A NOTE ON COMPACTIFICATIONS AND SEMI-NORMAL SPACES

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Recently Orrin Frink (see [2]) gave a neat internal characterization of Tychonoff or completely regular T_1 spaces. This characterization was given in terms of the notion of a *normal base* for the closed sets of a space X. A normal base $\mathscr{Z}$ for the closed sets of a space X is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members of $\mathscr{Z}$. In a normal space the ring of closed sets is a normal base.

To obtain the relationship between a normal base for a space X and Tychonoff spaces Frink considered a Hausdorff compactification of X. He showed that if X has a normal base, then the Wallman space $\omega(\mathscr{Z})$ consisting of the $\mathscr{Z}$-ultrafilters, is a Hausdorff compactification of X. It follows that X is Tychonoff. Conversely, if X is Tychonoff then the zero sets of real continuous functions over X form a normal base. Thus a space is Tychonoff if and only if it has a normal base.

By choosing different normal bases $\mathscr{Z}$ for a non-compact space X, different Hausdorff compactifications of X may be obtained in the form of Wallman spaces $\omega(\mathscr{Z})$. Frink raised the question as to whether every Hausdorff compactification may be obtained in this way. He showed that the Stone-Cech compactification always is a "Wallman-type" compactification.

Olav Njastad ([4]) gave sufficient conditions for a compactification to be of the *Wallman-type*. These were stated in terms of the embedding of the space into the compactification. He then used them to show that certain classes of compactifications are of the Wallman-type. In particular he showed that this is the case for the *Freudenthal compactification* and related compactifications (see [1]) and the *bounding system compactifications* of Gould ([3]). He also showed that a compactification is a Wallman-type compactification if and only if the corresponding (unique) proximity determined by the compactification has a productive base consisting of closed sets. This relates Wallman-type compactifications to the proximity aspects of the theory of compactifications and, in particular, to the *Smirnov compactification*.

In this note we give necessary and sufficient conditions for a Hausdorff compactification to be a Wallman-type compactification. These are given in terms of conditions imposed on the normal base $\mathscr{Z}$. These theorems will show, in an easy manner, that several compactifications are of the Wallman-type.

DEFINITIONS. A base $\mathscr{Z}$ for the closed sets of a T_1 space X is said to be *disjunctive* if, given any closed set F and any point x not in F, there exists a closed set A of $\mathscr{Z}$ which contains x and is disjoint from F. The base is said to be *separating* if any two disjoint members A and B of $\mathscr{Z}$ are subsets respectively of disjoint complements $X-C$ and $X-D$ of members C and D of $\mathscr{Z}$ (that is, $A \subset X-C$, $B \subset X-D$, and $(X-C) \cap (X-D) = \emptyset$).

A family of sets is called a *ring of sets* if it is closed under finite unions and intersections.

A base $\mathscr{Z}$ for the closed sets of a T_1 space X is a *normal base* if it is a disjunctive ring of closed sets that is also separating.

A proper subset of a normal base $\mathscr{Z}$ is called a *$\mathscr{Z}$-filter* if it is closed under finite intersections and contains every superset in $\mathscr{Z}$ of each of its members. We also assume that no $\mathscr{Z}$-filter contains the empty set. A *$\mathscr{Z}$-ultrafilter* is a maximal $\mathscr{Z}$-filter.

If $\mathscr{Z}$ is a normal base for X, the Wallman space $\omega(\mathscr{Z})$ is obtained in the following way. The points of $\omega(\mathscr{Z})$ are the $\mathscr{Z}$-ultrafilters of X. For each A in $\mathscr{Z}$ we define the set A^* to be the family of all $\mathscr{Z}$-ultrafilters having A as a member. The collection of sets A^* for A in $\mathscr{Z}$, is taken as a base for the closed sets of $\omega(Z)$. The space $\omega(\mathscr{Z})$ is a compact Hausdorff space. There is a natural embedding h of X into $\omega(\mathscr{Z})$ where $h(x)$ is the $\mathscr{Z}$-ultrafilter consisting of all $\mathscr{Z}$-sets that contain the element x. Equivalently we could take as a base for the open sets of $\omega(\mathscr{Z})$ the family of all sets U^* consisting of all $\mathscr{Z}$-ultrafilters having at least one subset of U as a member, where the complement of U is in $\mathscr{Z}$.

If A is a subset of X, we use $\mathrm{cl}_X A$ to denote the closure of A in X. When there is no chance of confusion, we write $\mathrm{cl}\, A$.

We first state three lemmas. The first two lemmas give some properties of a normal base $\mathscr{Z}$ on X and its corresponding Wallman space $\omega(\mathscr{Z})$. Then we state a characterization of a $\mathscr{Z}$-ultrafilter.

LEMMA 1. *If $\mathscr{Z}$ is a normal base for X, then*

(1) $(A \cap B)^* = A^* \cap B^*$ *for all A, B in $\mathscr{Z}$,*

(2) $h(A) = h(X) \cap A^*$ *for all A in $\mathscr{Z}$,*

(3) $\mathrm{cl}\, h(A) = A^*$.

PROOF. We omit the proofs of (1) and (2) which follow from the construction of $\omega(\mathscr{Z})$.

Since the collection of A^*, A in $\mathscr{Z}$, is a base for the closed sets in $\omega(\mathscr{Z})$, it follows from (2) that $\operatorname{cl} h(A)$ is included in A^*. If $\mathscr{F}$ is any member of A^* and U^* is any open set containing it then A is in $\mathscr{F}$ and there is a Z in $\mathscr{F}$ such that Z is included in U. Hence $A \cap Z$ is in $\mathscr{F}$ and we may choose a point x in X from $A \cap Z$. Then $h(x)$ is in $h(A)$ and also in U^* since Z is in $h(x)$ and Z is included in U. It follows that $\mathscr{F}$ is in $\operatorname{cl} h(A)$. This completes the proof.

LEMMA 2. *Let $\mathscr{Z}$ be a normal base for X and h the natural embedding of X into $\omega(\mathscr{Z})$. If $\mathscr{F}$ is any point of $\omega(\mathscr{Z})$ and if G^* is any open set containing it, then there is a Z in $\mathscr{Z}$ such that $\operatorname{cl} h(Z)$ is a neighbourhood of F and $\operatorname{cl} h(Z)$ is included in G^*.*

PROOF. Since $\omega(\mathscr{Z})$ is a compact Hausdorff space we can separate $\mathscr{F}$ and the complement of G^* by disjoint open sets V_1^* and V_2^*, each of which is a finite union of basic open sets. However, these basic open sets are just complements of basic closed sets Z^* where Z is in $\mathscr{Z}$. Let V_2^* be the finite union of basic open sets $\omega(\mathscr{Z})-Z_i^*$ that covers $\omega(\mathscr{Z})-G^*$. Since $\mathscr{Z}$ is a ring, Lemma 1.1 implies that V_2^* is just a set $\omega(\mathscr{Z})-Z^*$ for some Z in $\mathscr{Z}$, and Lemma 1.2 implies that this $Z^* = \operatorname{cl} h(Z)$. Hence $\mathscr{F}$ is in $V_1^* \subset \operatorname{cl} h(Z) \subset G^*$. It follows that $\operatorname{cl} h(Z)$ is a neighbourhood of $\mathscr{F}$.

LEMMA 3. *Suppose that $\mathscr{Z}$ is a normal base for X and that $\mathscr{F}$ is a $\mathscr{Z}$-filter on X. Then $\mathscr{F}$ is a $\mathscr{Z}$-ultrafilter if and only if for each Z in $\mathscr{Z}$ either Z is in $\mathscr{F}$ or there is an A in $\mathscr{F}$ such that A is included in the complement of Z.*

PROOF. Let $\mathscr{F}$ be a $\mathscr{Z}$-ultrafilter and let Z be any member of $\mathscr{Z}$. If for each A in $\mathscr{F}$, A is not included in the complement of Z, then Z meets each A. The $\mathscr{Z}$-ultrafilter $\mathscr{F}$ must then be equal to the $\mathscr{Z}$-filter generated by $\mathscr{F}$ and Z. Consequently Z must be in $\mathscr{F}$.

Conversely, if the conditions are satisfied, let $\mathscr{G}$ be a $\mathscr{Z}$-filter properly containing $\mathscr{F}$. If the $\mathscr{Z}$-set Z is in $\mathscr{G}$ and not in $\mathscr{F}$, then there is an A in $\mathscr{F}$, and therefore in $\mathscr{G}$, such that A is included in the complement of Z. Hence $A \cap Z = \emptyset$ and $A \cap Z$ is in $\mathscr{G}$ which is a contradiction.

THEOREM 1. *Let Y be a Hausdorff compactification of a T_1 space X, let g be the embedding of X into Y, and let $\mathscr{Z}$ be a normal base on X that satisfies the following property.*

(P) *For each y in Y and each neighborhood V of y there is a Z in $\mathscr{Z}$ such that $y \in \operatorname{cl} g(Z) \subset V$ and $\operatorname{cl} g(Z)$ is a neighborhood of y. Then there is a (closed) continuous map f of $\omega(\mathscr{Z})$ onto Y such that $f|h(X) = g \circ h^{-1}$.*

Conversely if f is a homeomorphism of $\omega(\mathscr{Z})$ onto Y, then condition P *is satisfied.*

PROOF. For each $\mathscr{F}$ in $\omega(\mathscr{Z})$ let $V(\mathscr{F})$ be the family of all basic open sets U^* of $\mathscr{F}$, and let $B(\mathscr{F})$ be the family of $\mathrm{cl}_Y\, g(U)$ for U^* in $V(\mathscr{F})$. $B(\mathscr{F})$ is a family of closed subsets of Y with the finite intersection property and hence $D = \cap\, B(\mathscr{F})$ is not empty.

Suppose that there are distinct points a and b of Y in D. Since Y is Hausdorff, the condition asserts that we can separate the points a and b by $\mathscr{Z}$-sets A and B whose closures $\mathrm{cl}_Y\, g(A)$ and $\mathrm{cl}_Y\, g(B)$ are disjoint and are neighborhoods of a and b respectively. Thus $A^* \cap B^* = \emptyset$. If U^* is any basic open set containing $\mathscr{F}$, then $\mathrm{cl}\, g(U)$ is in $B(\mathscr{F})$. Hence the intersection of $\mathrm{cl}\, g(A)$ with $g(U)$ and the intersection of $\mathrm{cl}\, g(B)$ with $g(U)$ are both non-empty. It follows that $A \cap U$ and $B \cap U$ are non-empty and therefore so are $A^* \cap U^*$ and $B^* \cap U^*$. Hence $\mathscr{F}$ is in $A^* \cap B^*$ which is a contradiction. Thus D consists of exactly one point $y(\mathscr{F})$.

We can define a map f of $\omega(\mathscr{Z})$ into Y by $f(\mathscr{F}) = y(\mathscr{F})$ for $\mathscr{F}$ in $\omega(\mathscr{Z})$. If $\mathscr{F}$ is in $h(X)$ then $\mathscr{F}$ is of the form $h(p) = \mathscr{F}_p$, the $\mathscr{Z}$-ultrafilter consisting of all $\mathscr{Z}$-sets which contain the point p. The point $g(p)$ is in $\mathrm{cl}\, g(U)$ for each U^* in $V(\mathscr{F}_p)$. Thus $f(\mathscr{F}_p) = g(p) = g(h^{-1}(\mathscr{F}_p))$ and $f|h(X) = g \circ h^{-1}$.

If f is continuous then it must also be a closed map since $\omega(\mathscr{Z})$ is compact and Y is Hausdorff. Let V be any open set in Y that contains a point $f(\mathscr{F})$. Since $B(\mathscr{F})$ is a filter base with a unique cluster point and since Y is compact, $B(\mathscr{F})$ converges to $f(\mathscr{F})$ and it follows that there is a U^* in $V(\mathscr{F})$ such that $\mathrm{cl}\, g(U) \subset V$. If $\mathscr{F}'$ is any $\mathscr{Z}$-ultrafilter in U^* then U^* is in $V(\mathscr{F}')$, so $f(\mathscr{F}')$ is in $\mathrm{cl}\, g(U) \subset V$. Thus $\mathscr{F}$ is in U^*, and $f(U^*) \subset V$, and therefore f is continuous.

Since $f(h(X)) = g(X)$ and since $\mathrm{cl}_Y(g(X)) = Y$, $\mathrm{cl}_Y(f(h(X))) = Y$. But f is a closed continuous map, so $Y = f(\mathrm{cl}_{\omega(\mathscr{Z})}(h(X))) = f(\omega(\mathscr{Z}))$ and f is an onto map.

Conversely suppose that f is a homeomorphism of $\omega(\mathscr{Z})$ onto Y and that $f|h(X) = g \circ h^{-1}$. Using an argument similar to that used to prove Lemma 2, for each y in Y and each open set V in Y containing y, we can obtain a set Z^* such that y is in $f(Z^*) \subset V$, Z in $\mathscr{Z}$. Since $Z^* = \mathrm{cl}\, h(Z)$ by Lemma 1.3, $f(Z^*) = \mathrm{cl}_Y(f(h(Z))) = \mathrm{cl}_Y\, g(Z)$. This completes the proof of the theorem.

THEOREM 2. *Let Y be a Hausdorff compactification of X. Then Y is homeomorphic to a Wallman-type compactification of X if and only if X has a normal base $\mathscr{Z}$ that satisfies*:

(a) $\mathrm{cl}_Y(A \cap B) = \mathrm{cl}_Y A \cap \mathrm{cl}_Y B$ *for all A, B in $\mathscr{Z}$.*

(b) *For each y in Y and each neighborhood V of y there is a Z in $\mathscr{Z}$ such that y is in $\mathrm{cl}_Y Z \subset V$.*

PROOF. The necessity of the conditions for a normal base $\mathscr{Z}$ of X

is immediate. Since, if f is a homeomorphism of Y onto $\omega(\mathscr{Z})$, then $f(\text{cl}_Y Z) = \text{cl}_{\omega(\mathscr{Z})}(f(Z)) = Z^*$ by Lemma 1.3. Then Lemma 1.1 yields condition (a) and Lemma 2 yields condition (b).

Conversely, suppose that $\mathscr{Z}$ is a normal base for X that satisfies conditions (a) and (b), and let $\mathscr{F}_p$ be the collection of all sets Z in $\mathscr{Z}$ such that p is in $\text{cl}_Y Z$. Clearly $\mathscr{F}_p$ cannot contain the empty set and contains every $\mathscr{Z}$-set that is a superset of any member of $\mathscr{F}_p$. That $\mathscr{F}_p$ is a $\mathscr{Z}$-filter follows from condition (a) which says that $\mathscr{F}_p$ is closed under finite intersections. If $\mathscr{F}$ is any $\mathscr{Z}$-filter that properly contains $\mathscr{F}_p$ then there is an A in $\mathscr{F}$ such that p is not in the $\text{cl}_Y A$. By condition (b) there is a Z in $\mathscr{Z}$ such that y is in $\text{cl}_Y Z \subset Y - \text{cl}_Y A$. Then Z is in $\mathscr{F}_p$ and therefore in $\mathscr{F}$. It follows that $A \cap Z$ is empty and is in $\mathscr{F}$ which is a contradiction. Hence $\mathscr{F}_p$ is a $\mathscr{Z}$-ultrafilter.

Now $\mathscr{F}_p$ are all the $\mathscr{Z}$-ultrafilters on X. For if $\mathscr{F}$ is any $\mathscr{Z}$-ultrafilter on X, then the collection of sets consisting of $\text{cl}_Y Z$, for Z in $\mathscr{F}$, is a family of sets closed in Y with the finite intersection property. Since Y is compact, there is a point p which is in $\text{cl}_Y Z$ for each Z in $\mathscr{F}$. It follows that $\mathscr{F} \subset \mathscr{F}_p$ and hence $\mathscr{F}$ must be equal to $\mathscr{F}_p$.

We can now define a map f from Y onto $\omega(\mathscr{Z})$ by $f(p) = \mathscr{F}_p$ for p in Y. Since Y is compact and since $\omega(\mathscr{Z})$ is Hausdorff, it follows that f is a closed map if f is continuous. To see that the mapping is continuous, let U^* be a basic open set containing $f(p) = \mathscr{F}_p$. Then $X - U$ is a member of $\mathscr{Z}$. Since $\mathscr{F}_p$ is in U^* there is an A in $\mathscr{F}_p$ such that $A \subset U$. If p were in the $\text{cl}_Y(X-U)$ then p would be in $\text{cl}_Y(X-U) \cap \text{cl}_Y A = \text{cl}_Y((X-U) \cap A)$. Thus $(X-U) \cap A$ would be in $\mathscr{F}_p$ which is a contradiction since the intersection is empty. Hence p is in the open set G which is the complement in Y of $\text{cl}_Y(X-U)$. We show that $f(G)$ is included in U^*. If g is any point of G then $X-U$ is not in $\mathscr{F}_g$. By Lemma 3 there is a Z in $\mathscr{F}_g$ such that Z is included in U. Consequently $\mathscr{F}_g$ is in U^* and $f(G) \subset U^*$.

The mapping is one-one since condition (b), in conjunction with the fact that Y is Hausdorff, asserts that we can separate any two distinct points a and b of Y by the closures in Y of $\mathscr{Z}$-sets A and B respectively. Thus A is in $\mathscr{F}_a$ and A is not in $\mathscr{F}_b$ by condition (a). It follows that $\mathscr{F}_a$ is not equal to $\mathscr{F}_b$ and f is one-one. This completes the proof.

Theorem 2 remains true if condition (b) is replaced by condition P. In fact the proof would be extremely easy since we could then use Theorem 1. *However the conditions on Theorem 2 give more insight into the nature of* a normal base $\mathscr{Z}$ on X. This we hope will lead to an answer to the question as to whether every Hausdorff compactification of X may be obtained as a Wallman space $\omega(\mathscr{Z})$.

It is interesting to note that in Theorem 1 we defined our map from $\omega(\mathscr{Z})$ into Y, whereas in Theorem 2 we defined it from Y into $\omega(\mathscr{Z})$. For the sake of clarification we should mention that in Theorem 2 we had to know that condition (a) held. Moreover, Theorem 2 gives us a representation for the $\mathscr{Z}$-ultrafilters in $\omega(\mathscr{Z})$.

If Y is the Alexandroff one point compactification of a locally compact Hausdorff space X, then a normal base $\mathscr{Z}$ for X is the collection of zero sets of those continuous functions on X that are constant on the complement of some compact subset of X.[1] That for this $\mathscr{Z}$, $\omega(\mathscr{Z})$ is homeomorphic to Y, follows immediately from our Theorem 2. In fact suppose p is the ideal point and p is in $\text{cl}_Y Z_1 \cap \text{cl}_Y Z_2$ but not in $\text{cl}_Y (Z_1 \cap Z_2)$ where Z_i is the zero set of the function f_i that is constant on the complement of the compact set K_i, $i = 1, 2$. Let k_i be the constant associated with K_i. If k_i is not zero then Z_i is included in the compact set K_i and must therefore be compact. Hence p is in $Z_i = \text{cl}_Y Z_i$ which is included in X and this is a contradiction. If $k_1 = k_2 = 0$ and if p is not in $\text{cl}_Y (Z_1 \cap Z_2)$ then there is an open set G containing p which is disjoint from $Z_1 \cap Z_2$. Since $K_1 \cup K_2$ is compact, its complement in Y intersected with G is an open set containing p. Then there is a point a in $(Y - K_1 \cup K_2) \cap G \cap Z_1$ and hence in $(Y - K_i) \cap X = X - K_i$ which is included in Z_i, $i = 1, 2$. Consequently a is in $G \cap Z_1 \cap Z_2$ which is a contradiction. Thus $\text{cl}_Y (Z_1 \cap Z_2) = \text{cl}_Y Z_1 \cap \text{cl}_Y Z_2$. If G is any open set in Y containing p then there is an open set V such that p is in $V \subset \text{cl}_Y V \subset G$. We can separate $\text{cl}_Y V$ and $X - G$ by a continuous function f which is zero on $\text{cl}_Y V$ and one on G. Since p is the ideal point, $Y - V = X - V$ is a compact subset of X. The restriction $f|X$ of f to X is constant on V. We take Z to be the zero set of $f|X$. If U is any open set containing p, then $U \cap V \cap X$ is not empty; hence p is in $\text{cl}_Y Z$ and $\text{cl}_Y Z$ is included in G.

Using Theorem 2 we can also show that any Hausdorff compactification of X which gives rise to a proximity that has a productive base (see Njastad [4]) can be obtained as a Wallman space $\omega(\mathscr{Z})$. Theorem 2 gives a very simple proof of this fact since we take as our normal base $\mathscr{Z}$ finite unions of members of the productive base for the proximity. It is immediately seen that $\mathscr{Z}$ satisfies the conditions of our Theorem.

In particular, if Y is any Hausdorff compactification of X then there is associated with it a (unique) proximity δ defined as: $A \; \delta \; B$ if and only if $\text{cl}_Y A \cap \text{cl}_Y B \neq \emptyset$ (see [4] and [5]). We say that two subsets of X are *far* if they are not members of the relation δ. A collection $\mathscr{P}$ of subsets of X is a *base for the proximity* δ if every two disjoint sets in $\mathscr{P}$ are far and if every two far sets are contained in subsets of $\mathscr{P}$ which are far.

If a proximity δ has a base $\mathscr{P}$ which is closed under finite intersections (Njastad [4] called such families *productive*) then Njastad has shown that the

[1] This example of a normal base is due to Ky Fan and N. Gottesman [7]. It also appears in a paper 'On Wallman compactifications' by R. M. Brooks.

ring $\mathscr{Z}$ generated by $\mathscr{P}$ is a normal base for X. We now show that $\mathscr{Z}$, which is just finite unions of members of $\mathscr{P}$, satisfies the conditions of Theorem 2.

If Z_1 and Z_2 are members of $\mathscr{Z}$ and if x is in $\text{cl}_Y (Z_1 \cap Z_2)$, then $x \,\delta\, (Z_1 \cap Z_2)$. Hence $x \,\delta\, Z_1$ and $x \,\delta\, Z_2$ which implies that x is in $\text{cl}_Y Z_1 \cap \text{cl}_Y Z_2$. It follows that $\text{cl}_Y (Z_1 \cap Z_2) = \text{cl}_Y Z_1 \cap \text{cl}_Y Z_2$.

If G is any open set in Y containing a point p then there are open sets V_1 and V_2 such that p is in V_1, $Y - G \subset V_2$, and $\text{cl}_Y V_1 \cap \text{cl}_Y V_2 = \emptyset$. Since X is dense in Y, it follows that $\text{cl}_Y (X \cap \text{cl}_Y V_i) = \text{cl}_Y V_i$, $i = 1, 2$. This implies that the sets $X \cap \text{cl}_Y V_i$, $i = 1, 2$, are subsets of X which are far. Since $\mathscr{Z}$ is a base for the proximity δ, let Z_1 and Z_2 be members of $\mathscr{Z}$ such that $X \cap \text{cl}_Y V_i \subset Z_i$ and $\text{cl}_Y Z_1 \cap \text{cl}_Y Z_2 = \emptyset$. Then p is in

$$\text{cl}_Y V_1 \subset \text{cl}_Y Z_1 \subset Y - \text{cl}_Y Z_2 \subset Y - \text{cl}_Y V_2 \subset G.$$

Thus there is a Z in $\mathscr{Z}$ such that p is in $\text{cl}_Y Z \subset G$ and the conditions of Theorem 2 are satisfied.

This can now be applied to the compactifications mentioned in the first part of this note. For example Njastad has shown that the compactification of Fan and Gottesman has a productive base of closed sets for the associated proximity. Our Theorem 2 is then applicable.

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