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# STRUCTURE THEOREM FOR $\mathcal{AN}$-OPERATORS

**G. RAMESH**

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**Abstract**

In this paper we prove a structure theorem for the class of $\mathcal{AN}$-operators between separable, complex Hilbert spaces which is similar to that of the singular value decomposition of a compact operator. Apart from this, we show that a bounded operator is $\mathcal{AN}$ if and only if it is either compact or a sum of a compact operator and scalar multiple of an isometry satisfying some condition. We obtain characterizations of these operators as a consequence of this structure theorem and deduce several properties which are similar to those of compact operators.

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## 1. Introduction

Carvajal and Neves [2] recently introduced a class of operators between Hilbert spaces, which generalizes the space of all compact operators.

Let $H_1, H_2$ be separable, complex Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Then $T$ is said to be *norm-attaining* if there exists a $x_0 \in H_1$ with $\|x_0\| = 1$ such that $T = Tx_0$. The operator $T$ is said to be of class $\mathcal{AN}$ if $T|_M$ (the restriction of $T$ to $M$) is norm attaining for any nonzero closed subspace $M$ of $H_1$. The class $\mathcal{AN}$ contains the class of compact operators and the class of isometries. Norm-attaining operators have received much attention in the literature (for details, see [12] and references therein). The authors in [2] characterized the norm-attaining operators in terms of eigenvalues and extreme points of the numerical range and gave several examples of operators which are in the class $\mathcal{AN}$.

The spectral theorem for compact self-adjoint operators ensures that every such operator is diagonalizable and has a representation in terms of eigenvalues and corresponding eigenvectors. The nonzero spectrum of a compact self-adjoint operator consists of eigenvalues only, with a possible limit point zero. Apart from these operators, operators of the form compact plus identity also have similar properties. But the spectral theorem is not valid in this case. In this paper we discuss representations of such operators as well as of a more general class of operators.

We prove that if $T$ is an $\mathcal{AN}$-operator between complex, separable Hilbert spaces $H_1$ and $H_2$, then:

(1) $Tx = \sum_{n=1}^{\infty} s_n\langle x, \phi_n\rangle\psi_n$ for all $x \in H_1$, where $s_j$ ($j \in \mathbb{N}$) is an eigenvalue of $|T|$ corresponding to the eigenvector $\phi_j \in H_1$ and $\psi_j \in H_2$ is an orthonormal vector, and the series above converges in the strong operator topology of $\mathcal{B}(H_2)$. If $(s_n)$ is infinite, then $s_n$ converges to $m(T)$, the minimum modulus of $T$.

(2) $T \in \mathcal{AN}$ if and only if either $T$ is compact or $T = K + \alpha V$, where $\alpha > 0$, $K$ is a compact operator and $V$ is an isometry such that $K^*K + 2\alpha\mathrm{Re}(V^*K)$ is positive.

We also obtain some properties of these operators analogous to those of compact operators and deduce several consequences.

This paper is organized as follows. In the rest of this section we introduce our terminology and notation. Then we prove that every self-adjoint $\mathcal{AN}$-operator can be represented as a compact self-adjoint operator, except that in our case the convergence is in the strong operator topology. Using this representation theorem, we characterize all $\mathcal{AN}$-operators and obtain some important properties of these operators. Finally, we illustrate our main results with examples.

**1.1. Notations.** Throughout the paper we consider infinite-dimensional separable, complex Hilbert spaces which will be denoted by $H, H_1, H_2, \ldots$. The inner product and the induced norm are denoted by $\langle , \rangle$ and $\|\cdot\|$ respectively. The unit sphere of a closed subspace $M$ of $H$ is denoted by $S_M := \{x \in M : x = 1\}$, and $P_M$ denotes the orthogonal projection $P_M : H \to H$ with range $M$. If $S$ is a subset of $H$, then the closed linear span of $S$ is denoted by $[S]$.

The space of all bounded (compact) operators between $H_1$ and $H_2$ is denoted by $\mathcal{B}(H_1, H_2)$ ($\mathcal{K}(H_1, H_2)$). If $H_1 = H_2 = H$, then $\mathcal{B}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$ are denoted by $\mathcal{B}(H)$ and $\mathcal{K}(H)$, respectively. The set of all norm-attaining operators between $H_1$ and $H_2$ is denoted by $\mathcal{N}(H_1, H_2)$ and $\mathcal{N}(H, H)$ by $\mathcal{N}(H)$. The adjoint of $T \in \mathcal{B}(H_1, H_2)$ is denoted by $T^*$. The null space and the range spaces of $T$ are denoted by $N(T)$ and $R(T)$, respectively.

If $T \in \mathcal{B}(H)$, then $T = (T + T^*)/2 + i\,((T - T^*)/2i)$. The operators $\mathrm{Re}(T) := (T + T^*)/2$ and $\mathrm{Im}(T) := (T - T^*)/2i$ are self-adjoint and called the *real* and the *imaginary* parts of $T$, respectively.

For a positive $T \in \mathcal{B}(H)$, the square root of $T$ is denoted by $T^{1/2}$. For $T \in \mathcal{B}(H)$, $(T^*T)^{1/2}$ is called the *modulus* of $T$ and is denoted by $|T|$. The *spectrum* and the *point spectrum* of $T$ are denoted by $\sigma(T)$ and $\sigma_p(T)$, respectively. All these basic concepts can be found in [4, 7, 8].

## 2. The class $\mathcal{AN}$

In this section we prove a representation theorem for self-adjoint $\mathcal{AN}$-operators and then extend it for general $\mathcal{AN}$-operators. The first one sharpens the result of Carvajal and Neves [2, Theorem 3.25] and the second one generalizes it. Using the second result, we will show that every $\mathcal{AN}$-operator can be written as a sum of a compact operator and a scalar multiple of an isometry satisfying some condition. We also prove the converse of this result. These two results combined give a characterization of all $\mathcal{AN}$-operators.

Recall that if $T \in \mathcal{B}(H_1, H_2)$, the quantity $m(T) := \inf \{\|Tx\| : x \in S_{H_1}\}$ is called the *minimum modulus* of $T$. We have the following proposition.

Proposition 2.1. *Let $T \in \mathcal{B}(H)$ be normal. Then:*

(1) $m(T) = d(0, \sigma(T))$*;*
(2) $m(T) = m(T^*)$*;*
(3) $m(T^n) = m(T)^n$ *for each* $n \in \mathbb{N}$*;*
(4) *if* $T \geq 0$*, then* $m(T) = m(T^{1/2})^2$.

Proof. The proof of (1) is given in [9, Theorem 4.4.7]. For the sake of completeness we give the details here. We prove (1) in two cases.

*Case 1.* $m(T) = 0$. In this case $T$ is not bounded below. That is, $T$ has no bounded inverse. Thus $0 \in \sigma(T)$, and consequently $d(0, \sigma(T)) = 0$.

*Case 2.* $m(T) > 0$. In this case $T$ has a bounded inverse and hence $d(0, \sigma(T)) > 0$.

Consider

$$\begin{aligned}
d(0, \sigma(T)) &= \inf \{|\lambda| : \lambda \in \sigma(T)\} \\
&= \frac{1}{\sup \{|\mu| : \mu \in \sigma(T^{-1})\}} \\
&= \frac{1}{\|T^{-1}\|} \\
&= \frac{1}{\sup \left\{ \frac{\|T^{-1}y\|}{\|y\|} : 0 \neq y \in H \right\}} \\
&= \frac{1}{\sup \left\{ \frac{\|x\|}{\|Tx\|} : 0 \neq x \in H \right\}} \\
&= \inf \left\{ \frac{\|Tx\|}{\|x\|} : 0 \neq x \in H \right\} \\
&= m(T).
\end{aligned}$$

The proof of (2) follows from the fact that $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$ and (1). The proof of (3) follows by the spectral mapping theorem and (1). □

Remark 2.2. Let $T \in \mathcal{B}(H)$. By using the fact that $m(T) = m(|T|)$, Proposition 2.1 and the spectral mapping theorem, it can be proved easily that $m(T^*T) = m(T)^2$.

Recall that $T \in \mathcal{B}(H_1, H_2)$ is called an $\mathcal{AN}$-operator, if for any nonzero closed subspace $M$ of $H_1$, $T|_M$ attains norm on the unit sphere of $M$. That is, there exists an $x_0 \in S_M$ such that $\|T|_M\| = \|(T|_M)(x_0)\|$.

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We denote the class of all $\mathcal{AN}$-operators between $H_1$ and $H_2$ by $\mathcal{AN}(H_1, H_2)$ and $\mathcal{AN}(H, H)$ by $\mathcal{AN}(H)$.

Theorem 2.3. *Let $T \in \mathcal{AN}(H)$ be positive. Then:*

(1) *There exists a sequence of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots$ of $T$ with corresponding orthonormal system of eigenvectors $\phi_1, \phi_2, \phi_3, \dots$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and*
$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n \quad \text{for all } x \in H. \tag{2.1}$$
*The above series converges in the strong operator topology of $\mathcal{B}(H)$. Moreover, if $\lambda_n$ is infinite, then $\lambda_n \to m(T)$.*

(2) *Either $T$ is compact or $T = K + m(T)I$ for some positive compact operator $K$.*

Proof. To prove the first part, we imitate the proof of the spectral theorem for compact self-adjoint operators (see [6, Theorem 5.1, page 178] or [10, Theorem VII.4, page 62] for details).

Let $H_1 := H$ and $T_1 := T$. Then by [2, Proposition 2.3], there exist a $\lambda_1 \in \mathbb{R}^+$ and $\phi_1 \in S_H$ such that $T_1\phi_1 = \lambda_1\phi_1$ and $\lambda_1 = \|T_1\|$. By the projection theorem, $H_1 = [\phi_1] \oplus^{\perp} [\phi_1]^{\perp}$. Let $H_2 = [\phi_1]^{\perp}$. Note that $H_2$ reduces $T_1$. Let $T_2 := T_1|_{H_2}$. The fact that $T_1 \in \mathcal{AN}(H)$ implies that $T_2 \in \mathcal{N}(H_2)$. Again, by [2, Proposition 2.3], there exists a $\lambda_2 \in \mathbb{R}^+$ and $\phi_2 \in S_{H_2}$ such that $T_2\phi_2 = \lambda_2\phi_2$ and $\lambda_2 = \|T_2| \leq |T_1\| = \lambda_1$. Clearly $\phi_1 \perp \phi_2$. Now, let $H_3 := [\phi_1, \phi_2]^{\perp}$ and $T_3 := T|_{H_3}$. If $T_3 = 0$, then there is nothing to prove. Otherwise, $H_3$ reduces $T$. Again, by [2, Proposition 2.3], there exist a $\lambda_3 \in \mathbb{R}^+$ and $\phi_3 \in S_{H_3}$ such that $T_3\phi_3 = \lambda_3\phi_3$, $\lambda_3 = \|T_3\| \leq \lambda_2 \leq \lambda_1$. By construction, $\phi_3 \perp \phi_j$, $j = 1, 2$.

Proceeding in this manner, either after some stage $n$, $T_n = 0$, or there exist a sequence $\lambda_n \in \mathbb{R}^+$ and corresponding vectors $\phi_n \in S_{H_n}$ such that $T_n\phi_n = \lambda_n\phi_n$ and $\lambda_n \geq \lambda_{n+1}$ for all $n \in \mathbb{N}$. Here we describe these two cases.

*Case 1.* $T_n = 0$ for some $n$. If $T_n = 0$ and $T_{n-1} \neq 0$, then $H_n \subseteq N(T)$. That is, $N(T)^{\perp} \subseteq H_n^{\perp} = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{n-1}\}$. Since, each $\phi_j \in N(T)^{\perp}$, it follows that $N(T)^{\perp} = H_n^{\perp}$. Hence $R(T) = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{n-1}\}$. Hence for every $x \in H$, we have $Tx = \sum_{j=1}^{n-1} \lambda_j \langle x, \phi_j \rangle \phi_j$.

*Case 2.* $T_n \neq 0$ for all $n \in \mathbb{N}$. As $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq m(T)$, $\lambda_n \to \alpha$ for some $\alpha \in \sigma(T)$ and $\alpha \geq m(T)$.

Let $G = \overline{\operatorname{span}}\{\phi_n : n \in \mathbb{N}\}$. If $y \in G$, then $y = \sum_{n=1}^{\infty} \langle y, \phi_n \rangle \phi_n$. By the continuity of $T$, we get $Ty = \sum_{n=1}^{\infty} \lambda_n \langle y, \phi_n \rangle \phi_n$.

Now, let $x \in H$. Write $y_n = x - \sum_{j=1}^{n-1} \langle x, \phi_j \rangle \phi_j$. Then $y_n \in H_n$ and $\|y_n\| \leq \|x\|$. Also,
$$\|(T - \alpha I)(y_n)\| \leq \|(T - \alpha I)|_{H_n}\| \, \|y_n\| \leq \|T_n - \alpha I_n\| \, \|x\|, \tag{2.2}$$
where $I_n : H_n \to H_n$ is the identity operator. Next, we show that $\|T_n - \alpha I_n\| = \lambda_n - \alpha$ for each $n \in \mathbb{N}$. By construction, $\|T_n\| = \lambda_n$ for each $n \in \mathbb{N}$. Hence by the triangle inequality,
$$|\lambda_n - \alpha| = \|\lambda_n| - \alpha| = |\, \|T_n\| - \|\alpha I_n\| \,| \leq \|T_n - \alpha I_n\|.$$

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On the other hand, as $T_n$ is self-adjoint for each $n \in \mathbb{N}$, we have that $T_n \le \|T_n\| I_n = \lambda_n I_n$ and $T_n - \alpha I_n \le (\lambda_n - \alpha) I_n$. Hence $\|T_n - \alpha I_n\| \le |\lambda_n - \alpha|$. By Equation (2.2), we can conclude that $(T - \alpha I)(y_n) \to 0$ as $n \to \infty$. Hence

$$(T - \alpha I)x = \sum_{n=1}^{\infty} (\lambda_n - \alpha)\langle x, \phi_n\rangle \phi_n, \quad \text{for each } x \in H. \tag{2.3}$$

By the converse of the spectral theorem for compact self-adjoint operators [6, Theorem 6.2, page 181], it follows that $T - \alpha I$ is compact. Let $K := T - \alpha I$. Clearly, $K = K^*$ and $\sigma(K) = \{\lambda_n - \alpha\} \cup \{0\}$. Hence $K \ge 0$. If $\alpha = 0$, then $T = K$.

Next, assume that $\alpha > 0$. In this case, $T = K + \alpha I > \alpha I$. This shows that $T^{-1} \in \mathcal{B}(H)$ and, by Proposition 2.1, we have $m(T) = d(0, \sigma(T)) = \alpha$.

As $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $H$, by Equation (2.3)), we have

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n\rangle \phi_n \quad \text{for all } x \in H. \qquad \square$$

Remark 2.4.

(1) Let $T \in \mathcal{AN}(H)$ be positive. Then $T$ is compact if and only if $m(T) = 0$.
(2) Let $P : H \to H$ be an orthogonal projection and $P \ne I$. Then $P \in \mathcal{AN}(H)$ if and only if $P$ is a finite-rank operator.
(3) The eigenspace corresponding to each $\lambda_n \ne m(T)$ is a reducing subspace for $T$.
(4) By the proof of Theorem 2.3, it follows that $\|T\| = \lambda_1$ and $\|K\| = \|T\| - m(T)$. Hence if $\|T\| = m(T)$, that is, $T$ is an isometry, then $K = 0$.
(5) If $T \in \mathcal{AN}(H)$ be positive and bounded below, then $H$ consists of a system of eigenvectors which form an orthonormal basis.
(6) Since $K$ is compact, each $\lambda_n - m(T) \in \sigma_p(K)$ is repeated finitely many times provided $\lambda_n - m(T) \ne 0$. This is true if and only if $\lambda_n$ ($\lambda_n \ne m(T)$) is repeated finitely many times.

Using (2) of Theorem 2.3 and properties of compact operators, we can prove the following theorem.

Theorem 2.5. *Let $T \in \mathcal{AN}(H)$ be positive. Then:*

(1) *eigenspaces of $T$ corresponding to $\lambda_j \in \sigma(T) \setminus \{m(T)\}$ are finite-dimensional;*
(2) *$R(T - \lambda_j I)$ is closed for each $\lambda_j \in \sigma(T) \setminus \{m(T)\}$.*

Proof. If $m(T) = 0$, then $T = K$ and all the above statements are well known for a compact operator. If $m(T) > 0$, then $T$ being self-adjoint, it is invertible. In this case, by Theorem 2.3, we have $T = K + m(T)I$. Now the statements follow by the corresponding statements for self-adjoint compact operators and the fact that $\sigma(K + m(T)I) = \{\mu + m(T) : \mu \in \sigma(K)\}$. $\square$

Next, we would like to obtain a representation of a self-adjoint $\mathcal{AN}$-operator as in Equation (2.1). For this purpose we need the following results.

Theorem 2.6 [2, Theorem 3.22]. *Let $K \in \mathcal{K}(H)$ be positive. Then $K + I \in \mathcal{AN}(H)$.*

Theorem 2.7. *Let $T \in \mathcal{B}(H)$ be positive. Then $T \in \mathcal{AN}(H) \Leftrightarrow T^{1/2} \in \mathcal{AN}(H)$.*

Proof. If $T \in \mathcal{AN}(H)$, then by Theorem 2.3,

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \quad \text{for all } x \in H,$$

where $\{\lambda_n\}, \{\phi_n\}$ are as in Theorem 2.3. Consequently,

$$T^{1/2}x = \sum_{n=1}^{\infty} \lambda_n^{1/2} \langle x, \phi_n \rangle \phi_n, \quad \text{for all } x \in H,$$

and if $(\lambda_n)$ is infinite, then $\lambda_n^{1/2} \to m(T^{1/2})$.

If $m(T) = 0$, then $T$ is compact and hence $T^{1/2}$ is compact.

If $m(T) > 0$, then $T^{1/2} = K + m(T^{\frac{1}{2}})I$, where

$$Kx = \sum_{n=1}^{\infty} (\lambda_n^{1/2} - m(T^{1/2}) \langle x, \phi_n \rangle \phi_n, \quad \text{for all } x \in H.$$

Note that $K \geq 0$. Hence by Theorem 2.6, $T^{1/2} \in \mathcal{AN}(H)$. The reverse implication follows along similar lines. □

As a consequence of the above theorem, we prove the following characterization of $\mathcal{AN}$-operators.

Theorem 2.8. *Let $T \in \mathcal{B}(H_1, H_2)$. Then $T \in \mathcal{AN}(H_1, H_2)$ if and only if $T^*T \in \mathcal{AN}(H_1)$.*

Proof. We know that $T \in \mathcal{AN}(H_1, H_2)$ if and only if $|T| \in \mathcal{AN}(H_1)$. This is true if and only if $T^*T = |T|^2 \in \mathcal{AN}(H_1)$ by Theorem 2.7. □

Theorem 2.9. *Let $T = T^* \in \mathcal{AN}(H)$. Then:*

(1) *There exists a sequence of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ of $T$ with corresponding orthonormal set of eigenvectors $\phi_1, \phi_2, \phi_3, \ldots$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n \quad \textit{for all } x \in H.$$

*The above series converges in the strong operator topology of $\mathcal{B}(H)$. Moreover, if $\lambda_n$ is infinite, then $|\lambda_n| \to m(T)$.*

(2) $\sigma(T) = \{\lambda_n\} \cup \{m(T)\}$.

Proof. We begin by proving (1). Following the proof of Theorem 2.3, we get a sequence of real numbers $(\lambda_n)$ such that $|\lambda_{n+1}| \leq |\lambda_n|$ with $|\lambda_n| = \|T_n\|$, and vectors $\phi_n$ such that $T_n\phi_n = T\phi_n = \lambda_n\phi_n$. Either $(\lambda_n)$ is finite, or if $(\lambda_n)$ is infinite, then $|\lambda_n| \to \beta$.

If $\beta = 0$, the proof is same as that of case (1) in Theorem 2.3.

Let $\beta > 0$. The operator $S = T^2$ is positive and $S \in \mathcal{AN}(H)$, by Theorem 2.8. By Theorem 2.3, we have $Sx = \sum_{n=1}^{\infty} \lambda_n^2 \langle x, \phi_n \rangle \phi_n$, where $\lambda_n^2$ is an eigenvalue of $S$ and $\phi_n$ is the corresponding eigenvector. Also $m(S) = m(T)^2 = \beta^2$. Hence $m(T) = \beta$. Since $T = T^*$, we can conclude that $T^{-1} \in \mathcal{B}(H)$. Also $T^{-1}\phi_n = \frac{1}{\lambda_n}\phi_n$. Thus for each $x \in H$,

$$Tx = T^{-1}Sx = \sum_{n=1}^{\infty} \lambda_n^2 \langle x, \phi_n \rangle T^{-1}\phi_n = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n.$$

The proof of part (2) follows by the above representation (1) and the proof of Theorem 2.3. □

Remark 2.10. Theorem 2.9 is not valid for normal $\mathcal{AN}$-operators. For example, the bilateral shift on $\ell^2(\mathbb{Z})$ is unitary, which is an $\mathcal{AN}$-operator, but it does not have representation as in (1) of Theorem 2.9. Also the spectrum of this operator is the unit circle and the point spectrum is empty. Hence statement (2) of Theorem 2.9 is not valid in this case.

Now by dropping the positivity of the operator we can obtain the above results for any $\mathcal{AN}$-operator as follows.

Theorem 2.11. *Let $T \in \mathcal{AN}(H_1, H_2)$ and $T = V|T|$ be the polar decomposition of $T$. Then the following statements are true.*

(1) *There exist orthonormal sets $\{\phi_j\}_{j=1}^{\infty} \subseteq H_1$, $\{\psi_j\}_{j=1}^{\infty} \subseteq H_2$ where $\phi_j$ is an eigenvector of $|T|$ corresponding to the eigenvalue $s_j$ of $|T|$ and $\psi_j = V\phi_j$, $j \in \mathbb{N}$, such that*

$$Tx = \sum_{n=1}^{\infty} s_n \langle x, \phi_n \rangle \psi_n \quad \textit{for all } x \in H.$$

*The above series converges in the strong operator topology of $\mathcal{B}(H)$. Moreover, if $s_n$ is infinite, then $s_n \to m(T)$.*

(2) *Either $T$ is compact or $T = K + m(T)V$, where $V$ is an isometry and $K \in \mathcal{K}(H_1, H_2)$.*

Proof. As $\||T|x\| = \|Tx\|$ for all $x \in H_1$, $|T| \in \mathcal{AN}(H_1)$. By Theorem 2.3, we have that either $|T|$ is compact or $|T| = K_1 + m(T)I$ where $K_1$ is a positive compact operator. Now by the polar decomposition of $T$ and the fact that $m(T) = m(|T|)$, we have $T = K + m(T)V$, where $K = VK_1 \in \mathcal{K}(H_1, H_2)$. □

Corollary 2.12. *Let $T \in \mathcal{AN}(H_1, H_2)$. If rank of $T$ is not finite, then either $R(T)$ is not closed or $T$ is bounded below.*

Proof. The corollary follows by Theorem 2.11. □

Note that in Theorem 2.11, the operators $K$ and $V$ satisfy the condition

$$K^*K + 2m(T)\,\mathrm{Re}(V^*K) \geq 0. \tag{2.4}$$

Next we prove the converse of Theorem 2.11.

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Theorem 2.13. *Let $K \in \mathcal{K}(H)$, $V$ be an isometry and $\alpha \geq 0$ satisfying the condition*

$$K^*K + 2\alpha \mathrm{Re}(V^*K) \geq 0. \tag{2.5}$$

*Let $T := K + \alpha V$. Then:*

(1) *$m(T) = \alpha$; and*
(2) *$T \in \mathcal{AN}(H)$.*

Proof. We have

$$|T|^2 = T^*T = K^*K + \alpha K^*V + \alpha V^*K + \alpha^2 I = K_1 + \alpha^2 I,$$

where $K_1 = K^*K + 2\alpha \mathrm{Re}(V^*K)$. By the hypothesis, $K_1 \geq 0$, and by Proposition 2.1 we have

$$m(T)^2 = m(|T|)^2 = m(T^*T) = d(0, \sigma(K_1 + \alpha^2 I)) = \alpha^2.$$

Hence $m(T) = \alpha$. Here we used the fact that $0 \in \sigma(K_1)$.

Now by Theorem 2.3, $|T|^2 \in \mathcal{AN}(H)$ and by Theorem 2.7, $|T| \in \mathcal{AN}(H)$. Hence $T \in \mathcal{AN}(H)$. □

Remark 2.14.

(1) Let $K$ be a one-to-one compact operator with the polar decomposition $K = V|K|$. Then $V$ is an isometry and satisfies the hypothesis of Theorem 2.13. Hence $K + V \in \mathcal{AN}(H)$.
(2) Let $K$ be a compact operator and $V$ be a unitary such that $V^*K \geq 0$. If $T = K + \alpha V (\alpha \geq 0)$, then $|K| = V^*K$ and in this case $T = V|T|$. Condition (2.5) is satisfied and hence $T \in \mathcal{AN}(H)$.

By the converse of the spectral theorem for compact self-adjoint operators and Theorem 2.13 we can prove the following result.

Corollary 2.15. *Let $(\lambda_n)$ be a sequence of real numbers such that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\{\phi_n : n \in \mathbb{N}\}$ be an orthonormal set in $H$. Define $T : H \to H$ by*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \quad \textit{for all } x \in H. \tag{2.6}$$

*Then $T = T^*, T \in \mathcal{AN}(H)$ and $m(T) = |\lambda|$.*

Example 2.16. Let $T : \ell^2 \to \ell^2$ be given by $Te_n = \frac{n+1}{n} e_{n+1}$, $n \in \mathbb{N}$. Let $Re_n = e_{n+1}$, the right shift operator on $\ell^2$ and $De_n = \frac{1}{n} e_n$, $n \in \mathbb{N}$. Then $T = RD + R = R(D + I) \in \mathcal{AN}(\ell^2)$ by [2, Proposition 3.22 and Proposition 3.2]. This can also be seen from Theorem 2.13.

Remark 2.17. Condition (2.5) cannot be dropped in Theorem 2.13. To see this, consider the operator $D$ in Example 2.16. The operator $T = D - I \notin \mathcal{N}(\ell^2)$. If $T \in \mathcal{N}(\ell^2)$, then $\|T\| = 1$ or $-\|T\| = -1$ should be an eigenvalue of $T$ by [2, Proposition 2.3], which is not true. Note that $m(T) = 0$, condition (2.5) is not satisfied in this case and the conclusion of the theorem fails.

If $T$ is normal, then $|T| = |T^*|$. Hence $T \in \mathcal{AN}(H)$ if and only if $T^* \in \mathcal{AN}(H)$.

Proposition 2.18. *Let $(\lambda_n)$ be a sequence of complex numbers such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and $|\lambda_n| \to \alpha$. Define the multiplication operator $M : \ell^2 \to \ell^2$ by*

$$M((x_n)) = (\lambda_n x_n) \quad \text{for all } (x_n) \in \ell^2.$$

*Then $m(M) = \alpha$ and $M \in \mathcal{AN}(\ell^2)$.*

Proof. It suffices to show that $|M| \in \mathcal{AN}(\ell^2)$. We have $|M|(x_n) = (|\lambda_n| x_n)$ for all $(x_n) \in \ell^2$. Then $(|M| - \alpha I) = (|\lambda_n| - \alpha)I$. Since $|\lambda_n| \to \alpha$, it follows that $(|M| - \alpha I) =: K$, where $K$ is a positive, compact operator. Hence by Theorem 2.13, $m(|M|) = \alpha$ and $|M| \in \mathcal{AN}(\ell^2)$. □

Corollary 2.19. *Let $T \in \mathcal{B}(H)$. Then $T \in \mathcal{AN}(H)$ if and only if $\lim_{n\to\infty}\langle |T|e_n, e_n\rangle = m(T)$ for every orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $H$.*

Proof. If $T$ is compact then $|T|$ is compact and the result is well known. If $T = K + m(T)V$, where $V$ is an isometry such $T = V|T|$, then $V^*T = |T| = V^*K + m(T)I$. Since $V^*K$ is compact, we have $\langle(|T| - m(T)I)e_n, e_n\rangle \to 0$ as $n \to \infty$, and the result follows by [1, 5]. For a similar reason the converse also holds. □

Proposition 2.20. *Let $T \in \mathcal{AN}(H)$ be such that $T^* \in \mathcal{AN}(H)$. If $T$ is not a finite-rank operator, then either $R(T)$ is not closed or $T^{-1} \in \mathcal{B}(H)$ .*

Proof. Assume that $T \in \mathcal{AN}(H)$. If $T \in \mathcal{K}(H)$, then $R(T)$ is not closed. If $T$ is not compact, then $T = K + m(T)V$, where $K$ and $V$ are as in Theorem 2.11. As $T^* = K^* + m(T)V^*$ and by the hypothesis, $T^* \in \mathcal{AN}(H)$, we must have $V^*$ an isometry, $KK^* + 2m(T)\mathrm{Re}(VK^*) \geq 0$ and $m(T^*) = m(T)$ by Theorem 2.13. That is both $T$ and $T^*$ are bounded below and hence $T^{-1} \in \mathcal{B}(H)$. □

Definition 2.21 [3, page 349]. Let $T \in \mathcal{B}(H_1, H_2)$. Then $T$ is called left *semi-Fredholm* if there exist a $B \in \mathcal{B}(H_2, H_1)$ and $K \in \mathcal{K}(H_1)$ such that $BT = K + I$, and *right semi-Fredholm* if there exist a $A \in \mathcal{B}(H_2, H_1)$ and $K' \in \mathcal{K}(H_2)$ such that $TA = K' + I$. If $T$ is both left semi-Fredholm and right semi-Fredholm, then $T$ is called Fredholm.

Remark 2.22. Note that $T$ is left semi-Fredholm if and only if $T^*$ is right semi-Fredholm (see [3, Section 2, page 349] for details).

Corollary 2.23. *Let $T \in \mathcal{AN}(H)$ be bounded below. Then $T$ is left semi-Fredholm.*

Proof. Let $T = K + m(T)V$ as in Theorem 2.11. Then $ST = K_1 + I$, where $K_1 = \frac{V^*K}{m(T)}$ and $S = \frac{V^*}{m(T)}$. Hence $T$ is left semi-Fredholm. □

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G. RAMESH, Department of Mathematics,
Indian Institute of Technology Hyderabad,
Ordnance Factory Estate, Yeddumailaram 502 205,
Andhra Pradesh, India
e-mail: rameshg@iith.ac.in