ \alpha_{14} = 10H,\ \alpha_{16} = 12H,\ \alpha_{18} = 15H,\ \alpha_{20} = 17H,$$

and recall that $\alpha_{2i+1} = -1 \cdot \alpha_{2i}$. Then the sets

$$S_1 = \cup\alpha_i, \quad i \in \{0, 3, 5, 6, 9, 10, 13, 14, 17, 18, 20\},$$
$$S_2 = \cup\alpha_i, \quad i \in \{0, 2, 4, 9, 11, 12, 13, 16, 19, 21\},$$
$$S_3 = \cup\alpha_i, \quad i \in \{1, 3, 6, 10, 11, 13, 14, 16, 20, 21\},$$
$$S_4 = \cup\alpha_i, \quad i \in \{2, 4, 6, 8, 9, 11, 14, 17, 19\},$$

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are $4-(67;33,30,30,27;53)$ sds with S_1 of skew type.

CASE. $n=113$. Here $H=\{1,16,28,30,49,106,109\}$ is the subgroup of G of order 7. We enumerate the cosets α_{2i} as follows :

$$\alpha_0=H,\ \alpha_2=2H,\ \alpha_4=3H,\ \alpha_6=5H,\ \alpha_8=6H,\ \alpha_{10}=9H,\ \alpha_{12}=10H,\ \alpha_{14}=13H.$$

The sets

$$\begin{aligned} S_1&=\cup\alpha_i, & i&\in\{0,3,4,6,8,10,13,14\},\\ S_2&=\cup\alpha_i, & i&\in\{1,3,8,9,10,11,12,13\},\\ S_3&=\cup\alpha_i, & i&\in\{0,2,3,5,6,7,12\},\\ S_4&=\cup\alpha_i, & i&\in\{1,2,3,5,8,9,15\},\end{aligned}$$

are $4-(113;56,56,49,49;97)$ sds with S_1 of skew type.

CASE. $n=127$. Here $H=\{1,2,4,8,16,32,64\}$ is the subgroup of G of order 7. We enumerate the cosets α_{2i} as follows :

$$\alpha_0=H,\ \alpha_2=3H,\ \alpha_4=5H,\ \alpha_6=7H,\ \alpha_8=9H,\ \alpha_{10}=11H,$$
$$\alpha_{12}=13H,\ \alpha_{14}=19H,\ \alpha_{16}=21H.$$

The sets

$$\begin{aligned} S_1&=\cup\alpha_i, & i&\in\{0,3,5,7,8,10,12,14,16\},\\ S_2&=\cup\alpha_i, & i&\in\{0,1,3,6,7,9,10,12,14,15\},\\ S_3&=\cup\alpha_i, & i&\in\{0,1,3,4,5,7,8,9,15,16\},\\ S_4&=\cup\alpha_i, & i&\in\{1,4,5,6,9,10,13,14,15,16\},\end{aligned}$$

are $4-(127;63,70,70,70;146)$ sds with S_1 of skew type. In this case S_1 is the (127,63,31) difference set consisting of the non-zero quadratic residues modulo 127 and consequently S_2,S_3,S_4 are $3-(127;70,70,70;115)$ sds.

CASE. $n=157$. Here $H=\{1,14,16,39,46,67,75,93,99,101,108,130,153\}$ is the subgroup of G of order 13. We enumerate the cosets α_{2i} as follows :

$$\alpha_0=H,\ \alpha_2=2H,\ \alpha_4=3H,\ \alpha_6=5H,\ \alpha_8=9H,\ \alpha_{10}=15H.$$

The sets

$$\begin{aligned} S_1&=\cup\alpha_i, & i&\in\{0,2,5,7,8,11\},\\ S_2&=\cup\alpha_i, & i&\in\{0,4,5,6,9,11\},\\ S_3&=\cup\alpha_i, & i&\in\{6,7,8,9,10,11\},\\ S_4&=\cup\alpha_i, & i&\in\{0,5,6,7,8,10,11\},\end{aligned}$$

are $4-(157;78,78,78,91;168)$ sds with S_1 of skew type.

CASE. $n = 163$. Here $H = \{1,38,40,53,58,85,104,133,140\}$ is the subgroup of G of order 9. We enumerate the cosets α_{2i} as follows :

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 5H,\ \alpha_8 = 6H,\ \alpha_{10} = 9H,$$
$$\alpha_{12} = 10H,\ \alpha_{14} = 15H,\ \alpha_{16} = 18H.$$

The sets

$$\begin{aligned} S_1 &= \cup\alpha_i, \quad i \in \{0,2,5,6,9,10,13,14,17\}, \\ S_2 &= \cup\alpha_i, \quad i \in \{0,1,7,10,12,15,16,17\}, \\ S_3 &= \cup\alpha_i, \quad i \in \{0,1,3,5,8,13,15,16,17\}, \\ S_4 &= \cup\alpha_i, \quad i \in \{3,6,7,8,11,12,13,14,16,17\}, \end{aligned}$$

are $4 - (163; 81,72,81,90; 161)$ sds with S_1 of skew type.

CASE. $n = 181$. Here $H = \{1,39,43,48,62,65,73,80,132\}$ is the subgroup of G of order 9. We enumerate the cosets α_{2i} as follows :

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 4H,\ \alpha_8 = 6H,\ \alpha_{10} = 7H,$$
$$\alpha_{12} = 8H,\ \alpha_{14} = 12H,\ \alpha_{16} = 13H,\ \alpha_{18} = 24H.$$

The sets

$$\begin{aligned} S_1 &= \cup\alpha_i, \quad i \in \{0,3,5,6,8,10,13,15,16,19\}, \\ S_2 &= \cup\alpha_i, \quad i \in \{4,5,7,8,11,14,15,16,18,19\}, \\ S_3 &= \cup\alpha_i, \quad i \in \{0,4,10,11,13,15,16,18,19\}, \\ S_4 &= \cup\alpha_i, \quad i \in \{2,4,5,7,11,13,15,17,19\}, \end{aligned}$$

are $4 - (181; 90,90,81,81; 161)$ sds with S_1 of skew type.

CASE. $n = 241$. Here

$$H = \{1,15,24,54,87,91,94,98,100,119,160,183,205,225,231\}$$

is the subgroup of G of order 15. We enumerate the cosets α_{2i} as follows :

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 4H,\ \alpha_6 = 5H,\ \alpha_8 = 7H,\ \alpha_{10} = 13H,$$
$$\alpha_{12} = 19H,\ \alpha_{14} = 35H.$$

The sets

$$\begin{aligned} S_1 &= \cup\alpha_i, \quad i \in \{0,2,4,6,8,11,12,14\}, \\ S_2 &= \cup\alpha_i, \quad i \in \{1,3,4,6,7,13,14,15\}, \\ S_3 &= \cup\alpha_i, \quad i \in \{6,8,9,10,12,13,14,15\}, \\ S_4 &= \cup\alpha_i, \quad i \in \{3,4,5,9,10,13,14\}, \end{aligned}$$

are $4 - (241; 120,120,120,105; 224)$ sds with S_1 of skew type.

This completes the proof. □

NOTE ADDED IN PROOF. If we interchange S_1 and S_4 in the sds given above for the case $n = 241$, then the matrix H is a Hadamard matrix of order 964 of maximal excess 29884.

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