Relativity in Fundamental Astronomy
Proceedings IAU Symposium No. 261, 2009
S. A. Klioner, P. K. Seidelman & M. H. Soffel, eds.
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doi:10.1017/S1743921309990202

Time ephemeris and general relativistic scale factor

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Abstract. Time ephemeris is the location-independent part of the transformation formula relating two time coordinates such as TCB and TCG (Fukushima 1995). It is computed from the corresponding (space) ephemerides providing the relative motion of two spatial coordinate origins such as the motion of geocenter relative to the solar system barycenter. The time ephemerides are inevitably needed in conducting precise four dimensional coordinate transformations among various spacetime coordinate systems such as the GCRS and BCRS (Soffel *et al.* 2003). Also, by means of the time average operation, they are used in determining the information on scale conversion between the pair of coordinate systems, especially the difference of the general relativistic scale factor from unity such as L_C. In 1995, we presented the first numerically-integrated time ephemeris, TE245, from JPL's planetary ephemeris DE245 (Fukushima 1995). It gave an estimate of L_C as $1.4808268457(10) \times 10^{-8}$, which was incorrect by around 2×10^{-16}. This was caused by taking the wrong sign of the post-Newtonian contribution in the final summation. Four years later, we updated TE245 to TE405 associated with DE405 (Irwin and Fukushima 1999). This time the renewed vale of L_C is $1.48082686741(200) \times 10^{-8}$ Another four years later, by using a precise technique of time average, we improved the estimate of Newtonian part of L_C for TE405 as $1.4808268559(6) \times 10^{-8}$ (Harada and Fukushima 2003). This leads to the value of L_C as $L_C = 1.48082686732(110) \times 10^{-8}$. If we combine this with the constant defining the mean rate of TCG-TT, $L_G = 6.969290134 \times 10^{-10}$ (IAU 2001), we estimate the numerical value of another general relativistic scale factor $L_B = 1.55051976763(110) \times 10^{-8}$, which has the meaning of the mean rate of TCB-TT. The main reasons of the uncertainties are the truncation effect in time average and the uncertainty of asteroids' perturbation. The former is a natural limitation caused by the finite length of numerical planetary ephemerides and the latter is due to the uncertainty of masses of some heavy asteroids. As a compact realization of the time ephemeris, we prepared HF2002, a Fortran routine to compute approximate harmonic series of TE405 with the RMS error of 0.446 ns for the period 1600 to 2200 (Harada and Fukushima 2003). It is included in the IERS Convention 2003 (McCarthy and Petit 2003) and available from the IERS web site; http://tai.bipm.org/iers/conv2003/conv2003_c10.html.

Keywords. general relativity, ephemerides, reference systems, time

1. Concept of Time Ephemeris

Consider a four dimensional spacetime coordinate transformation, $x^\mu = f^\mu (X^\alpha)$, where x^μ is the four dimensional coordinates of an event in a certain four dimensional coordinate system, which we call the background coordinate system, while X^α is the four dimensional coordinates of the same event in another four dimensional coordinate system, which we call the target coordinate system. If we expand the above coordinate transformation around the space coordinate origin of the target coordinate system, its time part is written as

$$t = f(T) + f_k(T)X^k + \cdots \tag{1.1}$$

https://doi.org/10.1017/S1743921309990202 Published online by Cambridge University Press

where t and T are the time coordinates of the background and target coordinate systems, respectively. Then, we define the time ephemeris as a function expressed (Fukushima 1995) as

$$\Delta\tau(t) \equiv t - f^{-1}(t). \tag{1.2}$$

In case of the Earth, the BCRS is the background coordinate system, the GCRS is the target coordinate system, TCB is t, and TCG is T (IAU 1992, Seidelmann and Fukushima 1992, IAU 2001). Namely the above time-time relation is rewritten (Fukushima 1995) as

$$\mathrm{TCB} - \mathrm{TCG} = \Delta\tau_E(\mathrm{TCB}) + \frac{\mathbf{v}_E \cdot \mathbf{X}}{c} + \cdots \tag{1.3}$$

where c is the speed of light in vacuum.

2. General Relativistic Scale Factor

The time average of the time ephemerides has an important meaning in the unit conversion between different coordinate systems (Fukushima *et al.* 1986). To make the description more understandable, hereafter, we deal with the case of the Earth only.

Assume that the background and target coordinate systems use different unit systems in length and in time as $[\mathrm{m}_B, \mathrm{s}_B]$ for the BCRS and $[\mathrm{m}_G, \mathrm{s}_G]$ for the GCRS. Adopt a convention such that the numerical values of c are the same in both coordinate systems as

$$c = 299792458\ \mathrm{m}_B/\mathrm{s}_B = 299792458\ \mathrm{m}_G/\mathrm{s}_G. \tag{2.1}$$

then the ratios of the length units and time units must be the same as

$$\frac{\mathrm{m}_G}{\mathrm{m}_B} = \frac{\mathrm{s}_G}{\mathrm{s}_B}. \tag{2.2}$$

The latter quantity is nothing but the time average of the differential ratio of the time scales, which is rewritten (Fukushima *et al.* 1986) as

$$\frac{\mathrm{s}_G}{\mathrm{s}_B} = \frac{< d\mathrm{TCG} >}{< d\mathrm{TCB} >} = 1 - L_C, \tag{2.3}$$

where

$$L_C = \left\langle \frac{d\Delta\tau_E}{dt} \right\rangle, \tag{2.4}$$

is the general relativistic scale factor of the Earth. Once this factor is obtained, the numerical values of all the physical quantities measured in each coordinate system have the proportional relations as

$$\frac{M_G}{M_B} = \frac{R_G}{R_B} = \frac{P_G}{P_B} = 1 - L_C, \tag{2.5}$$

where M, R, and P denote the mass, the radius, and the period of any kind.

3. Computation of Time Ephemeris

The relation between the two time scales are computed by assuming that the space coordinate origin of the target coordinate system, the geocenter in the case of the Earth, follows a geodesic in the background coordinate system, the solar system barycentric coordinate system in the case of the Earth. Adopt Einstein's general theory of relativity

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as the general relativistic theory to be based. Then, the time development equation of $\Delta\tau$ is obtained from the equation of geodesic, or that of proper time more specifically, as

$$\frac{d\Delta\tau_E}{dt} = \frac{\mathbf{v}_E^2 + U_E}{2c^2} + \frac{\mathbf{v}_E^4 + 12\mathbf{v}_E^2 U_E - 4U_E^2 - 32\mathbf{v}_E \cdot \mathbf{w}_E - 4W_E}{8c^4} + \cdots \tag{3.1}$$

where $\vec{v}_E$ is the velocity of the geocenter, U_E is the Newtonian gravitational potential acting on the Earth (and excluding the self-gravitational potential of the Earth itself), and $\vec{w}_E$ and W_E represent the post-Newtonian contributions in general (Soffel *et al.* 2003). In the EIH metric, they are expressed (Fukushima 1995) as

$$U_E \equiv \sum_{J\neq E} U_{EJ}, \quad \mathbf{w}_E \equiv \sum_{J\neq E} U_{EJ}\mathbf{v}_J,$$

$$W_E \equiv \sum_{J\neq E} U_{EJ}\left[4\mathbf{v}_J^2 - \left(\frac{\mathbf{r}_{EJ}\cdot\mathbf{v}_J}{r_{EJ}}\right)^2 + \sum_{K\neq J} U_{JK}\left(2 + \frac{\mathbf{r}_{EJ}\cdot\mathbf{r}_{JK}}{r_{JK}^2}\right)\right], \tag{3.2}$$

where

$$U_{JK} \equiv \frac{GM_K}{r_{JK}}, \quad \mathbf{r}_{JK} \equiv \mathbf{x}_J - \mathbf{x}_K, \quad r_{JK} \equiv |\mathbf{r}_{JK}|. \tag{3.3}$$

The right hand side of Equation (3.1) is independent on $\Delta\tau_E$ itself. Thus, it is a pure function of t if the motion of major celestial bodies in the solar system is known, i.e. if the planetary/lunar ephemerides is provided. In this sense, we may obtain the time ephemeris simply by the quadrature of the right hand side of the above equation.

4. Realizations of Time Ephemeris

There have been several analytical time ephemerides. Moyer's pioneer works (Moyer 1981a, Moyer 1981b) are based on the Keplerian approximation of planetary/lunar orbits. All of the later computations (Hirayama *et al.* 1987, Fairhead *et al.* 1988, Fairhead and Bretagnon 1990) are based on the analytical planetary ephemeris, VSOP82 (Bretagnon 1982) and the analytical lunar ephemeris, ELP2000 (Chapront-Touze and Chapront 1982). Since the ephemerides are expressed as Fourier series, it is easily to conduct the quadrature.

On the other hand, numerical time ephemerides are obtained twice (Fukushima 1995, Irwin and Fukushima 1999), all of which are based on the JPL numerical ephemerides, DE102, DE200, DE245, and DE405 (Newhall 1989, Standish *et al.* 1992, Standish 1998a, Standish 1998b). In this case, the quadrature was executed by the Romberg method (Press *et al.* 2007). Taking the same number of DE ephemerides used, we named the time ephemerides as TE102, TE200, TE245, and TE405, respectively.

The numerical representation of time ephemerides are, as the same as in case of numerical lunar/planetary ephemerides, usually the resulting numerical tables themselves or their Chebyshev polynomial representation. These are appropriate for fast computation. However, the periodic features are difficult to find out. Also, the full implementation requires an expertise on its installation and some disk storages.

In order to complement these weak points, we presented the harmonic decomposition of TE405, the latest time ephemeris of the Earth (Harada and Fukushima 2003). The used approach is a nonlinear method of harmonic analysis (Harada 2003), an excerpt of which is reported in Appendix B of our analysis of the planetary precession derived from

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Table 1. Main Terms of Fourier Series Expression of TE405

S (ns)	C (ns)	Period (days)
+505079.2018	−1551857.1407	365.2652622182
+21856.7326	−23134.7679	365.22102337
+20733.1083	−8526.5271	398.88401884
−11108.6620	−8369.7220	182.62982594
−3405.2830	−3354.5797	4333.21415

Note. Listed are the largest five Fourier terms of the harmonic decomposition of TE405 (Harada and Fukushima 2003).

DE405 (Harada and Fukushima 2004). The expression is in the form

$$\Delta\tau_E \approx \sum_{j=0}^{2} P_j \xi^j + \sum_{j=1}^{J} \left[S_j \sin(\omega_j \xi_j) + C_j \cos(\omega_j \xi_j)\right] + \sum_{j=1}^{K} \xi_j \left[S'_j \sin(\omega_j \xi_j) + C'_j \cos(\omega_j \xi_j)\right], \tag{4.1}$$

where

$$\xi \equiv \frac{\mathrm{JD} - 2414949.0}{54749.25}, \tag{4.2}$$

and P_j, S_j, C_j, ω_j, S'_j, and C'_j are certain constants. For the full period of TE405, i.e. from 1600 until 2200, the RMS of the residual of the approximation is 0.446 ns and the absolute maximum difference is 2.95 ns. For the shorter period from 1960 to 2020, the maximum difference reduces to 1.58 ns and most of the differences are less than 1 ns.

Table 1 shows the main Fourier terms. The full result contains a quadratic polynomial, 463 Fourier terms, and 36 mixed secular terms. Namely $J = 473$ and $K = 36$ in the above expression. The published article (Harada and Fukushima 2003) contains only the full coefficients of S'_j and C'_j, some of P_j, and the first five coefficients of Fourier terms.

For the full expression, refer the Fortran routine HF2002 and its parameter file included in the IERS Convention 2003 (McCarthy and Petit 2003). They are available from the IERS web site;

`http://tai.bipm.org/iers/conv2003/conv2003_c10.html`

5. Determination of General Relativistic Scale Factor

Let us return to the issue of scale factor. The factor L_C is split into the sum of three parts;

$$L_C = L_C^N + L_C^{PN} + L_C^A, \tag{5.1}$$

where the superscripts N and PN denote the Newtonian and the post-Newtonian contribution by the Earth's velocity in the BCRS and by the Newtonian gravitational potential of the Sun and major planets, while the superscript A does the Newtonian effect by asteroids (Fukushima 1995). Table 2 shows the dilated contribution of the first two parts for the case of TE245. Note that the asteroid part is too small to be listed.

The numerical value of L_C^N significantly differ ephemeris by ephemeris. See Table 3. The uncertainties shown here are basically caused by the finiteness of the effective period of the lunar/planetary ephemeris used. In fact, all the practical ephemerides whether being numerical or analytical are limited. Let us explain this situation more plainly.

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Table 2. Contribution to General Relativistic Scale Factor, L_C

Source	Contribution (10^{-17})
Sun	987062583
velocity	493530342
Jupiter	182856
Saturn	29647
Moon	14191
Venus	2877
Uranus	2250
Neptune	1741
Mars	240
Mercury	171
post-Newtonian	11

Note. Listed are the contribution of each source to the value of L_C for the case of TE245 (Fukushima 1995). Note that all the contributions including the post-Newtonian one are positive. The asteroids' contribution, which is dropped from the list, is 0.45 in the unit of table.

Table 3. Estimated Values of Main Newtonian Parts of General Relativistic Scale Factor, L_C^N

L_C^N (10^{-17})	Time Ephemeris	Reference
1480826869.80±0.5	TE102	Fukushima 1995
57.13±0.5	TE200	Fukushima 1995
56.21±0.5	TE245	Fukushima 1995
55.94±1.0	TE405	Irwin and Fukushima 1999
55.90±0.6	TE405	Harada and Fukushima 2003

Assume that the ephemeris contain a very long period term of the frequency Ω. The associated Fourier terms are expanded as

$$\cos \Omega t \approx 1 - \frac{(\Omega t)^2}{2} + \cdots, \quad \sin \Omega t \approx \Omega t - \frac{(\Omega t)^3}{6} + \cdots, \tag{5.2}$$

Therefore, for the finite time period such as $|t| < T$, we cannot discriminate the cosine and sine terms of the frequency Ω with a constant offset, 1, and a linear trend, Ωt, if the leading residual terms, $(\Omega T)^2/2$ or $(\Omega T)^3/6$, are sufficiently small. See the detailed discussion in our reports (Fukushima 1995, Irwin and Fukushima 1999, Harada and Fukushima 2003).

On the other hand, the values of the last two parts do not differ significantly so that we may fix them by the value of TE245 (Fukushima 1995) as

$$L_C^{PN} = (10.97 \pm 0.01) \times 10^{-17}, \quad L_C^A = (0.45 \pm 0.50) \times 10^{-17}. \tag{5.3}$$

The uncertainty of the post-Newtonian term comes from that of the PPN parameters, β and γ. Meanwhile, the large uncertainty of the asteroid effect is due to their mass uncertainty.

At any rate, let us calculate the final value. Table 4 shows the summed value of L_C for TE405 based on the latest determination of L_C^N (Harada and Fukushima 2003). Using this, we calculate another scale factor, L_B, as

$$L_B \equiv L_C + L_G - L_C L_G = (1550519767.63 \pm 1.1) \times 10^{-17}, \tag{5.4}$$

which determines the mean rate of TCB-TT and becomes a key factor to convert the numerical values of physical quantities obtained from the astronomical observations in the

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Table 4. General Relativistic Scale Factors

Constant	Meaning	Value (10^{-17})	σ (10^{-17})
L_C^N	Main Newtonian Part of L_C	1480826855.90	0.60
L_C^{PN}	post-Newtonian Part of L_C	10.97	0.01
L_C^A	Asteroid Part of L_C	0.45	0.50
L_C	Mean Rate of TCB-TCG	1480826867.32	1.1
L_G	Mean Rate of TCG-TT	69692901.34	
L_B	Mean Rate of TCB-TT	1550519767.63	1.1

solar system and those determined from the experimental measurements at laboratories on the Earth. Here

$$L_G \equiv 69692901.34 \times 10^{-17}, \tag{5.5}$$

is a defining constant to specify the mean rate of TCG-TT (McCarthy and Petit 2003).

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