Can. J. Math., Vol. XXXVII, No. 4, 1985, pp. 730-746

STRONG-Q-SEQUENCES AND VARIATIONS ON MARTIN'S AXIOM

JURIS STEPRĀNS

0. Introduction. As part of their study of $\beta\omega - \omega$ and $\beta\omega_1 - \omega_1$ A. Szymanski and H. X. Zhou [3] were able to exploit the following difference between ω_1 and ω: ω_1 contains uncountably many disjoint sets whereas any uncountable family of subsets of ω is, at best, almost disjoint. To translate this distinction between ω_1 and ω to a possible distinction between $\beta\omega_1 - \omega_1$ and $\beta\omega - \omega$ they used the fact that if $\mathscr{A}$ is a pairwise disjoint family of sets and a subset of each member of $\mathscr{A}$ is chosen then it is trivial to find a single set whose intersection with each member of $\mathscr{A}$ is the chosen set. However, they noticed, it is not clear that the same is true if $\mathscr{A}$ is only a pairwise almost disjoint family even if we only require equality except on a finite set. But any homeomorphism from $\beta\omega_1 - \omega_1$ to $\beta\omega - \omega$ would have to carry a disjoint family of subsets of ω_1 to an almost disjoint family of subsets of ω with this property. This observation should motivate the following definition.

Definition 1. Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$. Let

$$\mathscr{A}^* = \{f:\mathscr{A} \to \mathscr{P}(\omega); (\forall A \in \mathscr{A})(f(A) \subseteq A)\}.$$

Then $\mathscr{A}$ is a strong-Q-sequence if and only if

$$(\forall f \in \mathscr{A}^*)(\exists b \subseteq \omega)(\forall a \in \mathscr{A})(\exists n \in \omega)((b \cap a) \cup n$$
$$= f(a) \cup n).$$

It should be noted that this definition generalizes to arbitrary Boolean algebras as follows:

Definition 2. If B is a Boolean algebra and $\mathscr{A} \subseteq B$ then $\mathscr{A}$ is a strong-Q-sequence in B if and only if

$$(\forall f \in \mathscr{A}^*)(\exists b \subseteq B)(\forall a \in \mathscr{A})(b \cap a = f(a))$$

where $\mathscr{A}^*$ is defined as before.

Received January 19, 1983 and in revised form February 27, 1984. I would like to thank Jim Baumgartner for pointing out an error in an earlier version of this paper as well as for several helpful suggestions. Part of this paper formed part of my Ph.D. thesis at the University of Toronto.

Notice that for the Boolean algebra $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ the two notions of strong-Q-sequence coincide. It is also easy to see that a strong-Q-sequence in any Boolean algebra must be an antichain. The remarks at the end of the opening paragraph can now be paraphrased as follows: $\mathscr{P}(\omega_1)/[\omega]^{<\aleph_0}$ has an uncountable strong-Q-sequence but it is not clear that $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ does. It will be shown in this paper that the answer to the question of whether or not $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ has an uncountable strong-Q-sequence is independent of the usual axiom of set theory.

In particular it will be shown in Section 1 that MA_{ω_1} implies that there are no uncountable strong-Q-sequences. This result was also obtained, independently, by S. Shelah and answers a question of H. X. Zhou who had earlier shown that, assuming the existence of a large cardinal, it is consistent with MA_{ω_1} that there are no strong-Q-sequences. In fact, in Section 1 only MA_{ω_1} for σ-linked partial orders is required. (A σ-linked partial order is one which is the union of countably many subsets each of which has the property that any two elements in it are compatible.) In Section 2, however, it is shown that it is consistent with MA_{ω_1} for σ-centred partial orders that there is an uncountable strong-Q-sequence. This yields the known corollary, the proof of which can be found in [**2**], that MA_{ω_1} for σ-centred partial orders does not imply MA_{ω_1} for σ-linked partial orders. The usual method for proving this fact is to get a model where MA_{ω_1} for σ-centred partial orders holds, but MA_{ω_1} for the measure algebra does not. The result of Section 2 however yields a model where MA_{ω_1} for both σ-centred partial orders and probability measure algebras holds but MA_{ω_1} for σ-linked partial orders still does not hold. In Section 3 examples will be given for each $n \in \omega$ of partial orders which are $\sigma - n$-linked but not $\sigma - n + 1$-linked. The first such examples were found by Bell [**1**]. The results of Section 2 and Section 3 are generalized in Section 4 to the next higher cardinal to yield a model of BACH where P_1 holds but BACH fails answering a question in [**5**]. In that section, examples of $\aleph_1 - n$-linked but not $\aleph - n + 1$-linked partial orders are constructed. The original proof of the analogous result for MA does not generalize to a higher cardinal. Finally, in Section 5 some remarks are made about the relevance of strong-Q-sequences to the question of whether or not $\beta\omega_1 - \omega_1$ is homeomorphic to $\beta\omega - \omega$.

1. MA_{ω_1} implies that strong-Q-sequences are countable. The proof of the result in the title of this section will rely on the following lemma.

LEMMA 1. *Let $(T, \leqq_T)$ be a tree of height ω such that each node branches at most twice. Let $\mathscr{B}$ be any uncountable collection of branches through $(T, \leqq_T)$. (Throughout this paper a branch will always refer to a maximal branch of maximal height.) Then $\mathscr{B}$ is not a strong-Q-sequence on $\mathscr{P}(T)/[T]^{<\aleph_0}$.*

Proof. Let $\{\sigma_0^\tau, \sigma_1^\tau\}$ list the immediate successors of τ in $(T, \leqq_T)$. For $B \in \mathscr{B}$ let

$$f(B) = \{\tau \in B:\sigma_0^\tau \in B\}.$$

If $\mathscr{B}$ is a strong-Q-sequence then there is $A \subseteq T$ and, for each $B \in \mathscr{B}$, $\tau_B \in B$ such that

$$A \cap \{\tau \in B:\tau \geqq_T \tau_B\} = \{\tau \in f(B):\tau \geqq_T \tau_B\}.$$

Choose $\{B_0, B_1\} \in [\mathscr{B}]^2$ such that $\tau_{B_0} = \tau_{B_1}$. Let $\tau \in B_0 \cap B_1$ such that, without loss of generality, $\sigma_i^\tau \in B_i$. Then

$$\tau \geqq \tau_{B_0} = \tau_{B_1}$$

but

$$\tau \in f(B_0) \text{ and } \tau \notin f(B_1).$$

Lemma 1 says that no almost disjoint family obtained from the branches of a binary tree can be a strong-Q-sequence. The remainder of the proof will concentrate on showing that, assuming MA_{ω_1}, every uncountable almost disjoint family must have embedded in it just such an almost disjoint family. The next lemma will make precise what is meant by "embedded".

LEMMA 2. *If $\mathscr{A}$ is a strong-Q-sequence then*
i) *if $\mathscr{B} \subseteq \mathscr{A}$ then $\mathscr{B}$ is a strong-Q-sequence*
ii) *if $X \subseteq \cup \mathscr{A}$ then $\{A \cap X:A \in \mathscr{A}\}$ is a strong-Q-sequence.*

Proof. This is clear.

LEMMA 3. *Assume MA_{ω_1} for σ-centred partial orders. Let $\mathscr{A}$ be an uncountable almost disjoint family of subsets of ω then there is $T \subseteq \omega$ and $\leqq_T$, a finite branching tree order of height ω on T, such that there are uncountably many $A \in \mathscr{A}$ such that $A \cap T$ is a branch through $(T, \leqq_T)$.*

Proof. Define a partial order $(\mathbf{P}, \leqq_{\mathbf{P}})$ by $p \in \mathbf{P}$ if and only if:
(a) $p = (T_p, \leqq_p, \mathscr{A}_p)$
(b) $T_p \in [\omega]^{<\aleph_0}$ and $\mathscr{A}_p \in [\mathscr{A}]^{<\aleph_0}$
(c) $\leqq_p$ is a tree order on T_p
(d) if $A \in \mathscr{A}_p$ then $A \cap T_p$ is a branch through $(T_p, \leqq_p)$.
Define $p \leqq_{\mathbf{P}} q$ if and only if
(e) $T_p \supseteq T_q$ and $\mathscr{A}_p \supseteq \mathscr{A}_q$
(f) $\leqq_p$ is an end-extension of $\leqq_q$.
Now, for $p \in \mathbf{P}$, let

$$\mathscr{B}(p) = \{A \in \mathscr{A}:A \text{ is a branch through } (T_p, \leqq_p)\}.$$

Let

$$\bar{\mathscr{A}} = \mathscr{A} \setminus \cup \{\mathscr{B}(p): p \in \mathbf{P} \text{ and } |\mathscr{B}(p)| < \aleph_1\}$$

and notice that $|\bar{\mathscr{A}}| = \aleph_1$. Now let

$$\bar{\mathbf{P}} = \{p \in \mathbf{P}: \mathscr{A}_p \subseteq \bar{\mathscr{A}} \text{ and } \mathscr{A}_p \neq 0\}.$$

MA_{ω_1} will be applied to $(\bar{\mathbf{P}}, \leqq_{\mathbf{P}} \cap (\bar{\mathbf{P}} \times \bar{\mathbf{P}}))$.

It is easy to see that this partial order is σ-centred. Furthermore, if $\{A_\alpha: \alpha \in \omega_1\}$ enumerates $\mathscr{A}$ then for each $\alpha \in \omega_1$,

$$\{p \in \bar{\mathbf{P}}: (\exists \beta > \alpha)(A_\beta \in \mathscr{A}_p)\}$$

is dense. To see this choose $p \in \bar{\mathbf{P}}$. It suffices to show that $|\mathscr{B}(p)| = \aleph_1$. But if $|\mathscr{B}(p)| < \aleph_1$ then

$$\bar{\mathscr{A}} \cap \mathscr{B}(p) = 0.$$

However $\mathscr{A}_p \neq 0$ and $\mathscr{A}_p \subseteq \bar{\mathscr{A}}$ and, by definition, $\mathscr{A}_p \subseteq \mathscr{B}(p)$.

To finish the proof it suffices to show that for each $p \in \bar{\mathbf{P}}$ there is $q \in \bar{\mathbf{P}}$ such that

$$q \leqq_{\mathbf{P}} p \quad \text{and}$$

$$\text{height}(T_q, \leqq_q) = \text{height}(T_p, \leqq_p) + 1.$$

To see this choose $A \in \mathscr{A} \setminus \mathscr{A}_p$ and $k \in \omega$ such that $\{A\} \cup \mathscr{A}_p$ is a disjoint family above k. For each maximal node of T_p, n, and each $B \in \mathscr{A}_p$ such that $n \in B$ choose $m \in B \setminus k$. If there is no B containing n then choose $m \in A \setminus k$. In either case put m in T_q and set $n \leqq_q m$. Because of the choice of k it is now easy to verify that $(T_q, \leqq_q, \mathscr{A}_p)$ is in $\bar{\mathbf{P}}$ and it clearly extends p.

While the hypothesis of Lemma 1 requires a binary tree, Lemma 3 only gives us a finite branching tree. This difficulty will be rectified by the next two lemmas.

LEMMA 4. *If $(T, \leqq_T)$ is a finite branching tree of height ω and $\mathscr{B}$ is an uncountable collection of branches through $(T, \leqq_T)$ then there is $S \subseteq T$ such that, letting $\leqq_S = \leqq_T \cap S \times S$,*

a) *height $(S, \leqq_S) = \omega$ and S is an initial segment of $(T, \leqq_T)$.*

b) *if $s \in S$ then, letting $\mathscr{B}_s = \{B \in \mathscr{B}: s \in B\}$,*

$$|\{B \in \mathscr{B}_s: |B \cap S| = \aleph_0\}| > \aleph_0.$$

Proof. Let

$$S = \{s \in T: |\mathscr{B}_s| > \aleph_0\}.$$

Clearly a) is satisfied.

To see that b) is satisfied choose $s \in S$ and notice that

$$|\{B \in \mathscr{B}_s; |B \cap S| < \aleph_0\}| \leqq \aleph_0.$$

LEMMA 5. *Assume* MA_{ω_1} *for* σ*-linked partial orders. If* $(T, \leqq_T)$ *is a finite branching tree of height* ω *and* $\mathscr{B}$ *is an uncountable set of branches through* $(T, \leqq_T)$ *then there is* $S \subseteq T$ *such that*

a) $(S, \leqq_T \cap S \times S)$ *is an initial segment of* $(T, \leqq_T)$ *and has height* ω

b) $(S, \leqq_T \cap S \times S)$ *is binary (i.e., each node splits at most twice)*

c) $|\{B \in \mathscr{B}: |B \cap S| = \aleph_0\}| > \aleph_0$.

Proof. First choose $T' \subseteq T$ satisfying the conclusion of Lemma 4 and let

$$\mathscr{B}' = \{b \in \mathscr{B}; b \subseteq T\}.$$

Let $\mathbf{P}$ be defined by $p \in \mathbf{P}$ if and only if $p = (T_p, \mathscr{B}_p)$

d) T_p is an initial segment of T'

e) $\mathscr{B}_p \in [\mathscr{B}]^{<\aleph_0}$

f) $\{B_0, B_1\} \in [\mathscr{B}_p]^2$ then $B_0 \cap T_p \neq B_1 \cap T_p$

g) $(T_p, \leqq_T \cap T_p \times T_p)$ is binary.

Define $p \leqq q$ if and only if T_p is an end-extension of T_q and $\mathscr{B}_p \supseteq \mathscr{B}_q$.

It is easy to see that $\{p \in \mathbf{P}: T_p$ has height at least $n\}$ is dense in $\mathbf{P}$ for each $n \in \omega$. Also, from the conclusion of Lemma 4 it follows that if $\{b_\alpha: \alpha \in \omega_1\}$ enumerates $\mathscr{B}'$ then for each $\alpha \in \omega_1$,

$$\{p \in \mathbf{P}: (\exists \eta \supseteq \alpha)(b_\eta \in \mathscr{B}_p)\}$$

is dense in $\mathbf{P}$. Hence it suffices to show that $(\mathbf{P}, \leqq)$ is σ-linked.

To see this let $\{p, q\} \subseteq \mathbf{P}$ and suppose that $T_p = T_q$. For each maximal branch of T_p, b, there are at most two branches in $\mathscr{B}_p \cup \mathscr{B}_q$, C_p and C_q, such that

$$C_p \cap T_p = C_q \cap T_p = b.$$

Hence it is easy to find a binary end-extension, T, of T_p such that

$$(T, \mathscr{B}_p \cup \mathscr{B}_q) \in \mathbf{P}.$$

This completes the proof.

THEOREM 6. MA_{ω_1} *for* σ*-linked partial orders implies that there are no uncountable strong-*Q*-sequences.*

Proof. Apply Lemmas 1, 2, 3, 4 and 5.

2. A model which has an uncountable strong-Q-sequence. In this section it will be shown that it is consistent with a certain version of MA_{ω_1} that there is a strong-Q-sequence. The version of MA_{ω_1} will require the following definition.

Definition 3. If $F: \omega \to {}^\omega\omega$ and $(\mathbf{P}, \leqq)$ is a partial order then $(\mathbf{P}, \leqq)$ is F-centred if and only if

$$\mathbf{P} = \cup \{\mathbf{P}_n; n \in \omega\}$$

and for each $n \in \omega$ the following condition holds:

$$(\forall m \in \omega)(\forall \Gamma \in [\mathbf{P}_n]^{F(n)(m)})(\exists \Lambda \in [\Gamma]^m)(\Lambda \text{ is centred}).$$

So, for example, a partial order is σ-centred if and only if it is I-centred where $I(n) = \mathrm{id}_\omega$ for each $n \in \omega$. A less trivial example of an F-centred partial order is the partial order associated with the measure algebra associated with a probability measure. The following lemma will be used in showing this.

LEMMA 7. *Let μ be a probability measure on X. If $\{A_i:i \in n + 1\}$ are subsets of X and $\mu(A_i) > 1/n$ for each $i \in n + 1$ then there is $\{i, j\} \in [n + 1]^2$ such that*

$$\mu(A_i \cap A_j) > 1/n^3.$$

Proof. Suppose not. Then

$$\mu(A_i \cap A_j) \leqq 1/n^3 \text{ for } \{i, j\} \in [n]^2.$$

Hence

$$\begin{aligned}\mu(\cup \{A_i:i \in n\}) &\geqq \sum \{\mu(A_i); i \in n\} \\ &\quad - \sum \{\mu(A_i \cap A_j); \{i, j\} \in [n]^2\} \\ &\geqq 1 - n(n-1)\left(\frac{1}{n^3}\right).\end{aligned}$$

Hence

$$\sum \{\mu(A_n \cap A_i); i \in n\} \geqq \mu(A_n \cap (\cup \{A_i:i \in n\}))$$
$$> \frac{1}{n} - n(n-1)\left(\frac{1}{n^3}\right) = n\left(\frac{1}{n^3}\right).$$

Hence there is some $j \in n$ such that

$$\mu(A_n \cap A_j) > \frac{1}{n^3}.$$

This is a contradiction.

PROPOSITION 8. *Define $\Phi:\omega \to {}^\omega\omega$ by*

$$\Phi(n)(k) = \prod_{j=1}^{\ulcorner 1n_2k \urcorner} (n^{3^{(j-1)}} + 1).$$

If $(\mathbf{P}, \subseteq)$ is the partial order associated with a probability measure algebra (μ, X) then $(\mathbf{P}, \subseteq)$ is Φ-centred.

Proof. Let

$$\mathbf{P}_n = \left\{ A \subseteq X;\ \mu(A) > \frac{1}{n} \right\}.$$

It must be shown that if $\Gamma \in [\mathbf{P}_n]^{\Phi(n)(k)}$ then there is $\Lambda \in [\Gamma]^k$ such that $\mu(\cap\ \Lambda) > 0$. This will be shown by induction on $\ulcorner 1n_2 k \urcorner$. Let the induction hypothesis be that if $\Gamma \in [\mathbf{P}]^{\Phi(n)(k)}$ then there is $\Lambda \in [\Gamma]^k$ such that

$$\mu(\cap\ \Lambda) > n^{-(3^{\ulcorner 1n_2 k \urcorner})}$$

and notice that, by lemma 7, this is true when $k = 2$.

Now suppose that the induction hypothesis is true for $k = 2^m$. We will prove that it also holds for 2^{m+1}. Let

$$\Gamma \in [\mathbf{P}_n]^{\Phi(n)(2^{m+1})}.$$

Since

$$\Phi(n)(2^{m+1}) = (n^{3^m} + 1)\Phi(n)(k)$$

it is possible to partition Γ so that

$$\Gamma = \cup\ \{\Gamma_l : l \in (n^{3^m} + 1)\} \quad \text{and}$$

$$|\Gamma_l| = \Phi(n)(k) \quad \text{for } l \in (n^{3^m} + 1).$$

Using the induction hypothesis it is possible to choose $\Lambda_l \in [\Gamma]^k$ so that

$$\mu(\cap\ \Lambda_l) > n^{-(3^m)} \quad \text{for each } l \text{ in } n^{3^m} + 1.$$

But now it is possible to apply Lemma 2 to $\{\cap\ \Lambda_l : l \in n^{3^m} + 1\}$ to get $\{i, j\} \in [n^{3^m} + 1]^2$ such that

$$\mu(\ (\cap\ \Lambda_i) \cap (\cap\ \Lambda j)\ ) > \frac{1}{(n^{3^m})^3} = n^{-(3^{m+1})}.$$

Definition 4. If $\mathscr{F} \subseteq {}^{\omega}({}^{\omega}\omega)$ then $\mathrm{MA}_{\omega_1}(\mathscr{F})$ is the assertion that MA_{ω_1} holds for any F-centred partial order where $F \in \mathscr{F}$.

Clearly MA_{ω_1} implies $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega))$ and $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega))$ implies MA_{ω_1} for σ-centred partial orders. The relationship between $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega))$ and MA_{ω_1} for σ-linked partial orders, however, is not clear. It will now be shown that $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ is consistent with the existence of a strong-Q-sequence. This will show that $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ is strictly weaker than MA_{ω_1} for σ-linked partial orders even though $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ implies MA_{ω_1} for probability measure algebras since the function Φ is obviously constructible.

The obvious strategy for creating an uncountable strong-Q-sequence is to take some family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ and for each $f \in \mathscr{A}^*$ to generically add the appropriate subset of ω. The following definition introduces a partial order for doing this.

https://doi.org/10.4153/CJM-1985-039-6 Published online by Cambridge University Press

Definition 5. If $\psi:\omega_1 \to \mathscr{P}(\omega)$ indexes a family of subsets of ω and $f \in (\psi''\omega_1)^*$ then let

$$\mathbf{K}(f, \psi) \subseteq \cup \{{}^{\Lambda}\omega:\Lambda \in [\omega_1]^{<\aleph_0}\}$$

be defined by $g \in \mathbf{K}(f, \psi)$ if and only if

$$(\forall\{\alpha, \beta\} \in [D(g)]^2)(f(\psi(\alpha)) \cap X^g_{\alpha,\beta} = f(\psi(\beta)) \cap X^g_{\alpha,\beta})$$

where

$$X^g_{\alpha,\beta} = \psi(\alpha) \cap \psi(\beta) \setminus (g(\alpha) \cup g(\beta)).$$

LEMMA 9. *If G is $\mathbf{K}(f, \psi)$ generic and*

$$E_G = \cup \{f(\psi(\alpha)) \setminus (\cup G)(\alpha);\ \alpha \in \omega_1\}$$

then $A_\alpha \cap E$ is almost equal to $f(\psi(\alpha))$ for each $\alpha \in \omega_1$.

Proof. First note that $|D(\cup G)| = \omega_1$ since if $g \in \mathbf{K}(f, \psi)$ and $\alpha \notin D(g)$ then it is possible to choose $M \in \omega$ large enough so that

$$\psi(\beta) \cap \psi(\alpha) \subseteq M \quad \text{for each } \beta \in D(g).$$

Then clearly $g \cup \{(\alpha, M)\} \in \mathbf{K}(f, \psi)$.

Now it suffices to show that

$$(E_G \cap A_\alpha) \setminus (\cup G)(\alpha) = f(\psi(\alpha)) \setminus (\cup G)(\alpha).$$

Clearly

$$f(\psi(\alpha)) \setminus (\cup G)(\alpha) \subseteq E_G \setminus (\cup G)(\alpha) \quad \text{and} \quad f(\psi(\alpha)) \subseteq \psi(\alpha).$$

Now suppose that $n \in E_G \cap A_\alpha \setminus (\cup G)(\alpha)$. Then

$$n \in f(\psi(\beta)) \setminus (\cup G)(\beta) \quad \text{for some } \beta.$$

Choose $g \in G$ such that $\{\alpha, \beta\} \subseteq D(g)$. Then, since $n \in f(\psi(\beta)) \subseteq \psi(\beta)$, it follows that

$$n \in \psi(\beta) \cap \psi(\alpha) \setminus (g(\beta) \cup g(\alpha)) = X^g_{\alpha,\beta}.$$

But then

$$n \in f(\psi(\beta)) \cap X^g_{\alpha,\beta}$$

and, by Definition 5,

$$n \in f(\psi(\alpha)) \cap X^g_{\alpha,\beta} \subseteq f(\psi(\alpha)) \setminus (\cup G)(\alpha).$$

The problem with the partial order $\mathbf{K}(f, \psi)$ is that it may not satisfy the countable chain condition. This problem can be solved, however, by obtaining ψ generically. The following definition will explain the notation to be used when referring to the obvious partial for doing this.

Definition. From now on $(\mathbf{A}, \geqq)$ will refer to the usual partial order for adding an uncountable family of almost disjoint subsets of ω with finite conditions. In particular $p \in \mathbf{A}$ if and only if:

1. $D(p) = n_p \times \Gamma_p$ where $n_p \in \omega$ and $\Gamma_p \in [\omega_1]^{<\aleph_0}$.
2. $p''D(p) \subseteq 2$.

If $\{p, q\} \subseteq \mathbf{A}$ then $p \leqq q$ if and only if $p \supseteq q$ and

$$(\forall i \in n_p \setminus n_q)(\,|\{\alpha \in \Gamma_q; p(i, \alpha) = 1\}| \leqq 1).$$

If G is $(\mathbf{A}, \geqq)$ generic then let $\psi_G:\omega_1 \rightarrow \mathscr{P}(\omega)$ be defined by

$$\psi_G(\alpha) = \{n \in \omega; (\exists p \in G)(p(n, \alpha) = 1)\}.$$

Clearly $(\mathbf{A}, \geqq)$ satisfies the countable chain condition. (In fact, forcing with $(\mathbf{A}, \geqq)$ is the same thing as adding $\aleph_1$ Cohen reals.) Another obvious fact, which will be used later on, is that if $p \in \mathbf{A}$ and $\{\alpha, \beta\} \in [\Gamma_p]^2$ then

$$p \Vdash ''\psi_G(\alpha) \cap \psi_G(\beta) \subseteq n_p''.$$

LEMMA 10. *Let G be $(\mathbf{A}, \geqq)$ generic over V. Suppose that in $V[G]$ $\mathbf{Q}$ is a finite support iteration of length ω_2. Suppose further that*:

1) *if $\alpha \in \omega_2$ is even then*

$$\mathbf{Q}_{\alpha+1} = \mathbf{Q}_\alpha{}^*\mathbf{K}(f_\alpha, \psi_G)$$

where

$$1 \Vdash_{\mathbf{Q}_\alpha} ''f_\alpha \in (\psi_G''\omega_1)^{*}''$$

2) *if $\alpha \in \omega_2$ is odd then*

$$\mathbf{Q}_{\alpha+1} = \mathbf{Q}_\alpha{}^*\mathbf{P}_\alpha \quad \textit{and}$$

$$1 \Vdash_{\mathbf{Q}_\alpha} ''\mathbf{P}_\alpha \textit{ is } F_\alpha\textit{-centred where } F_\alpha \in {}^\omega({}^\omega\omega \cap L)''$$

then $V[G] \mid= ''\mathbf{Q}$ has the countable chain condition''.

Proof. The proof of the countable chain condition will rely on the fact that there is a dense set of conditions with which it is somewhat easier to work. In order to isolate these define a condition $q \in \mathbf{Q}$ to be determined if:

3) if $\sigma \in$ support (q) and σ is even then

$$q \upharpoonright \sigma \Vdash_{\mathbf{Q}_\sigma} ''q(\sigma) = g_q^\sigma{}''$$

4) if $\sigma \in$ support (q) and σ is odd then

$$q \upharpoonright \sigma \Vdash_{\mathbf{Q}_\sigma}$$

$''q(\sigma) \in \mathbf{P}_\sigma^{m_q^\sigma}$ where $\mathbf{P}_\alpha = \cup \{\mathbf{P}_\alpha^n : n \in \omega\}$ witnesses that $\mathbf{P}_\alpha$ is F_α-centred'' and furthermore

$$q\ \sigma \Vdash_{\mathbf{Q}_\sigma} ''F(m_q^\sigma) = h_q^\sigma.$$

A standard argument by induction on the support shows that the set of determined conditions is dense in $\mathbf{Q}$. Hence if there is an uncountable antichain then there is one which consists of determined conditions.

Now suppose that $1 \Vdash_{\mathbf{A}} "\{q_\eta : \eta \in \omega_1\}$ is an antichain of determined conditions". In order to exploit the fact that G is a generic family, choose $p_\eta \in \mathbf{A}$ for $\eta \in \omega_1$ such that (p_η, q_η) is as similar to (p_ξ, q_ξ) as possible and p_η decides everything relevant about q_η. In particular:

5) $p_\eta \Vdash$"support $(q_\eta) = \check{\Sigma}_\eta$"

6) if $\sigma \in \Sigma_\eta$ and σ is even then

$$p_\eta \Vdash "g^\sigma_{q_\eta} = \check{g}^\sigma_\eta"$$

7) if $\sigma \in \Sigma_\eta$ and σ is odd then

$$p_\eta \Vdash "m^\sigma_{q_\eta} = \check{m}^\sigma_\eta" \quad \text{and} \quad p_\eta \Vdash "h^\sigma_{q_\eta} = \check{h}^\sigma_\eta".$$

By extending p_η if necessary we may also assume that if $\sigma \in \Sigma_\eta$ is even then

$$D(g^\sigma_\eta) \subseteq \Gamma_{p_\eta}.$$

Furthermore, it may be assumed that

8) $\{\Gamma_{p_\eta} : \eta \in \omega_1\}$ form a Δ-system with root Γ

9) $\{\Sigma_\eta : \eta \in \omega_1\}$ form a Δ-system with root Σ

10) $n_{p_\eta} = k$ for $\eta \in \omega_1$

11) if $\{\eta, \zeta\} \subseteq \omega_1$ then there is $\varphi_{\eta,\zeta} : \Gamma_{p_\eta} \to \Gamma_{p_\zeta}$ such that:

a) $\varphi_{\eta,\zeta} \upharpoonright \Gamma = \mathrm{id}_\Gamma$

b) if $\alpha \in \beta$ then $\varphi_{\eta,\zeta}(\alpha) \in \alpha_{\eta,\zeta}(\beta)$

c) if $\eta = \zeta$ then $\varphi_{\eta,\zeta} = \mathrm{id}_{\Gamma_\eta}$

d) if $\eta \in \zeta$ and $m \in k$ and $\alpha \in \Gamma_{p_\eta}$ then

$$p_\eta(m, \alpha) = p_\zeta(m, \varphi_{\eta,\zeta}(\alpha)).$$

Let $\{\sigma_0, \dots, \sigma_j\}$ enumerate Σ in increasing order. We may also assume that:

12) if $i \in j + 1$ and σ_i is even and $\eta \in \zeta$ then

$$D(g^{\sigma_i}_\zeta) = \varphi_{\eta,\zeta}"D(g^{\sigma_i}_\eta)$$

and if $\alpha \in D(g^{\sigma_i}_\eta)$ then

$$g^{\sigma_i}_\eta(\alpha) = g^{\sigma_i}_\zeta(\varphi_{\eta,\zeta}(\alpha))$$

13) if $i \in j + 1$ and σ_i is odd and $\eta \in \omega_1$ then $m^{\sigma_i}_\eta = m_i$ and $h^{\sigma_i}_\eta \upharpoonright l_i + 1 = h_i$ will now be determined. Define l_i inductively as follows. Set $l_j = 2$. If l_i has been defined, $i \in j$ and σ_i is odd, then set

$$l_{i-1} = h_i(l_i).$$

If σ_i is even then set

$$l_{i-1} = (l_i - 1)(2^{k-|\Gamma_\eta|}) + 1.$$

(Notice that by (12) $|\Gamma_\eta|$ is independent of η.)

Now let $p = \cup \{p_i; i \in l_{-1}\}$. From (8) and (12) it follows that $p \in \mathbf{A}$. The condition p has been constructed by amalgamating so many similar conditions that, as will be shown, two of them must force two members of the antichain to be compatible. In particular define $r_i \in \mathbf{A} * \mathbf{Q}$ and $\Omega_i \in [\Omega_{i-1}]^{l_i}$ inductively for $i \in j + 1$ as follows.

We define

$$r_0 = (p, \cup \{q_i \upharpoonright \sigma_0; i \in l_{i-1}\}) \quad \text{and} \quad \Omega_0 = l_{-1}.$$

Now if r_i has been defined so that

$$r_i \in \mathbf{A} * \mathbf{Q}_{\sigma_i} \quad \text{and} \quad |\Omega_i| = l_{i-1}$$

then proceed as follows.

If σ_i is even find $\hat{r}_i \leqq r_i$ such that $\hat{r}_i \in \mathbf{A} * \mathbf{Q}_{\sigma_i}$ and

$$\hat{r}_i \Vdash \text{"}f_{\sigma_i}(\psi_G(\gamma)) \cap k = H^i_\gamma \quad \text{for each } \gamma \in \cup \{\Gamma_{p_\lambda} : \lambda \in \Omega_i\}.$$

Then choose $\Omega_{i+1} \in [\Omega_i]^{l_{i+1}}$ such that if $\{\lambda, \lambda'\} \in [\Omega_{i+1}]^2$ and $\gamma \in \Gamma_{p_\lambda}$ then

14) $n \in H^i_\gamma$ if and only if $n \in H^i_{\varphi_{\lambda,\lambda'}(\gamma)}$. Then

$$\hat{r}_i \Vdash \text{"} \cup \{g^{\sigma_i}_\eta; \eta \in \Omega_{i+1}\} \in \mathbf{K}(f_{\sigma_i}, \psi_G)\text{"}.$$

To see this note that by (11) and (12)

$$g_i = \cup \{g^{\sigma_i}_\eta; \eta \in \Omega_{i+1}\}$$

is a function. Furthermore, if $\{\alpha, \beta\} \in [D(g_i)]^2$ then there are λ and λ' such that

$$\{\varphi_{\lambda,\lambda'}(\alpha), \beta\} \subseteq \Gamma_p.$$

But then, since

$$\hat{r}_i \Vdash \text{"}g^{\sigma_i}_{\lambda'} \in \mathbf{K}(f_{\sigma_i}, \psi_G)\text{"}$$

it follows that

$$f_{\sigma_i}(\psi(\varphi_{\lambda,\lambda'}(\alpha)) \cap X^{g_i}_{\varphi_{\lambda,\lambda'}(\alpha),\beta} = f_{\sigma_i}(\psi(\beta)) \cap X^{g_i}_{\varphi_{\lambda,\lambda'}(\alpha),\beta}.$$

(Note that $X^{g_i}_{\varphi_{\lambda,\lambda'}(\alpha),\beta}$ can be determined exactly by using (6) and the fact that

$$p \Vdash \text{"}\psi(\rho) \cap \psi(\rho') \subseteq k\text{"}$$

for $\{\rho, \rho'\} \in [\Gamma_p]^2$). From (15) it now follows that

$$f_{\sigma_i}(\psi(\alpha)) \cap X^{g_i}_{\varphi_{\lambda,\lambda'}(\alpha),\beta} = f_{\sigma_i}(\psi(\beta)) \cap X^{g_i}_{\varphi_{\lambda,\lambda'}(\alpha),\beta}.$$

But from (12) it follows that

$$f_{\sigma_i}(\psi(\alpha)) \cap X^{g_i}_{\alpha,\beta} = f_{\sigma_i}(\psi(\beta)) \cap X^{g_i}_{\alpha,\beta}$$

and hence

$$\hat{r}_i \Vdash ``\sigma_i \in \mathbf{K}(f_{\sigma_i}, \psi_G).$$

If $\hat{r}_i = (\hat{p}, \hat{q})$ let

$$\bar{r}_i = (\hat{p}, \hat{q} \cup \{(\sigma_i, \check{g}_i)\}).$$

If, on the other hand, σ_i is odd then

$$p \Vdash ``F_{\sigma_i}(m_i, l_i) = l_{i-1}".$$

Hence there is $\hat{r}_i \leqq r_i$ such that $\hat{r}_i \in \mathbf{A} * \mathbf{Q}_{\sigma_i}$ and there is $\Omega_{i+1} \in [\Omega_i]^{l_i}$ such that

$$\hat{r}_i \Vdash ``\{q_\lambda(\sigma_i):\lambda \in \check{\Omega}_{i+1}\} \text{ is centred}".$$

Hence it may be assumed that there is a term q_i such that

$$\hat{r}_i \Vdash ``(\forall\lambda \in \check{\Omega}_{i+1})(q_\lambda(\sigma_i) \geqq q_i)".$$

If $\hat{r}_i = (\hat{p}, \hat{q})$ then let

$$\bar{r}_i = (\hat{p}, \hat{q} \cup \{(\sigma_i, q_i)\}).$$

In either case let s_i be a $\mathbf{A} * \mathbf{Q}_{\sigma_i}$ term for

$$\cup\{q_\lambda \restriction (\sigma_{i+1} \setminus \sigma_i):\lambda \in \Omega_{i+1}\}$$

and let r_i be the element of $\mathbf{A} * \mathbf{Q}_{\sigma_{i+1}}$ corresponding to $(\hat{r}_i, s_i)$ under the canonical map from $\mathbf{A} * \mathbf{Q}_{\sigma_i} * \mathbf{Q}^{\sigma_i,\sigma_{i+1}}$ to $\mathbf{A} * \mathbf{Q}_{\sigma_{i+1}}$.

It is clear from the construction that $|\Omega j| = 2$. Also, if $r_j = (p^*, q^*)$ then

$$p^* \Vdash_{\mathbf{A}} ``q^* \leqq q_\lambda \text{ for } \lambda \in \Omega j".$$

This shows that $\mathbf{Q}$ has the countable chain condition in $V[G]$.

THEOREM 11. *If* ZF *is consistent then so are* ZFC *and* $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ *and the existence of an uncountable strong-Q-sequence.*

Proof. Let $V \Vdash$ GCH and let G be $(\mathbf{A}, \leqq)$ generic over V. In $V[G]$ set up an iteration $\mathbf{Q}$ as in Lemma 10. Do the usual sort of enumeration to ensure that $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ will hold and also that if $f \in \psi''_G\omega_1$ then $f_\alpha = f$ for some $\alpha \in \omega_2$. This will ensure that if H is $\mathbf{Q}$ generic over $V[G]$ then $\psi''_G\omega_1$ will be an uncountable strong-Q-sequence in $V[G][H]$.

3. Examples of $\sigma - n$-linked partial orders. Note that Theorems 11 and 6 show that $\mathrm{MA}_{\omega_1}({}^{\omega}({}^{\omega}\omega \cap L))$ does not imply MA_{ω_1} for σ-linked partial orders. In particular, Proposition 8 shows that MA_{ω_1} for σ-centred partial orders and for probability measure algebras does not imply MA_{ω_1} for

σ-linked partial orders. Hence the partial order of Lemma 5 is an example of a σ-linked but not σ-centred partial order which is demonstrably different from the partial order associated with any probability measure algebra. It turns out that this partial order can be easily modified to display another unusual property.

THEOREM 12. *For each $n \in \omega$ there is a partial order which is $\sigma - n$-linked but not $\sigma - n + 1$-linked. Moreover this partial order can be as small as the least cardinality of a non-meager set of reals.*

Proof. Let $(T, \leqq_T)$ be the complete ω-ary tree of height ω.

For $t \in T$ let $\mathscr{E}_t = \{b \subseteq T;\ b$ is a maximal branch through T and $t \in b\}$.

Then the branches of T under the topology generated by $\{\mathscr{E}_t;\ t \in T\}$ are homeomorphic to the irrationals. Choose $\mathscr{C}$, a collection of maximal branches through T, such that $\mathscr{C}$ is not meagre.

Let $\mathbf{P}_n$ consist of those pairs $(S, \mathscr{A})$ such that:

1) S is a finite initial segment of T.

2) $\mathscr{A} \in [\mathscr{C}]^{>\aleph_0}$.

3) Each node of S branches at most n-times in S.

4) If $\{b_0, b_1\} \in [\mathscr{A}]^2$ then $b_0 \cap S \neq b_1 \cap S$.

Define $(S, \mathscr{A}) \leqq (S, \mathscr{A}')$ if and only if S is an end extension of S' and $\mathscr{A} \supseteq \mathscr{A}'$.

The argument of Lemma 5 shows that $(\mathbf{P}_n, \leqq)$ is $\sigma - n$-linked. To see that it is not $\sigma - n + 1$-linked suppose that $\mathbf{P}_n$ is the union of countably many $n + 1$-linked subsets. Then in particular

$$\{ (S, \mathscr{A}) \in \mathbf{P}_n;\ S = 0\} = \cup \{\mathbf{Q}_i;\ i \in \omega\}$$

where each $\mathbf{Q}_i$ is $n + 1$-linked. Let

$$\mathscr{B}_i = \cup \{\mathscr{A}:(0, \mathscr{A}) \in \mathbf{Q}_i\}.$$

Then $\mathscr{B}_{i_0}$ is not meager for some $i_0 \in \omega$ since

$$\cup \{\mathscr{B}_i : i \in \omega\} = \mathscr{C}.$$

Hence there is some $t \in T$ such that $\mathscr{B}_{i_0} \cap \mathscr{E}_t$ is dense in $\mathscr{E}_t$. Now let $\{t_i : i \in n + 1\}$ be distinct successors of t. Then, for each $i \in n + 1$, since $\mathscr{B}_{i_0} \cap \mathscr{E}_{t_i} \neq 0$ it is possible to choose $(0, \mathscr{A}_i) \in \mathbf{Q}_{i_0}$ such that $\mathscr{A}_i \cap \mathscr{E}_{t_i} \neq 0$. But if $(S, \mathscr{A}) < (0, \mathscr{A}_i)$ for each $i \in n + 1$ then

$$\{t_i;\ i \in n + 1\} \subseteq S.$$

This contradicts the fact that S must be only n-branching.

Finally note that some restriction on the cardinality of the counter example in Theorem 12 is necessary. For example there is no absolute example of a $\sigma - n$-linked but not $\sigma - n + 1$-linked partial order of size $\aleph_1$ since MA_{ω_1} implies that all such partial orders are, in fact, σ-centred.

4. Generalizations to BACH. It has already been mentioned that it is possible to show that MA_{ω_1} for σ-centred partial orders does not imply MA_{ω_1} for σ-linked partial orders by finding a model where MA_{ω_1} for σ-centred partial orders holds but in which the real line is the union of fewer than $2^{\aleph_0}$ many measure zero sets. The problem with this proof is that it does not generalize to the next higher cardinal since there is no appropriate generalization of the measure algebra to the next higher cardinal (such a generalization, for example, would have to be countably complete). In this section it will be shown that the arguments of Sections 1 and 2 easily generalize up one cardinal.

The generalization of MA to be considered in this section is J. Baumgartner's BACH [4][5]. BACH says that CH holds and that if $\mathbf{P}$ is a partial order satisfying the following three properties:

1) $\mathbf{P}$ is countably closed.

2) Any two compatible elements of $\mathbf{P}$ have a greatest lower bound (i.e., $\mathbf{P}$ is well-met).

3) $\mathbf{P}$ is $\aleph_1$-linked (i.e., $\mathbf{P}$ is the union of $\aleph_1$ many linked subsets) and $\mathscr{D}$ is a collection of less than $2^{\aleph_1}$ many dense subsets of $\mathbf{P}$ then it is possible to find a generic set meeting each member of $\mathscr{D}$. BACH for σ-centred partials orders is the same statement as BACH except that in (3) "linked" is changed to "centred". It will now be indicated how to modify the arguments of Sections 1 and 2 to show that BACH for σ-centred partial orders does not imply BACH.

It is easy to see that the results of Section 2 generalize to show that BACH for σ-centred partial orders is consistent with existence of a strong-Q-sequence in $\mathscr{P}(\omega_1)/[\omega_1]^{<\aleph_1}$ of size $\aleph_2$. In fact, the countable closure of the analog of $\mathbf{A} * \mathbf{Q}$ can be used to significantly simplify the proof in this case. Hence it suffices to show that BACH implies that there are no strong-Q-sequences of size $\aleph_2$ in $\mathscr{P}(\omega_1)/[\omega_1]^{<\aleph_1}$.

Lemmas 1, 2 and 4 of Section 1 generalize to the next cardinal in a straightforward way. The analog of the partial order of Lemma 3 is easily seen to be countably closed and well met in addition to being σ-centred.

The generalization of Lemma 5 will require some modifications to the proof, however. By using the analog of Lemma 4, as in Section 1, it suffices to prove the following.

LEMMA 13. *Assume* BACH *and* $2^{\aleph_1} > \aleph_2$. *Let* $(T, \leqq_T)$ *be a normal* $\aleph_0$ *branching tree of height* ω_1 *and* $\mathscr{B}$ *be an* $\aleph_1$ *sized collection of branches through* $(T, \leqq_T)$ *such that*

$$|\{b \in \mathscr{B}: t \in b\}| = \aleph_2 \quad \textit{for each } t \in T.$$

Then there is an initial segment of T, S *such that*:

a) $(S, \leqq_T \cap S \times S)$ *has height* ω_1.

b) $(S, \leqq_T \cap S \times S)$ *is binary* (*i.e., each node splits at most twice*).

c) $|\{b \in \mathscr{B}; |b \cap S| = \aleph_1\}| > \aleph_1$.

Proof. Let $\mathbf{P}$ be defined by $p \in \mathbf{P}$ if and only if $p = (S_p, \mathscr{B}_p)$ where

1) $S_p \in [T]^{<\aleph_1}$ and S_p is an antichain.

2) $\mathscr{B}_p \in [\mathscr{B}]^{<\aleph_1}$.

3) There is a bijection $\varphi: S_p \to \mathscr{B}_p$ satisfying

$$(\forall S \in S_p)(\varphi(S) \cap S_p = \{S\}).$$

This bijection is of course canonical and need not be named. The idea here is that the T_p of Lemma 5 corresponds to

$$S_p = \{t \in T: t \leqq_T s \text{ for some } s \in S_p\}.$$

Hence we must include the following condition:

4) $(S_p, \leqq_T \cap S_p \times S_p)$ is binary.

Since, in Section 1, $p \leqq q$ entailed that T_p be an end extension of T_q the elements of T_q without an extension to the top level might as well not be there. This results in difficulties when trying to prove that the partial order is well-met. Hence define $p \leqq q$ if and only if:

5) $\mathscr{B}_p \supseteq \mathscr{B}_q$

6) for each $t \in S_q$ there is some $t' \in S_p$ such that $t \leqq_T t'$.

It is easy to see that the necessary sets are dense and that if G is generic for $(\mathbf{P}, \leqq)$ then

$$\{t \in S; (\exists p \in G)(\exists s \in S_p)(t \leqq_T s)\}$$

is a binary subtree. The argument of Lemma 5 combined with CH shows that $(\mathbf{P}, \leqq)$ is $\aleph_1$-linked.

To see that $(\mathbf{P}, \leqq)$ is countably closed let $\{p_n; n \in \omega)$ be a descending sequence in $\mathbf{P}$. Let

$$\mathscr{B}_\omega = \cup \{\mathscr{B}_{p_n}: n \in \omega\}.$$

For each b in $\mathscr{B}_\omega$ choose $t(b) \in b$ such that $t(b) \geqq_T s$ for each s in $\cup \{S_{p_n}; n \in \omega\}$. Let

$$S_\omega = \{t(b); b \in \mathscr{B}_\omega\}.$$

It is easy to check that $(S_\omega, \mathscr{B}_\omega) \in \mathbf{P}$. Also,

$$p_n \geqq (S_\omega, \mathscr{B}_\omega) \quad \text{for each } n \in \omega.$$

To see this it suffices to check (6) and this is easily seen to follow from (3).

Finally, it must be verified that $(\mathbf{P}, \leqq)$ is well-met. Let p and q be compatible conditions. Let $\mathscr{B} = \mathscr{B}_p \cup \mathscr{B}_q$ and let S' be the maximal elements of $S_p \cup S_q$. Let

$$S'' = \{s \in S'; (\exists\{b_0^s, b_1^s\} \in [\mathscr{B}]^2)(b_0^s \cap S' = b_1^s \cap S' = \{s\})\}.$$

If $s \in S''$ and let s_0 and s_1 be the minimal elements of b_0^s and b_1^s respectively such that s_0 and s_1 are incomparable. Let

$$S = (S' \setminus S'') \cup \{s_i;\ i \in 2 \text{ and } s \in S''\}.$$

Then $(S, \mathscr{B})$ is the greatest lower bound of p and q.

It is easy to generalize the results of Section 3 to the next higher cardinal by using CH. In particular, the topology on the branches of the complete ω-ary tree of height ω_1 is the same as the countable support product on ${}^{\omega_1}\omega$. The Baire category theorem generalizes in this case.

5. $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ and $\mathscr{P}(\omega_1)/[\omega_1]^{<\aleph_0}$. In this section it will be shown that the model of Section 2 is at least an approximation to the sort of model which would be required to prove the consistency of the assertion that $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ is isomorphic to $\mathscr{P}(\omega_1)/[\omega_1]^{<\aleph_0}$. In particular it will be shown that in the model of Section 2 there is an ideal $\mathscr{I}$ on ω and an ideal $\mathscr{J}$ on ω_1 such that $\mathscr{P}(\omega_1)/\mathscr{J}$ is isomorphic to $\mathscr{P}(\omega)/\mathscr{I}$. If the ideal $\mathscr{J}$ happens to contain a co-countable set or is the product of countably many ultrafilters on ω_1 then the result is trivial. However the ideal $\mathscr{J}$ constructed will not be of this form and in fact can be thought of as the ideal on $\omega_1 \times \omega$ generated by all the sets bounded by a function from ω_1 to ω. From now on let $\mathscr{J}$ denote this ideal. Notice that $\mathscr{P}(\omega_1)/\mathscr{J}$ has cardinality $2^{\aleph_1}$ and so if $2^{\aleph_1} \neq 2^{\aleph_0}$ then there is no ideal $\mathscr{I}$ on ω such that $\mathscr{P}(\omega)/\mathscr{I}$ is isomorphic to $\mathscr{P}(\omega_1)/\mathscr{J}$. The result will follow from the following general result.

LEMMA 14. *Let A and B be Boolean algebras. Suppose that $\mathscr{A} \subseteq A$ and $\mathscr{B} \subseteq B$ are strong-Q-sequences. Suppose furthermore that there is a bijection $\Phi: \mathscr{A} \to \mathscr{B}$ such that $A \upharpoonright a$ is isomorphic to $B \upharpoonright \Phi(a)$ for each $a \in A$. Then $A/\bar{\mathscr{A}}$ is isomorphic to $B/\bar{\mathscr{B}}$ where*

$$\bar{\mathscr{A}} = \{x \in A;\ (\forall a \in \mathscr{A})(a \cap x = 0)\}$$

and the ideal $\bar{\mathscr{B}}$ is defined similarly.

Proof. For each $a \in \mathscr{A}$ let

$$\psi_a : A \upharpoonright a \to B \upharpoonright \Phi(a)$$

be an isomorphism.

If $x \in B$ let $f_x \in \mathscr{B}^*$ be defined by $f_x(b) = b \cap x$. If $f \in \mathscr{B}^*$ let $g_f \in \mathscr{A}^*$ be defined by

$$g_f(a) = \psi_a^{-1}(f(\Phi(a))).$$

Define $\theta: B/\bar{\mathscr{B}} \to A/\bar{\mathscr{A}}$ by the rule

$$\theta(x) \cap a = g_{f_x}(a) \quad \text{for each } a \in \mathscr{A}.$$

This is easily seen to be an isomorphism.

THEOREM 15. *If* ZF *is consistent then so is* ZFC *and $\mathscr{P}(\omega_1)/\mathscr{J}$ is isomorphic to $\mathscr{P}(\omega)/\mathscr{I}$ for some ideal $\mathscr{I}$.*

Proof. Use the model of Section 2 and Lemma 14. Also note that

$$\{ (\alpha + \omega) \setminus \alpha : \alpha \in \omega_1 \text{ is a limit}\}$$

is an uncountable strong-Q-sequence in $\mathscr{P}(\omega_1)/[\omega_1]^{<\aleph_0}$ consisting of countable sets. Hence any bijection between it and a strong-Q-sequence in $\mathscr{P}(\omega)/[\omega]^{<\aleph_0}$ will satisfy the hypothesis of Lemma 14. The ideal defined in Lemma 14 clearly is isomorphic to $\mathscr{J}$.

References

1. M. Bell, *Two Boolean algebras with extreme cellular and compactness properties*, to appear in Can. J. Math.
2. C. Herink, *Some applications of iterated forcing*, Ph.D. dissertation, University of Wisconsin, Madison.
3. A. Szymanski and H. X. Zhou, preprint.
4. F. Tall, *Some applications of a generalized Martin's Axiom*, to appear in Trans. A.M.S.
5. W. Weiss, *Versions of Martin's Axiom*, to appear in The Handbook of Set Theoretic Topology.

York University,
Downsview, Ontario