Y. Ikeda
Nagoya Math. J.
Vol. 60 (1976), 151–172

REGULARITY OF SOLUTIONS FOR QUASI-LINEAR PARABOLIC EQUATIONS

YOSHIAKI IKEDA

§ 1. Introduction.

Let Ω be a bounded domain in n-dimensional Euclidian space E^n ($n \geqq 2$), and consider the space-time cylinder $Q = \Omega \times (0, T]$ for some fixed $T > 0$. In this paper we deal with the Cauchy and Dirichlet problem for a second order quasi-linear equation

$$(1.1) \qquad u_t - \operatorname{div} \mathscr{A}(x, t, u, u_x) + B(x, t, u, u_x) = 0 \qquad \text{for } (x, t) \in Q ,$$

$$(1.2) \qquad u(x, 0) = \phi(x) \quad \text{in } \Omega \text{ and } \quad u(x, t) = \psi(x, t) \quad \text{for } (x, t) \in \Gamma = \partial\Omega \times (0, T] ,$$

where $\partial\Omega$ is a boundary of Ω which satisfies the following condition (A):
Condition (A). There exist constants ρ_0 and λ_0 both in $(0, 1)$ such that, for any sphere $K(\rho)$ with center on $\partial\Omega$ and radius $\rho \leqq \rho_0$, the inequality $\operatorname{meas}[K(\rho) \cap \Omega] \leqq (1 - \lambda_0) \times \operatorname{meas} K(\rho)$ holds, where meas E means the measure of a measurable set E.

In the equation $\mathscr{A} = (\mathscr{A}_1, \cdots, \mathscr{A}_n)$ is a given vector function of (x, t, u, u_x), B is a given scalar function of the same variables, and $u_x = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$ denotes the spatial gradient of the dependent variable $u = u(x, t)$. Also $\operatorname{div} \mathscr{A}$ refers to the divergence of the vector $\mathscr{A}(x, t, u, u_x)$ with respect to the variables $x = x(x_1, \cdots, x_n)$. The functions $\phi(x)$ and $\psi(x, t)$ in (1.2) are bounded, measurable and belong to the spaces $L^2(\Omega)$ and $L^\infty[0, T; L^2(\tilde{\Omega})] \cap L^\alpha[0, T; H^{1,\alpha}(\tilde{\Omega})]$ respectively, where $\tilde{\Omega}$ is a domain containing Ω.

Throughout the paper we assume that $\mathscr{A}$ and B satisfy inequalities of the form

$$(1.3) \qquad \begin{cases} p \cdot \mathscr{A}(x, t, u, p) \geqq a_0 |p|^\alpha - c(x, t) |u|^\alpha - f(x, t) , \\ |B(x, t, u, p)| \leqq b(x, t) |p|^{\alpha-1} + d(x, t) |u|^{\alpha-1} + g(x, t) , \\ |\mathscr{A}(x, t, u, p)| \leqq \bar{a} |p|^{\alpha-1} + e(x, t) |u|^{\alpha-1} + h(x, t) , \end{cases}$$

Received November 18, 1974.

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

for any n-dimensional real vector p and for any real number $\alpha > 2$. Here a_0 and $\bar{a}$ are positive constants and the coefficients b, c, d, e, f, g, h are non-negative functions of (x, t) and $b^\alpha, c, d, e^{\alpha/(\alpha-1)}, f, g, h^{\alpha/(\alpha-1)}$ belong to some space $L^{p,q}(Q)$, where p and q are non-negative real numbers satisfying

$$\frac{n}{\alpha p} + \frac{\alpha n - 2n + 2\alpha}{2\alpha q} < 1 \qquad \text{when } \alpha < n \tag{1.4}$$

and

$$\frac{1}{p} + \frac{\alpha}{2q} < 1 - \varepsilon_0 , \qquad p > 1 \tag{1.5}$$

for any sufficiently small $\varepsilon_0 > 0$ when $\alpha \geqq n$.

A function $w = w(x, t)$ which is measurable on Q will be said to belong to the class $L^{p,q}(Q)$ if the iterated integral

$$\|w\|_{p,q} = \left\{\int_0^T \left(\int_\Omega |w|^p \, dx\right)^{q/p} dt\right\}^{1/q}$$

is finite. If a function $w(x)$ which is measurable on Ω possesses a distribution derivative $(u_{x_1}, \cdots, u_{x_n})$ and if $\|w\|_{L^p(\Omega)} + \|w_x\|_{L^p(\Omega)} < \infty$, then $w(x)$ is said to belong to $H^{1,p}(\Omega)$, where $\|w_x\|^p_{L^p(\Omega)} = \sum_{i=1}^n \|u_{x_i}\|^p_{L^p(\Omega)}$.

The space $H_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to this norm.

We denote by $L^q[0, T; H^{1,p}(\Omega)]$ the space of functions $w(x, t)$ with the following properties:

(i) $w(x, t)$ is measurable on Q,
(ii) for almost all $t \in (0, T]$, $w(x, t) \in H^{1,p}(\Omega)$,
(iii) $\|w(x, t)\|_{H^{1,p}(\Omega)} \in L^q[0, T]$.

The function u is said to be a weak solution of the problem (1.1), (1.2) if u belongs to the space $H^{1,2}[0, T; L^2(\Omega)] \cap L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H^{1,\alpha}(\Omega)]$ and if u satisfies the following conditions:

$$\int_{t_0}^{t_1}\int_\Omega \{u_t\Phi(x, t) + \mathscr{A}(x, t, u, u_x)\Phi_x + B(x, t, u, u_x)\Phi\}dxdt = 0 \tag{1.6}$$

for any t_0, t_1 $(0 \leqq t_0 < t_1 \leqq T)$ and

$$\lim_{t\to 0}\int_\Omega u(x, t)\Phi(x, t)dx = \int_\Omega \phi(x)\Phi(x, 0)dx \tag{1.7}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

for any continuously differentiable function $\Phi = \Phi(x, t)$ with compact support in Ω. That the boundary value of u is equal to $\psi(x, t)$ on Γ in (1.2) means that $u(x, t) - \psi(x, T) \in L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H_0^{1,\alpha}(\Omega)]$ for $\psi(x, t) \in L^\infty[0, T; L^2(\tilde{\Omega})] \cap L^\alpha[0, T; H^{1,\alpha}(\tilde{\Omega})]$, where $\tilde{\Omega} \supset \bar{\Omega}$.

In section 4 we shall prove the boundedness of the solution of the problem (1.1), (1.2) when $\phi(x)$ and $\psi(x, t)$ are bounded. The same result was obtained by D. G. Aronson and J. Serrin [2] for non-linear parabolic equation (1.1) under the condition

$$\begin{cases} p \cdot \mathscr{A}(x, t, u, p) \geqq a\,|p|^\alpha - e^\alpha\,|u|^\alpha - h^\alpha\,, \\ |B(x, t, u, p)| \leqq b\,|p|^{\alpha-1} + d^{\alpha-1}\,|u|^{\alpha-1} + g^{\alpha-1}\,, \end{cases}$$

where coefficients $a, b, \cdots, g$ are non-negative constants.

In section 5 our main theorem states that if u is a weak solution of the problem (1.1), (1.2), then u is Hölder continuous in Q and that, moreover if the boundary value $\psi(x, t)$ of u is Hölder continuous then u is Hölder continuous on $\bar{Q} = \bar{\Omega} \times (0, T]$.

This result extends theorems proved by Ladyzenskaya and Uralceva [3] on some linear and quasi-linear parabolic equations, theorems proved by Serrin [4] on quasi-linear elliptic equations, and those given by Aronson and Serrin [1] on the quasi-linear parabolic equations

$$u_t = \operatorname{div} \mathscr{A}(x, t, u, u_x) + B(x, t, u, u_x)$$

under the conditions

$$\begin{cases} p \cdot \mathscr{A}(x, t, u, p) \geqq a\,|p|^2 - c^2\,|u|^2 - f^2\,, \\ |B(x, t, u, p)| \leqq b\,|p| + d\,|u| + g\,, \\ |\mathscr{A}(x, t, u, p)| \leqq \bar{a}\,|p| + e\,|u| + h\,, \end{cases}$$

where a and $\bar{a}$ are positive constants, while the coefficients $b, c, \cdots, h$ are non-negative functions of (x, t) and each coefficient is contained in some space $L^{p,q}(Q)$, where

$$p > 2 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < \frac{1}{2} \quad \text{for } b, c, e, f, h$$

and

$$p > 1 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{p} < 1 \quad \text{for } d, g\,.$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

§ 2. Preliminaries.

In this section we shall state and prove several lemmas which are often used later.

Using the Hölder's inequality we can easily prove the following lemma:

LEMMA 2.1 (Aronson-Serrin [1]). *If w is contained in $L^{q,q_1} \cap L^{r,r_1}$, then w is contained in L^{p,p_1}, where*

$$\frac{1}{p} = \frac{\lambda}{q} + \frac{\mu}{r}, \quad \frac{1}{p_1} = \frac{\lambda}{q_1} + \frac{\mu}{r_1} \quad (\lambda, \mu \geqq 0,\ \lambda + \mu = 1) .$$

Moreover

$$\|w\|_{p,p_1} \leqq \|w\|^{\lambda}_{q,q_1} \cdot \|w\|^{\mu}_{r,r_1} ,$$

where

$$\|w\|_{p,q} = \left(\int_0^T \left(\int_\Omega |w|^p \, dx\right)^{q/p} dt\right)^{1/q} .$$

LEMMA 2.2 (Aronson-Serrin [1]). *Let w belong to the space $L^\alpha[0, T; H_0^{1,\alpha}(\Omega)]$. Then*

$$\|w\|_{\alpha^*,\alpha} \leqq K \|w_x\|_{\alpha,\alpha} ,$$

where $\dfrac{1}{\alpha^*} = \dfrac{1}{\alpha} - \dfrac{1}{n}$ when $n > \alpha$, and α^* is any finite number when $\alpha \geqq n$. The constant K depends only on α, n and Ω. If $n \leqq \alpha$, then K depends on the choice of α^*.

LEMMA 2.3. *If w belongs to the space $L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H_0^{1,\alpha}(\Omega)]$, then w belongs to the space $L^{\alpha p', \alpha q'}$ for all exponents pairs (p', q') whose Hölder conjugate (p, q) satisfies*

$$\frac{n}{\alpha p} + \frac{\alpha n - 2n + 2\alpha}{2\alpha q} < 1 \qquad \text{when } \alpha < n$$

and

$$\frac{1}{p} + \frac{\alpha}{2q} < 1 - \varepsilon_0 \quad \text{for any sufficiently small } \varepsilon_0 > 0 \text{ when } \alpha \geqq n .$$

Moreover

$$\|w\|^{\alpha}_{\alpha p', \alpha q'} \leqq K T^{\nu} \{ \|w\|^{\alpha}_{2,\infty} + \|w_x\|^{\alpha}_{\alpha,\alpha} \} \tag{2.1}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

and for any $\varepsilon > 0$

$$\|w\|^{\alpha}_{\alpha p', \alpha q'} \leqq \varepsilon \|w_x\|^{\alpha}_{\alpha,\alpha} + C(\varepsilon) T^{\nu\alpha/(\alpha-1)} \|w\|^{\alpha}_{2,\infty} , \tag{2.2}$$

where $\nu = \left(1 - \frac{1}{\kappa}\right)\frac{1}{q'}$, $\kappa = \frac{2}{p'} - \frac{2}{\alpha^* q'} + \frac{1}{q'} > 1$, K *depends only on* α, n *and* meas Ω, *and* $C(\varepsilon)$ *depends only on* ε, α, n *and* meas Ω.

Proof. Let κ be a real number >1. Then by Hölder's inequality and Lemma 2.1,

$$\|w\|^{\alpha}_{\alpha p', \alpha q'} \leqq \|w\|^{\alpha}_{\alpha p'\kappa, \alpha q'\kappa} \{T^{1/q'} (\text{meas } \Omega)^{1/p'}\}^{1-1/\kappa}$$

and

$$\|w\|^{\alpha}_{\alpha p'\kappa, \alpha q'\kappa} \leqq \|w\|^{\lambda}_{\alpha^*, \alpha} \|w\|^{\alpha-\lambda}_{2,\infty}$$

provided that

$$0 \leqq \lambda \leqq \alpha \quad \text{and} \quad \frac{1}{\kappa p'} = \frac{\lambda}{\alpha^*} + \frac{\alpha - \lambda}{2}, \quad \frac{1}{\kappa q'} = \frac{\lambda}{\alpha} .$$

These relations imply

$$\lambda = \frac{\alpha}{\kappa q'} , \qquad \kappa = \frac{2}{\alpha p'} - \frac{2}{\alpha^* q'} + \frac{1}{q'} > 1 .$$

From Young's inequality and Lemma 2.2 we have (2.1) and (2.2).

LEMMA 2.4. *If the function* $u(x)$ *belongs to the space* $H_0^{1,\alpha}(\Omega)$, *then it holds*

$$\int_\Omega |u|^\alpha \, dx \leqq K \int_\Omega |u_x|^\alpha \, dx \cdot [\text{meas } \Omega]^{\alpha/n} .$$

Proof. By Hölder's inequality, it is clear that

$$\int_\Omega |u|^\alpha \, dx \leqq \left(\int_\Omega |u|^{\alpha^*} dx\right)^{\alpha/\alpha^*} \cdot (\text{meas } \Omega)^{1-\alpha/\alpha^*} ,$$

where $\alpha^* = \frac{n-\alpha}{\alpha n}$ when $\alpha < n$ and when $\alpha \geqq n$, α^* is any number $>\alpha$.

If $\alpha < n$, then $1 - \frac{\alpha}{\alpha^*} = \frac{\alpha}{n}$ and from Sobolev's lemma we have our lemma.

If $\alpha \geqq n$, we take $\beta < n$ such that $\beta^* = \frac{n-\beta}{\beta n} = \alpha^*$. Then

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\left(\int_\Omega |u|^{\alpha^*} dx\right)^{\alpha/\alpha^*}\cdot(\text{meas }\Omega)^{1-\alpha/\alpha^*} \leqq K\left(\int_\Omega |u_x|^\beta dx\right)^{\alpha/\beta}(\text{meas }\Omega)^{1-\alpha/\beta^*}$$
$$\leqq K\left(\int_\Omega |u_x|^\alpha dx\right)(\text{meas }\Omega)^{\alpha/\beta(1-\beta/\alpha)+1-\alpha/\beta^*}$$
$$= K\left(\int_\Omega |u_x|^\alpha dx\right)(\text{meas }\Omega)^{\alpha/n} .$$

§ 3. Fundamental inequalities.

In this section we shall derive some fundamental inequalities for weak solutions of the problem (1.1), (1.2), which are used in the following sections.

Let u be a weak solution of the problem (1.1), (1.2) and for a real number k, put

$$A_k(t) = \{x \in \Omega \mid u(x, t) \geqq k\} \quad \text{and} \quad B_k(t) = \{x \in \Omega \mid u(x, t) < k\} .$$

We assume that the boundary value $\psi(x, t)$ and the initial value $\phi(x)$ belong to the spaces $L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H^{1,\alpha}(\Omega)]$ and $L^2(\Omega)$ respectively and they are bounded, i.e. there exists a positive constant M_0 such that

(3.1) $$|\psi(x, t)| \leqq M_0 , \qquad |\phi(x)| \leqq M_0 .$$

We put $M = \max_{0\leqq t\leqq T}\left(\int_\Omega u^2 dx\right)^{1/2} = \|u\|_{2,\infty}$, and $U = \dfrac{u}{M}$.

Then, since u is a weak solution of (1.1), we have

(3.2) $$U_t - \frac{1}{M}\operatorname{div}\mathscr{A}(x, t, MU, MU_x) + \frac{1}{M}B(x, t, MU, MU_x) = 0 .$$

Thus, it holds that

(3.3) $$\int_{t_0}^{t_1}\int_\Omega \Big\{U_t\Phi + \frac{1}{M}\mathscr{A}(x, t, MU, MU_x)\Phi_x + \frac{1}{M}B(x, t, MU, MU_x)\Phi\Big\}dxdt = 0$$

for any differentiable function $\Phi(x, t)$ with compact support in Ω.

It is clear that (3.3) is valid for $\Phi \in L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H_0^{1,\alpha}(\Omega)]$.

Now we put $u^{(k)} = \max(u, k) - k$.

If $k \geqq M_0$, then $u^{(k)} \in L^\infty[0, T; L^2(\Omega)] \cap L^\alpha[0, T; H_0^{1,\alpha}(\Omega)]$. Hence, taking $\Phi = u^{(k)}$ in (3.3), we have

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$(3.4)\qquad \int_{t_0}^{t_1}\int_{A_k(t)}\left(U_t u^{(k)} + \frac{1}{M}\mathscr{A}\cdot u_x^{(k)} + \frac{1}{M}B\cdot u^{(k)}\right)dxdt = 0\ .$$

If we put $U^{(k)} = \dfrac{u^{(k)}}{M}$, then, letting $t_0 \to 0$, we see,

$$\int_{t_0}^{t_1}\int_{A_k(t)} U_t u^{(k)}dxdt = M\int_{t_0}^{t_1}\int_{A_k(t)}\frac{1}{2}\frac{\partial}{\partial t}\{(U^{(k)})^2\}dxdt$$
$$\longrightarrow \frac{M}{2}\int_{A_k(t)}(U^{(k)})^2dx \qquad \text{as } t_0 \to 0\ ,$$

because of $U^{(k)}(x,0) = 0$.

It is obvious from the condition (1.3) that

$$\int_0^{t_1}\int_{A_k(t)}\frac{1}{M}\mathscr{A}\cdot u_x^{(k)}dxdt = \int_0^{t_1}\int_{A_k(t)}\mathscr{A}\cdot U_x^{(k)}dxdt$$
$$\geqq \frac{a_0}{M}\int_0^{t_1}\int_{A_k(t)} M^\alpha |U_x^{(k)}|^\alpha\, dxdt - \frac{1}{M}\int_0^{t_1}\int_{A_k(t)} c(x,t)\,|MU|^\alpha\, dxdt$$
$$- \frac{1}{M}\int_0^{t_1}\int_{A_k(t)} f(x,t)dxdt$$

and

$$\int_0^{t_1}\int_{A_k(t)}\frac{1}{M}B\cdot u^{(k)}dxdt = \int_0^{t_1}\int_{A_k(t)} BU^{(k)}dxdt$$
$$\leqq \int_0^{t_1}\int_{A_k(t)}\{b(x,t)M^{\alpha-1}|U_x^{(k)}|^{\alpha-1}|U^{(k)}| + d(x,t)M^{\alpha-1}|U|^{\alpha-1}|U^{(k)}|$$
$$+ g(x,t)\,|U^{(k)}|\}dxdt\ .$$

Thus we obtain

$$\frac{M}{2}\|U^{(k)}\|_{2,\infty}^2 + a_0M^{\alpha-1}\|U_x^{(k)}\|_{\alpha,\alpha}^\alpha$$
$$(3.5)\qquad \leqq \int_0^{t_1}\int_{A_k(t)}\Big\{M^{\alpha-1}b\,|U_x^{(k)}|^{\alpha-1}|U^{(k)}| + cM^{\alpha-1}\,|U|^\alpha + dM^{\alpha-1}|U^{\alpha-1}|\,|U^{(k)}|$$
$$+ \frac{1}{M}f + g\,|U^{(k)}|\Big\}dxdt\ ,$$

where $\|U^{(k)}\|_{2,\infty}^2 = \max\limits_{0\leqq t\leqq t_1}\displaystyle\int_{A_k(t)}(u^{(k)})^2dx$

and

$$\|U_x^{(k)}\|_{\alpha,\alpha}^\alpha = \int_0^{t_1}\int_{A_k(t)}|U_x^{(k)}|^\alpha\, dxdt\ .$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

Using Young's inequality, we see

$$M^{\alpha-1}b\,|U_x^{(k)}|^{\alpha-1}|U^{(k)}| \leqq \frac{1}{2}a_0M^{\alpha-1}|U_x^{(k)}|^{\alpha} + C_0b^{\alpha}M^{\alpha-1}|U^{(k)}|^{\alpha} \tag{3.6}$$

and

$$M^{\alpha-1}d\,|U^{\alpha-1}||U^{(k)}| \leqq C_1M^{\alpha-1}d\{|U|^{\alpha} + |U^{(k)}|^{\alpha}\} \tag{3.7}$$

where C_0 and C_1 are positive constants depending only on a_0 and α.

Since $U = U^{(k)} + \dfrac{k}{M}$ in $A_k(t)$, it follows that

$$|U|^{\alpha} \leqq C_2\left\{|U^{(k)}|^{\alpha} + \left(\frac{k}{M}\right)^{\alpha}\right\}, \tag{3.8}$$

where C_2 is a positive constant depending only on α.

Moreover, since $\|U^{(k)}\|_{2,\infty} \leqq 1$, it is clear that

$$\|U^{(k)}\|_{2,\infty}^{\alpha} \leqq \|U^{(k)}\|_{2,\infty}^{2}\,. \tag{3.9}$$

Thus we have from (3.5)~(3.9),

$$\begin{aligned} a_1(\|U^{(k)}\|_{2,\infty}^{\alpha} &+ \|U_x^{(k)}\|_{\alpha,\alpha}^{\alpha} \\ &\leqq C\left\{\int_0^{t_1}\int_{A_k(t)}\{(b^{\alpha} + c + d + 1)|U^{(k)}|^{\alpha} + (1 + k^{\alpha})(c + d + f)\right. \\ &\qquad\qquad + g\,|U^{(k)}|\}dxdt\,, \end{aligned} \tag{3.10}$$

where $a_1 = \min\left(\dfrac{M}{2}, \dfrac{a_0}{2}M^{\alpha-1}\right)$ and C is a positive constant depending only on α and M.

If we put $\theta_1 = b^{\alpha} + c + d + 1$, then θ_1 belongs to the space $L^{p,q}(Q)$ with p and q satisfying the inequality (1.5). Thus from Lemma 2.3, we see

$$\begin{aligned} \int_0^{t_1}\int_{A_k(t)} &\theta_1|U^{(k)}|^{\alpha}\,dxdt \\ &\leqq \|\theta_1\|_{p,q}\|U^{(k)}\|_{\alpha p',\alpha q'}^{\alpha} \\ &\leqq K\,\|\theta_1\|_{p,q}t_1^{\nu}(\|U_x^{(k)}\|_{\alpha,\alpha}^{\alpha} + \|U^{(k)}\|_{2,\infty}^{\alpha})\,. \end{aligned} \tag{3.11}$$

Similarly if we put $\theta_2 = c + d + f$, then $\theta_2 \in L^{p,q}$. Thus we see

$$\begin{aligned} \int_0^{t_1}\int_{A_k(t)} &\theta_2(1 + k^{\alpha})dxdt \\ &\leqq (1 + k^{\alpha})\,\|\theta_2\|_{p,q}\left(\int_0^{t_1}(\text{meas}\,A_k(t))^{q'/p'}dt\right)^{1/q'}, \end{aligned} \tag{3.12}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

and

$$\text{(3.13)}\qquad \int_0^{t_1}\int_{A_k(t)} g\,|U^{(k)}|\,dxdt$$
$$\leqq \|g\|_{p,q}\,\|U^{(k)}\|_{\alpha p',\alpha q'}\left(\int_0^{t_1}(\text{meas}\,A_k(t))^{q'/p'}dt\right)^{((\alpha-1)/\alpha)\times(1/q')}$$
$$\leqq Kt_1^{\nu}(\|U_x^{(k)}\|_{\alpha,\alpha}^{\alpha}+\|U^{(k)}\|_{2,\infty}^{\alpha})$$
$$+\|g\|_{p,q}^{\alpha/(\alpha-1)}\left(\int_0^{t_1}(\text{meas}\,A_k(t))^{q'/p'}dt\right)^{1/q'}.$$

If we take t_1 sufficiently so small that

$$Kt_1^{\nu}(\|\theta_1\|_{p,q}+1)<a_1\,,$$

then from (3.10)~(3.13) we have

$$\text{(3.14)}\qquad \|U^{(k)}\|_{2,\infty}^{\alpha}+\|U_x\|_{\alpha,\alpha}^{\alpha}\leqq C(1+k^{\alpha})\left(\int_0^{t_1}(\text{meas}\,A_k(t))^{q'/p'}dt\right)^{1/q'}$$

where C is a positive constant depending only on α, M, a_0, $\|b\|$, $\|c\|$, $\|d\|$, $\|f\|$ and $\|g\|$.

The following analogous inequality is obtained by the same calucu-lation as above:

$$\text{(3.15)}\qquad \|U^{(k)}\|_{2,\infty}^{\alpha}+\|U_x^{(k)}\|_{\alpha,\alpha}^{\alpha}\leqq C(1+k^{\alpha})\left(\int_0^{t_1}(\text{meas}\,B_k(t))^{q'/p'}dt\right)^{1/q'}$$

for $k\leqq -M_0$.

The inequalities (3.14) and (3.15) are used to prove boundedness of weak solutions u (see §4).

In the following, we derive other inequalities for weak solutions which will be used in §5.

Let u be a bounded weak solution of (1.1), (1.2) and put

$$\|u\|_{\infty,Q}=M_1\,,\quad c(x,t)M_1^{\alpha}+f(x,t)=f_1(x,t)\,,\quad d(x,t)M_1^{\alpha-1}+g(x,t)=g_1(x,t)$$

and $e(x,t)M_1^{\alpha-1}+h(x,t)=h_1(x,t)$.

Then from the condition (1.3), we have

$$\text{(3.16)}\qquad \begin{cases} p\cdot\mathscr{A}(x,t,u,p)\geqq a_0|p|^{\alpha}-f_1\,,\\ |B(x,t,u,p)|\leqq b\,|p|^{\alpha-1}+g_1\,,\\ |\mathscr{A}(x,t,u,p)|\leqq \bar{a}\,|p|^{\alpha-1}+h_1\,.\end{cases}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

We introduce the notation

$$K(\rho)=\{x \mid |x-x_0|<\rho,\ x_0\in\Omega\}\,,\qquad \Gamma_\rho=K(\rho)\cap\partial\Omega\,,$$
$$A_{k,\rho}(t)=\{x\in K(\rho)\mid u(x,t)\geqq k\}\,,$$
$$B_{k,\rho}(t)=\{x\in K(\rho)\mid u(x,t)\leqq k\}\,,$$

and for $\rho>\rho'$

$$\zeta=\zeta(x;\rho,\rho')=\begin{cases}1 & \text{for } x\in K(\rho-\rho')\,,\\ \dfrac{\rho-|x-x_0|}{\rho-\rho'} & \text{for } x\in K(\rho)-K(\rho')\,,\\ 0 & \text{outside } K(\rho)\,,\end{cases}$$

where $K(\rho')$ is a concentric cube with $K(\rho)$.

If we put $\Phi(x,t)=u^{(k)}\zeta^\alpha$ for $k\geqq\max\limits_{\Gamma_\rho\times[t_0,t_1]}u$, then

$\Phi\in L^\infty[0,T;L^2(\Omega)]\cap L^\alpha[0,T;H_0^{1,\alpha}(\Omega)]$. (When $K(\rho)\subset\Omega$, k is an arbitrary number.) Since u is a weak solution of (1.1), (1.2), the equality (1.7) is valid for $\Phi=u^{(k)}\zeta^\alpha$, that is for any t_0,t_1 $(0\leqq t_0<t_1\leqq T)$,

$$\text{(3.17)}\qquad \int_{t_0}^{t_1}\int_{A_{k,\rho}(t)}\{u_t u^{(k)}\zeta^\alpha+(u_x^{(k)}\zeta^\alpha+\alpha\zeta^{\alpha-1}\zeta_x u^{(k)})\cdot\mathscr{A}+u^{(k)}\zeta^\alpha B\}dxdt=0\,.$$

Since ζ^α is independent of the variable t, it follows that

$$\text{(3.18)}\qquad u_t u^{(k)}\zeta^\alpha=\frac{1}{2}\{(u^{(k)})^2\}_t \qquad \text{in } A_{k,\rho}(t)\,.$$

From the condition (3.16), we see

$$\text{(3.19)}\qquad u_x^{(k)}\zeta^\alpha\cdot\mathscr{A}\geqq a_0|u_x^{(k)}|^\alpha\zeta^\alpha-f_1\,,$$

$$\text{(3.20)}\qquad \begin{aligned}\alpha u^{(k)}\zeta^{\alpha-1}\zeta_x\cdot\mathscr{A}&\leqq\alpha\bar{a}|u_x^{(k)}|^{\alpha-1}|u^{(k)}|\zeta^{\alpha-1}|\zeta_x|+d|u^{(k)}|^{\alpha-1}\zeta^{\alpha-1}|\zeta_x|h_1\\ &\leqq\varepsilon|u_x^{(k)}|^\alpha\zeta^\alpha+C_0|u^{(k)}|^\alpha|\zeta_x|^\alpha\\ &\quad+C_1(|u^{(k)}|^\alpha|\zeta_x|^\alpha+h_1^{\alpha/(\alpha-1)}\zeta^\alpha)\,,\end{aligned}$$

and

$$\text{(3.21)}\qquad \begin{aligned}u^{(k)}\zeta^\alpha B&\leqq b|u_x^{(k)}|^{\alpha-1}\zeta^\alpha|u^{(k)}|+g_1|u^{(k)}|\zeta^\alpha\\ &\leqq\varepsilon|u_x^{(k)}|^\alpha\zeta^\alpha+C_2b^\alpha|u^{(k)}|^\alpha\zeta^\alpha+g_1|u^{(k)}|\zeta^\alpha\end{aligned}$$

for an arbitrary positive number ε, where C_0, C_1 and C_2 are constants depending only on α and ε.

Taking $\varepsilon=\dfrac{a_0}{4}$, we have from (3.17)~(3.21),

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$(3.22)\quad \frac{1}{2}\int_{A_{k,\rho}(t)}(u^{(k)})^2\zeta^\alpha dx - \frac{1}{2}\int_{A_{k,\rho}(t_0)}(u^{(k)}(x,t_0))^2\zeta^\alpha dx + \frac{a_0}{2}\int_{t_0}^{t}\int_{A_{k,\rho}(t)}|u_x|^\alpha\zeta^\alpha dxdt$$
$$\leqq C_3\Big\{\int_{t_0}^{t}\int_{A_{k,\rho}(t)}\{b^\alpha|u^{(k)}|^\alpha\zeta^\alpha + g_1|u^{(k)}|\zeta^\alpha + (f_1 + h_1^{\alpha/(\alpha-1)})\zeta^\alpha + |u^{(k)}|^\alpha|\zeta_x|^\alpha\}dxdt\}$$

for any t $(0 \leqq t_0 \leqq t \leqq t_1 \leqq T)$.

First, we see from Lemma 3.3,

$$(3.23)\quad \int_{t_0}^{t}\int_{A_{k,\rho}(t)} b^\alpha|u^{(k)}|^\alpha\zeta^\alpha dxdt \leqq \|b^\alpha\|_{p,q}\{(t-t_0)^{\nu\alpha/(\alpha-1)}\|u^{(k)}\zeta\|^\alpha_{2,\infty} + \varepsilon\|(u^{(k)}\zeta)_x\|^\alpha_{\alpha,\alpha}$$

where $\|u^{(k)}\zeta\|^\alpha_{2,\infty} = \max_{t_0\leqq t\leqq t_1}\left(\int(u^{(k)})^2dx\right)^{\alpha/2}$.

Similarly we obtain

$$(3.24)\quad \int_{t_0}^{t}\int_{A_{k,\rho}(t)} g_1|u^{(k)}|\zeta^\alpha dxdt$$
$$\leqq \|g_1\|_{p,q}\|u^{(k)}\zeta\|_{\alpha p',\alpha q'}\left(\int_{t_0}^{t}(\text{meas}\,A_{k,\rho}(t))^{q'/p'}dt\right)^{(\alpha-1)/\alpha q'}$$
$$\leqq \varepsilon(\|u_x^{(k)}\zeta\|^\alpha_{\alpha,\alpha} + \|u^{(k)}\zeta_x\|^\alpha_{\alpha,\alpha}) + C_4(t-t_0)^{\nu\alpha/(\alpha-1)}\|u^{(k)}\zeta\|^\alpha_{2,\infty} + C_5\|g_1\|^{\alpha/(\alpha-1)}_{p,q}\left(\int_{t_0}^{t}(\text{meas}\,A_{k,\rho}(t))^{q'/p'}dt\right)^{1/q'}$$

and

$$(3.25)\quad \int_{t_0}^{t}\int_{A_{k,\rho}(t)}(f_1 + h_1^{\alpha/(\alpha-1)})\zeta^\alpha dxdt$$
$$\leqq \|f_1 + h_1^{\alpha/(\alpha-1)}\|_{p,q}\left(\int_{t_0}^{t}(\text{meas}\,A_{k,\rho}(t))^{q'/p'}dt\right)^{1/q'}.$$

From (3.22)~(3.25), by putting $\varepsilon = \dfrac{a_0}{4(1+\|b\|_{p,q})}$, it holds

$$(3.26)\quad \frac{1}{2}\int_{A_{k,\rho}(t)}(u^{(k)})^2\zeta^\alpha dx - \frac{1}{2}\int_{A_{k,\rho}(t_0)}(u^{(k)}(x,t_0))^2\zeta^\alpha dx + \frac{a_0}{4}\int_{t_0}^{t}\int_{A_{k,\rho}(t)}|u_x^{(k)}|^\alpha\zeta^\alpha dxdt$$
$$\leqq \mathscr{C}\Big\{\int_{t_0}^{t}\int_{A_{k,\rho}(t)}|u^{(k)}|^\alpha|\zeta_x|^\alpha dxdt + \max_{t_0\leqq t\leqq t_1}\left(\int_{A_{k,\rho}(t)}|u^{(k)}|^2\zeta^2dx\right)^{\alpha/2}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\times (t - t_0)^{\nu\alpha/(\alpha-1)} + \left(\int_{t_0}^{t} (\text{meas}\, A_{k,\rho}(t))^{q'/p'} dt\right)^{1/q'}\Bigg\} = I(t)$$

for any t $(t_0 \leqq t \leqq t_1)$. From this we have the following two inequalities

$$\max_{t_0 \leqq t \leqq t_1} \int_{A_{k,\rho}(t)} (u^{(k)}(x, t))^2 \zeta^\alpha dx \leqq I(t_1) + \int_{A_{k,\rho}(t_0)} (u^{(k)}(x, t_0))^2 \zeta^\alpha dx \,, \tag{3.27}$$

$$\int_{t_0}^{t_1}\int_{A_{k,\rho}(t)} |u_x^{(k)}|^\alpha \zeta^\alpha dx dt \leqq I(t_1) \tag{3.28}$$

for any t_1 $(0 \leqq t_0 < t_1 \leqq T)$.

§ 4. Boundedness of weak solutions.

In this section we concern with boundedness of a weak solution u when u is bounded on the parabolic boundary $\partial Q = \partial \Omega \times (0, T] \cup \Omega \times \{t = 0\}$, that is, when $\psi(x, t)$ and $\phi(x)$ are bounded.

LEMMA 4.1 (Stampacchia [5]). *Let $\Xi(k)$ be a non-negative and non-increasing function defined for $k \geqq k_0$. If the inequality*

$$\Xi(h) \leqq \frac{C}{(h - k)^s}[\Xi(k)]^\beta$$

holds for $h > k \geqq k_0$ and $\beta > 1$, then

$$\Xi(k_0 + d^s) = 0 \,,$$

where $d^s = C[\Xi(k_0)]^{\beta-1} 2^{s\beta/(\beta-1)}$.

Now we can prove the following.

THEOREM 4.1. *Suppose that $\psi(x, t)$ and $\phi(x)$ are bounded. Then a weak solution of the problem* (1.1), (1.2) *is bounded in Q.*

Proof. Let M_0 be a positive constant such that

$$|\psi(x, t)| \leqq M_0 \quad \text{and} \quad |\phi(x)| \leqq M_0 \quad (M_0 > 1)$$

and let

$$U = \frac{u}{M} \,, \qquad \text{where } M = \max_{0 \leqq t \leqq T} \left(\int_\Omega u^2 dx\right)^{1/2} .$$

Then the inequality (3.14) and (3.15) hold for U.

Now, put $k_h = M_0\left(2 - \dfrac{1}{2^h}\right) \qquad (h = 0, 1, 2, \cdots)$ and

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\mu(k) = \int_{t_0}^{t_1} (\text{meas } A_k(t))^{q'/p'} dt .$$

Then it follows that

$$(k_{h+1} - k_h)^\alpha \mu(k_{h+1})^{\alpha/q'\kappa} \leqq \left(\int_0^{t_1} \left(\int_{A_{k_h}(t)} (u_h^{(k_h)})^{\alpha\kappa p'} dx\right)^{q'/p'} dt\right)^{\alpha/\alpha\kappa q'}$$
$$= \|u^{(k_h)}\|^\alpha_{\alpha\kappa p', \alpha\kappa q'} \leqq Kt^\nu(\|u^{(k_h)}\|^\alpha_{2,\infty} + \|u_x^{(k_h)}\|^\alpha_{\alpha,\alpha}) \leqq Ck_h^\alpha \mu(k_h)^{\alpha/q'} ,$$

where C is a positive constant depending only on α, M_0, M, a_0, $\|b\|$, $\|c\|$, $\|d\|$, $\|f\|$ and $\|g\|$.

If we put $\Xi(k) = \mu(k)^{\alpha/q'\kappa}$, then

$$(4.1) \qquad (k_{h+1} - k_h)\Xi(k_{h+1}) \leqq Ck_h[\Xi(k_h)]^\kappa .$$

Since $\kappa > 1$, from the preceding lemma 4.1 we have

$$\Xi(k_0 + d^s) = 0 ,$$

that is, $u(x, t)$ is bounded from above in $\Omega \times (0, t_1]$.

Similarly, from the inequality (3.15) we see that $u(x, t)$ is bounded from below in $\Omega \times (0, t_1]$.

Repeating the same argument on $\Omega \times (Nt_1, (N+1)t_1]$ inductively, we conclude that u is bounded in Q.

§ 5. Hölder continuity of weak solutions.

In this section we prove Hölder continuity of a weak solution u of the problem (1.1), (1.2). The method presented here is based on the idea of [3].

Throughout this section, we assume that there is a positive constant M_1 such that $|u| \leqq M_1$ in Q.

First we shall state some lemmas.

LEMMA 5.1 (Theorem 6.3 in [5]). *Let* $u(x) \in H^{1,2}(K(\rho))$ *and let* $A(k, \rho) = \{x \in K(\rho) \mid u(x) \geqq k\}$. *If there exist two constants* k_0 *and* θ *with* $0 \leqq \theta < 1$ *such that* meas $A(k_0, \rho) < \theta$ meas $K(\rho)$, *then the following inequality holds:*

$$(5.1) \qquad (h - k)[\text{meas } A(h, \rho)]^{1-1/n}\, C \int_{[A(k,\rho)-A(h,\rho)]} |u_x(t)|\, dt$$

for $h > k > k_0$, *where* C *is a positive constant depending only on* θ *and* n.

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

LEMMA 5.2. *Suppose that* meas $A_{k,\rho}(t_0) \leqq \frac{1}{2}\kappa_n\rho^n$, *where* $\kappa_n = $ meas $K(1)$. *Then for any* β *in* $\left(\frac{1}{\sqrt{2}}, 1\right)$, *there exist positive numbers* a *and* θ $(0 \leqq \theta < 1)$ *depending only on* β *such that if*

$$k \geqq \max_{\substack{x \in \partial\Omega \cap K(\rho) \\ t \in [t_0, t_0 + a\rho^\alpha]}} u(x,t) \quad \text{and} \quad 2M_1 \geqq H = \max_{\substack{x \in A_{k,\rho}(t) \\ t \in [t_0, t_0 + a\rho^\alpha]}} (u(x,t) - k) > \rho^\gamma,$$

where $\gamma = 1 - \left(\frac{n}{\alpha p} + \frac{\alpha n - 2n + 2\alpha}{2\alpha q}\right)$ *when* $\alpha < n$ *and* $\gamma = 1 - \left(\frac{1}{p} + \frac{\alpha}{2q}\right)$ *when* $\alpha \geqq n$, *then*

$$\text{meas } A_{k+\beta H,\rho}(t) < \theta \text{ meas } K(\rho)$$

for $t \in [t_0, t_0 + a\rho^\alpha]$.

Proof. We choose $\zeta(x)$ as follows:

$$\zeta(x; \rho, \rho - \sigma\rho) = \begin{cases} 1 & \text{for } x \in K(\rho - \sigma\rho), \\ \dfrac{\rho - |x - x_0|}{\sigma\rho} & \text{for } x \in K(\rho) - K(\rho - \sigma\rho), \\ 0 & \text{outside of } K(\rho), \end{cases}$$

where σ is any number in the interval $(0, 1)$. For such a ζ and $t \in [t_0, t_0 + a\rho^\alpha]$, it follows from the inequality (3.27) that

$$\begin{aligned}
&(\beta H)^2(\text{meas } A_{k+\beta H,\rho-\sigma\rho}(t)) \\
&\leqq \int_{A_{k,\rho-\sigma\rho}(t)} (u-k)^2 dx \leqq \int_{A_{k,\rho}(t)} (u^{(k)})^2 \zeta^\alpha dx \\
&\leqq \mathscr{C}\left\{\int_{t_0}^t \int_{A_{k,\rho}(t)} |u^{(k)}|^\alpha |\zeta_x|^\alpha \, dxdt + \max_t \left(\int_{A_{k,\rho}(t)} |u^{(k)}|^2 \zeta^2 dx\right)^{\alpha/2} (t - t_0)^{\alpha\nu/(\alpha-1)} \right. \\
&\quad \left. + \left(\int_{t_0}^t (\text{meas } A_{k,\rho}(t))^{q'/p'} dt\right)^{1/q'}\right\} + \int_{A_{k,\rho}(t_0)} (u^{(k)}(x, t_0))^2 \zeta^\alpha dx .
\end{aligned}$$

Since, from the hypotheses,

$$\int_{t_0}^t \int_{A_{k,\rho}(t)} (u^{(k)})^\alpha |\zeta_x|^\alpha \, dx \leqq \frac{H^\alpha}{(\sigma\rho)^\alpha}(t - t_0)\kappa_n\rho^n ,$$

$$(t - t_0)^{\alpha\nu/(\alpha-1)} \|u^{(k)}\zeta\|_{2,\infty}^\alpha \leqq H^\alpha (t - t_0)^{\alpha\nu/(\alpha-1)} \kappa_n^{\alpha/2} \rho^{\alpha n/2} \leqq H^\alpha \kappa_n^{\alpha/2} \rho^n a^{\alpha\nu/(\alpha-1)} ,$$

$$\begin{aligned}
\left(\int_{t_0}^t (\text{meas } A_{k,\rho}(t))^{q'/p'} dt\right)^{1/q'} &\leqq (t - t_0)^{1/q'}(\kappa_n\rho^n)^{1/p'} \\
&\leqq (t - t_0)^{1/q'} H^\alpha \kappa_n^{1/p'} \rho^{n/p' - \alpha\gamma} \leqq a^{1/q'} H^\alpha \kappa_n^{1/p'} \rho^n
\end{aligned}$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

and

$$\int_{A_{k,\rho}(t_0)} \{u^{(k)}(x, t_0)\}^2 \zeta^\alpha dx \leqq \frac{1}{2}\kappa_n \rho^n H^2 ,$$

it follows that

$$\text{meas } A_{k+\beta H, \rho-\sigma\rho}(t)$$

$$\leqq \frac{\mathscr{C}}{\beta^2} H^{\alpha-2}\left\{\frac{a}{\sigma^\alpha} + a^{1/q'}\kappa_n^{1/p'-1} + a^{\alpha\nu/(\alpha-1)}\kappa_n^{-1}\right\}\kappa_n\rho^n + \frac{1}{2\beta^2}\kappa_n\rho^n . \tag{5.2}$$

Now we take $\beta \in \left(\frac{1}{\sqrt{2}}, 1\right)$ and choose θ $(0 \leqq \theta < 1)$ and $\sigma > 0$ such that the inequality

$$\frac{1}{2\beta^2} < \theta(1-\sigma)^n$$

holds. Then if we choose the number a sufficiently small, the right hand side of (5.2) is smaller than $\theta\kappa_n(1-\sigma)^n\rho^n$. Hence we obtain

$$\text{meas } A_{k+\beta H, \rho-\sigma\rho}(t) \leqq \theta \text{ meas } K((1-\sigma)\rho) \qquad \text{for } t \in [t_0, t_0 + a\rho^\alpha] , \tag{5.3}$$

from which we have the lemma.

In what follows, we take $\beta = \frac{3}{4}$.

We introduce standard cylinders $Q(r\rho)$ whose bases are the ball $K(r\rho)$ with heights equal to $a(r\rho)^\alpha$, where a is a positive constant chosen in Lemma 5.2, that is,

$$Q(r\rho) = K(r\rho) \times [t_1 - a(r\rho)^\alpha, t_1] , \qquad t_1 > a(r\rho)^\alpha .$$

Write

$$\mu_1 = \max_{Q(8\rho)} u , \quad \mu_2 = \min_{Q(8\rho)} u \quad \text{and} \quad \omega = \mu_1 - \mu_2 .$$

LEMMA 5.3. *For any $\theta_1 > 0$ and for any $\rho < 1$, there exists an $s(\theta_1) > 0$ such that for any cylinder $Q(8\rho) \subset Q$, either*

$$\omega < 2^s \rho^\gamma \tag{5.4}$$

where $\gamma = 1 - \left(\frac{n}{\alpha p} + \frac{\alpha n - 2n + 2\alpha}{2\alpha q}\right)$ *when* $n > \alpha$, *and* $\gamma = 1 - \left(\frac{1}{p} + \frac{\alpha}{2q}\right)$ *when* $\alpha \geqq n$, *or*

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

(5.5) $$\int_{t_1-a(4\rho)^\alpha}^{t_1} \operatorname{meas} A_{\mu_1-(\omega/2^{s+1}),4\rho}(t)dt \leqq \theta_1\rho^{n+\alpha},$$

or

(5.6) $$\int_{t_1-a(4\rho)^\alpha}^{t_1} \operatorname{meas} B_{\mu_2+(\omega/2^{s+1}),4\rho}(t)dt \leqq \theta_1\rho^{n+\alpha}.$$

Proof. Let r be an integer >2. Since $\mu_2 + \frac{\omega}{2^r} < \mu_1 - \frac{\omega}{2^r}$, it is obvious that at least one of the following inequalities holds:

$$\operatorname{meas} A_{\mu_1-(\omega/2^r),4\rho}(t_1 - a(4\rho)^\alpha) \leqq \frac{1}{2}\kappa_n(4\rho)^n$$

and

$$\operatorname{meas} B_{\mu_2+(\omega/2^r),4\rho}(t_1 - a(4\rho)^\alpha) \leqq \frac{1}{2}\kappa_n(4\rho)^n.$$

Suppose for example that the first one holds. We shall prove that then (5.5) will be satisfied if $\omega > 2^s\rho^r$.

From Lemma 5.2, for all $t \in [t_1 - a(4\rho)^\alpha, t_1]$

$$\operatorname{meas} A_{\mu_1-(\omega/2^{r+2}),4\rho}(t) < \theta\kappa_n(4\rho)^n,$$

so that, for such a t, Lemma 5.1 may be applied on account of the fact that

$$h > k \geqq u_1 - \frac{\omega}{2^{r+2}}.$$

We denote by $D_{\lambda_\ell}(t)$ the set

$$A_{\mu_1-(\omega/2^\ell),4\rho}(t) - A_{\mu_1-(\omega/2^{\ell+1}),4\rho}(t), \qquad r + 2 \leqq \ell \leqq s.$$

Using Lemma 5.1, we have

$$\frac{\omega}{\kappa_n^{1/n}(4\rho)2^{\ell+1}} \operatorname{meas} A_{\mu_1-(\omega/2^{\ell+1}),4\rho}(t) \leqq \frac{\omega}{2^{\rho+1}}[\operatorname{meas} A_{\mu_1-(\omega/2^{\ell+1}),4\rho}(t)]^{1-1/n}$$
$$\leqq \mathscr{C}\int_{D_{\lambda_\ell}(t)} |u_x|\, dx.$$

From this we have, putting $t_0 = t_1 - a(4\rho)^\alpha$,

(5.7) $$\frac{\omega^\alpha}{2^{\alpha(\ell+3)}\kappa_n^{\alpha/n}\rho^\alpha}\left\{\int_{t_0}^{t_1} (\operatorname{meas} A_{\mu_1-(\omega/2^{\ell+1}),4\rho}(t))dt\right\}^\alpha$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\leqq \mathscr{C}^{\alpha}\Big(\int_{t_0}^{t_1}\int_{D_{\lambda\ell}} |u_x|^{\alpha}\, dxdt\Big)\Big(\int_{t_0}^{t_1} \text{meas}\, D_{\lambda\ell}(t)dt\Big)^{\alpha-1} .$$

On the other hand, if we take $\zeta(x) = \zeta(x; 8\rho, 4\rho)$ in (3.28) with $t_0 = t_1 - a(4\rho)^{\alpha}$, then we obtain

$$\int_{t_0}^{t_1}\int_{D_{\lambda\ell}} |u_x|^{\alpha}\, dxdt \leqq \int_{t_0}^{t_1}\int_{A_{\mu_1-(\omega/2^{\ell}),8\rho}(t)} |u_x|^{\alpha}\zeta^{\alpha} dxdt$$

(5.8)
$$\leqq \mathscr{C}\Big\{a(4\rho)^{\alpha}\Big[\frac{\omega^{\alpha}}{2^{\alpha\ell}\rho^{\alpha}}\kappa_n(8\rho)^n\Big] + (a4^{\alpha}\rho^{\alpha})^{\alpha\nu/(\alpha-1)}\frac{\omega^{\alpha}}{2^{\alpha\ell}}(8\rho)^{n\alpha/2}\kappa_n^{n/2}$$
$$+ (a4^{\alpha}\rho^{\alpha})^{1/q'}(8^n\rho^n\kappa_n)^{1/p'}\Big\}$$
$$\leqq C_1\omega^{\alpha}\{\rho^n + \rho^{\alpha(\alpha\nu/\alpha-1)+\alpha n/2} + \rho^{\alpha-(\alpha/q)+n-(n/q)-\alpha\gamma}\} \leqq C_1\omega^{\alpha}\rho^n ,$$

where C_1 is a positive constant depending only on a, κ_n and $\mathscr{C}$ in (3.28), and we used the fact that

$$\alpha\Big(\frac{\alpha\nu}{\alpha-1}\Big) + \frac{n\alpha}{2} \geqq n , \qquad \alpha - \frac{\alpha}{q} + n - \frac{n}{p} - \alpha\gamma \geqq n .$$

Therefore the inequalities (5.7) and (5.8) yield

(5.9)
$$\Big(\int_{t_0}^{t_1} \text{meas}\, A_{\mu_1-(\omega/2^{s+1}),4\rho}(t)dt\Big)^{\alpha/\alpha-1} \leqq C_2(\rho^{n+\alpha})^{\alpha/(\alpha-1)}\int_{t_0}^{t_1} \text{meas}\, D_{\lambda\ell}(t)dt .$$

We sum up these inequalities with respect to ℓ from $r+2$ to s and obtain

$$(s-r-1)\Big(\int_{t_0}^{t_1} \text{mean}\, A_{\mu_1-(\omega/2^{s+1}),4\rho}(t)dt\Big)^{\alpha/(\alpha-1)}$$
$$\leqq C_2(\rho^{n+\alpha})^{1/(\alpha-1)}\int_{t_0}^{t_1} K(4\rho)dt = C_2 2^{2n+2\alpha}a(\rho^{n+\alpha})^{1/(\alpha-1)}\rho^{n+\alpha} = C_3(\rho^{n+\alpha})^{\alpha/(\alpha-1)} .$$

Hence we have

(5.10)
$$\int_{t_0}^{t_1} \text{meas}\, A_{\mu_1-(\omega/2^{s+1}),4\rho}(t)dt \leqq \Big(\frac{C_3}{s-r+1}\Big)^{(\alpha-1)/\alpha}\rho^{n+\alpha} .$$

Therefore we have the inequality (5.5) by choosing s such that

$$\Big(\frac{C_3}{s-r+1}\Big)^{(\alpha-1)/\alpha} = \theta_1 .$$

LEMMA 5.3′. *Suppose that the oscillation $\omega_1 = \text{osc}\,\{u, Q(8\rho)\}$ of u on the intersection $\Gamma(8\rho)$ of the cylinder $Q(8\rho)$ with Γ satisfies $\omega_1 \leqq L\rho^{\varepsilon}$, for some positive number ε.*

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

Then for any $\theta_1 > 0$ *one can find an* $s(\theta_1) > 0$ *such that for any pair of coaxial cylinders* $Q(4\rho)$ *and* $Q(8\rho)$ *satisfying the condition*

$$\text{meas}\,[K(4\rho) - K(4\rho) \cap \Omega] \geqq b_1\rho^n\ ,$$

at least one of the three inequalities $\omega = \text{osc}\,\{u, Q(8\rho)\} < 2^s\rho^{\varepsilon_1}$ $(\varepsilon_1 = \min \gamma, \varepsilon)$, (5.5) *and* (5.6) *holds.*

The proof is analogous to the proof of Lemma 5.3, so we omit it here.

LEMMA 5.4. *There exists a* $\theta_2 > 0$ *such that if*

$$\max_{t\in[t_1 - a(2\rho)^\alpha, t_1]} \text{meas}\, A_{k,2\rho}(t) < \theta_2\rho^n \quad in \quad Q(2\rho)$$

and if

$$k \geqq \max_{\Gamma(2\rho)} u(x,t)\ , \qquad H = \max_{Q(2\rho)} (u - k) > \rho^\gamma\ ,$$

then

$$\text{meas}\, A_{k+H/2,\rho}(t) = 0\ , \qquad t \in [t_1 - a\rho^\alpha, t_1]\ .$$

Proof. We introduce the notation

$$k_h = k + \frac{H}{2} - \frac{H}{2^{h+1}}\ , \qquad t_h = t_1 - a\rho^\alpha - \frac{a\rho^\alpha}{2^h}\ , \qquad \rho_h = \rho + \frac{\rho}{2^h}\ ,$$

$$\mu_h = \max_{t\in[t_h, t_1]} (\text{meas}\, A_{k_h,\rho_h}(t))\ , \qquad \zeta_h = \zeta(x\,;\rho_h, \rho_{h+1})\ , \qquad (h = 0, 1, 2, \cdots)$$

Evidently, for any h.

$$\begin{aligned}(k_{h+1} - k_h)^\alpha\, \text{meas}\, A_{k_{h+1},\rho_{h+1}}(t) &\leqq \int_{A_{k_h,\rho_{h+1}}(t)} (u - k_h)^\alpha dx \\ &\leqq \int_{A_{k_h,\rho_h}(t)} (u^{(k_h)})^\alpha \zeta_h^\alpha dx\ .\end{aligned}$$

Integrating by t and using Lemma 2.4 and (3.28) we have

$$\begin{aligned}(k_{h+1} - k_h)^\alpha \int_{t_h}^t \text{meas}\, A_{k_{h+1},\rho_{h+1}}(t) &\leqq \int_{t_h}^t \int_{A_{k_h,\rho_h}(t)} (u^{(k_h)})^\alpha \zeta_h^\alpha dxdt \\ &\leqq K\Big(\int_{t_h}^t \int_{A_{k_h,\rho_h}(t)} (|u_x^{(k_h)}|^\alpha \zeta_h^\alpha + |u^{(k_h)}|^\alpha\, |(\zeta_h)_x|^\alpha) dxdt\Big)\mu_h^{\alpha/n} \\ &\leqq C_1\Big\{\frac{t - t_h}{(\rho_h - \rho_{h+1})^\alpha} H^\alpha \mu_h + H^\alpha \mu_h^{\alpha/2} + H^\alpha \frac{(t - t_h)^{1/q'} \mu_h^{1/p'}}{\rho^{\alpha\gamma}}\Big\}\mu_h^{\alpha/n}\end{aligned}$$

for any $t > t_h$. Choose $t = t_{h+1}$. Then we obtain

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\mu_{h+1} \leqq \frac{C_1}{(k_{h+1}-k_h)^\alpha}\left\{\frac{\mu_h^{1+\alpha/n}}{(t_{h+1}-t_h)} + \frac{\mu_h^{\alpha/2+\alpha/n}}{(t_{h+1}-t_h)} + \frac{\mu_h^{1+\alpha/n-1/p}}{\rho^{\alpha r}(t_{h+1}-t_h)^{1/q}}\right\},$$

from which, taking account of the definition of k_h, ρ_h, t_h we arrive at the inequality

$$y_{h+1} \leqq C_2 2^{\alpha h} y_h^{1+\varepsilon}$$

where $\varepsilon = \dfrac{\alpha}{n} - \dfrac{1}{p} > 0, y_h = \dfrac{\mu_h}{\rho^n}$ and C_2 is a positive constant depending only on $\mathscr{C}$ in (3.28).

Now we choose θ_2 such as

(5.11) $$\theta_2 \leqq \frac{1}{C_2 2^{2\alpha/\varepsilon}}.$$

Then if $y_0 \leqq \theta_2$, we have

$$y_h \leqq \theta_2 2^{-\alpha h/\varepsilon}.$$

Taking such a θ_2 and letting h tend to $+\infty$, we have that $y_h \to 0$, i.e., that

$$\operatorname{meas} A_{k+H/2,\rho}(t) = 0 \qquad \text{for} \qquad t \in [t_1 - a\rho^\alpha, t_1].$$

In what follows we fix $\theta_2\,(1 > \theta_2 > 0)$ satisfying condition (5.11) and a sufficiently small number ρ_0 such that

$$\mathscr{C}(2M_1)^{\alpha-2} a^{\alpha\nu/(\alpha-1)} (4\rho_0)^{\alpha 2\nu/(\alpha-1)+(n\alpha/2)} \rho_0^{-n} = \frac{\theta_2}{2},$$

where $\mathscr{C}$ is a positive constant in (3.27) of (3.28).

LEMMA 5.5. *For $\theta_2 > 0$, there exists a $\theta_1 > 0$ such that if*

$$k > \max_{\Gamma(4\rho)} u(x,t), \qquad H = \max_{Q(4\rho)} (u-k) > \rho^r, \qquad \rho \leqq \rho_0,$$

then inequality

(5.12) $$\int_{t_1-a(4\rho)^\alpha}^{t_1} \operatorname{meas} A_{k,4\rho}(t)\,dt < \theta_1 \rho^{n+\alpha}$$

implies

(5.13) $$\operatorname{meas} A_{k+H/2,2\rho}(t) \leqq \theta_2 \rho^n, \qquad t \in [t_1 - a(2\rho)^\alpha, t_1].$$

Proof. Put $\zeta = \zeta(x; 4\rho, 2\rho)$. Then we have from (3.27)

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

$$\left(\frac{H}{2}\right)^2 \text{meas } A_{k+H/2,\rho}(t) \leqq \mathscr{C}\left\{\frac{H^\alpha}{\rho^\alpha}\int_\tau^t \text{meas } A_{k,4\rho}(t)dt + (t-\tau)^{\alpha\nu/(\alpha-1)}H^\alpha\left(\max_t \text{meas } A_{k,4\rho}(t)\right)^{\alpha/2} + \left(\int_\tau^t (\text{meas } A_{k,4\rho}(t))^{q'/p'}dt\right)^{1/q'}\right\} + H^2 \text{ meas } A_{k,4\rho}(\tau)\,, \qquad t_1 - a(4\rho)^\alpha \leqq \tau \leqq t \leqq t_1\,. \tag{5.14}$$

From (5.12), it is clear that

$$\frac{1}{\rho^\alpha}\int_\tau^t \text{meas } A_{k,4\rho}(t)dt \leqq \theta_1\rho^n\,. \tag{5.15}$$

Since $t - \tau < a(4\rho)^\alpha$ and $\rho \leqq \rho_0$, it holds that

$$(t-\tau)^{\alpha\nu/(\alpha-1)}\left(\max_t \text{meas } A_{k,4\rho}(t)\right)^{\alpha/2} \leqq a^{\alpha\nu/(\alpha-1)}(4\rho)^{\alpha\nu/(\alpha-1)+\alpha n/2} \leqq \frac{1}{2}\theta_2\rho^n(\mathscr{C}(2M)^{\alpha-2})^{-1}\,. \tag{5.16}$$

If $q' \geqq p'$, then

$$\left(\int_\tau^t (\text{meas } A_{k,4\rho}(t))^{q'/p'}dt\right)^{1/q'} \leqq \left(\int_\tau^t \text{meas } A_{k,4\rho}(t)dt\right)^{1/q'}(4\rho)^{n/p'-n/q'} \leqq 4^{n/p'-n/q'}\theta_1^{1/q'}\rho^{n+\alpha\gamma'}\,,$$

where $\gamma' = 1 - \left(\dfrac{n}{\alpha p} + \dfrac{1}{q}\right)$.

On the other hand, if $p' > q'$, then the Hölder's inequality yields

$$\left(\int_\tau^t (\text{meas } A_{k,4\rho}(t))^{q'/p'}dt\right)^{1/q'} \leqq \left(\int_\tau^t (\text{meas } A_{k,4\rho}(t))dt^{1/p'}(t-\tau)^{1/q'-1/p'}\right. \leqq \{a(4\rho)^\alpha\}^{1/q'-1/p'}\theta_1^{1/p'}\rho^{(n+\alpha)/p'} \leqq (4^\alpha)^{1/q'-1/p'}\theta_1^{1/p'}\rho^{n+\alpha\gamma'}\,.$$

Thus, putting $\theta_1^\gamma = \max(\theta_1^{1/p'}, \theta_1^{1/q'})$ and $C_1 = \max(4^{\alpha(1/q'-1/p')}, 4^{n(1/p'-1/q')})$, we obtain

$$\left(\int_\tau^t (\text{meas } A_{k,4\rho}(t))^{q'/p'}dt\right)^{1/q'} \leqq C_1\theta_1^\gamma\rho^{n+\alpha\gamma'}\,. \tag{5.17}$$

Finally we choose τ in the interval $[t_1 - a(4\rho)^\alpha, t_1 - a(2\rho)^\alpha]$ such that

$$\text{meas } A_{k,4\rho}(\tau) \leqq \frac{\theta_1\rho^n}{(4^\alpha - 2^\alpha)a}\,. \tag{5.18}$$

Then, from (5.13) ~ (5.18) we have

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

(5.19) $$\text{meas } A_{k+H/2,2\rho}(t) \leqq \mathscr{C}(2M)^{\alpha-2}\left[\theta_1 + C_1\theta_1^r + \frac{\theta_1}{(4^\alpha - 2^\alpha)a} + \frac{\theta_2}{2(2M)^{\alpha-2}}\right]\rho^n .$$

From (5.19), we obtain the lemma, while θ_1 satisfies

$$\mathscr{C}(2M)^{\alpha-1}\left[\theta_1 + C_1\theta_1^r + \frac{\theta_1}{(4^\alpha - 2^\alpha)a}\right] \leqq \frac{1}{2}\theta_2 .$$

We put $\mu_1(\rho) = \max_{Q(\rho)} u$, $\mu_2(\rho) = \min_{Q(\rho)} u$ and $\omega(\rho) = \mu_1(\rho) - \mu_2(\rho)$. Then the following Lemma was proved by G. Stampacchia [5]:

LEMMA 5.6. *If* $\omega(\rho) \leqq \eta\omega(8\rho)$ *with* $0 < \eta < 1$, *then there exist a constant* λ *in interval* $(0, 1)$ *and positive constant* K *such that*

$$\omega(\rho) \leqq K\rho^\lambda .$$

Now we can prove the main theorem:

THEOREM 5.1. *A weak solution* u *of the problem* (1.1), (1.2) *is Hölder continuous in* Q.

Proof. Let (x_0, t_1) be any point of Q and choose $\rho_0 > 0$ so small that $Q(8\rho_0)$ is contained in Q, where $Q(8\rho_0) = K(8\rho_0) \times (t_1 - a(8\rho_0)^\alpha, t_1]$ and $K(8\rho_0) = \{x \in \Omega \mid |x - x_0| < 8\rho_0\}$.

First we choose θ_2 as in Lemma 5.4 and we choose θ_1 as in Lemma 5.5. Then we take $s(\theta_1)$ as in Lemma 5.3.

Now suppose that $\omega(8\rho) \geqq 2^{s+2}\rho^r$. Then either the inequality (5.5) or (5.6) in Lemma 5.3 holds. If the inequality (5.5) is valid, then from Lemma 5.5, we have

$$\text{meas } A_{\mu_1-\omega/2^{s+2},2\rho}(t) \leqq \theta_2\rho^n \qquad \text{for} \qquad t \in [t_1 - a(2\rho)^\alpha, t_1] .$$

Therefore Lemma 5.4 gives

$$u(x, t) \leqq \theta_1 - \frac{\omega}{2^{s+3}} \qquad \text{in } Q(\rho) ,$$

so that

(5.20) $$\omega(\rho) \leqq \left(1 - \frac{1}{2^{s+3}}\right)\omega(8\rho) .$$

This and Lemma 5.6 imply

$$\omega(\rho) \leqq K\rho^\lambda :$$

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press

If the inequality (5.5) does not hold, then (5.6) is valid and, considering $-u$ instead of u, we have (5.20) by the similar argument to the above.

THEOREM 5.2. *Let u be a weak solution of the problem* (1.1), (1.2). *If the boundary value $\psi(x,t)$ belongs to the class $C^{\varepsilon,\varepsilon/2}(\partial\Omega)$, then u is Hölder continuous on $\bar{Q} = \bar{\Omega} \times (0,T]$.*

The proof is analogous to the proof of the preceding theorem, with the sole difference that Lemma 5.3′ is used instead of Lemma 5.3.

REFERENCES

[1] D. G. Aronson and J. Serrin: Local behavior of solutions of quasi-linear parabolic equations, Archive for Rational Mechanics and Analysis, **25** (1967), 81–122.

[2] D. G. Aronson and J. Serrin: A maximum principle for non-linear parabolic equations., Annali Scuola Normale Superiore di Pisa, **21** (1967), 291–305.

[3] O. A. Ladyzenskaya and N. N. Ural'ceva: A boundary value problem for linear and quasi-linear parabolic equations, Dokl. Akad. Nauk. SSSR, **139** (1964), 544–547.

[4] J. Serrin: Local behavior of solutions of quasi-linear equations, Acta Math., **111** (1964), 247–302.

[5] G. Stampacchia: Le problème de Dirichlet pour les équations elliptiques de second ordre à coéfficients discontinus, Ann. Inst. Fourier, **15** (1965), 189–258.

Aichi University of Education

https://doi.org/10.1017/S0027763000017207 Published online by Cambridge University Press