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ON UNIFORMLY DISTRIBUTED SEQUENCES OF INTEGERS AND POINCARÉ RECURRENCE III

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Let S be a semigroup contained in a locally compact Abelian group G. Let $\widehat{G}$ denote the Bohr compactification of G. We say that a sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ contained in S is Hartman uniform distributed on G if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(k_n) = 0,$$

for any character χ in $\widehat{G}$. Suppose that $(T_g)_{g\in S}$ is a semigroup of measurable measure preserving transformations of a probability space (X,β,μ) and B is an element of the σ-algebra β of positive μ measure. For a map $T : X \to X$ and a set $A \subseteq X$ let $T^{-1}A$ denote $\{x \in X : Tx \in A\}$. In an earlier paper, the author showed that if $\mathbf{k}$ is Hartman uniform distributed then

$$\lim_{M\to\infty} \frac{1}{M}\sum_{n=1}^{M} \mu\big(B \cap (T_{k_n})^{-1}B\big) \geqslant \mu(B)^2.$$

In this paper we show that $\geqslant$ cannot be replaced by $=$. A more detailed discussion of this situation ensues.

Let G be a locally compact Abelian group and let $\mathbf{k} = (k_n)_{n=1}^{\infty}$ be a sequence contained in a semigroup S contained in G. We say that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is Hartman uniform distributed if for each non-trivial character χ in the dual group $\widehat{G}$ of G we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(k_n) = 0.$$

For a set X let β denote a σ-algebra of its subsets and let μ be a probability measure defined on them. We say that a measurable map T from X to itself is measure preserving if for any element A of β, denoting by $T^{-1}A$ the set $\{x \in X : Tx \in A\}$, we have $\mu(T^{-1}A) = \mu(A)$ for all A in β. In [1] the following is shown.

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Theorem A. *Suppose that* $\mathbf{k} = (k_j)_{j=1}^{\infty}$ *is Hartman uniformly distributed on a locally compact Abelian group* G, *containing a semigroup* S *containing* $\mathbf{k}$. *Suppose that* $(T_g)_{g \in S}$ *is a semigroup of measurable measure preserving transformations of a probability space* (X, β, μ) *and that* B *is an element of* β *of positive* μ *measure. Then*

$$\lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} \mu\Big(B \cap (T_{k_n})^{-1} B\Big) \geqslant \mu(B)^2.$$

The existence of the limit is part of the conclusion to Theorem A. We however have the following theorem.

Theorem B. *The* $\geqslant$ *in the statement of Theorem A can't be replaced by* $=$.

Proof: Suppose otherwise and we shall specialise to the case $G = \mathbf{Z}$. Recall that a sequence of real numbers $(x_n)_{n=1}^{\infty}$ is said to be uniformly distributed modulo one, if for each interval I that is closed on the left and open on the right we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(\{x_n\}) = |I|.$$

Here $\{x\}$ denotes the fractional part of the real number x, χ_I denotes the characteristic function of the interval I and $|I|$ denotes its Lebesgue measure. As no ambiguity should arise, for a finite set F we denote its cardinality also by $|F|$. We say that a sequence of natural numbers $\mathbf{k} = (k_j)_{j=1}^{\infty}$ is uniformly distributed on $\mathbf{Z}$ if for each integer m in $\mathbf{N}$ and each integer l in $[0, m-1] \cap \mathbf{Z}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \Big| \{ j \in [0, N-1) \cap \mathbf{Z} : k_j \equiv l \bmod m \} \Big| = \frac{1}{m}.$$

In [**1**] it is shown that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is Hartman uniform distributed on $\mathbf{Z}$ if $\mathbf{k}$ is uniformly distributed on $\mathbf{Z}$ and for each irrational number α, the sequence $(k_n \alpha)_{n=1}^{\infty}$ is uniformly distributed modulo one. Also in [**1**] it is shown that there are many sequences with this property. Suppose (X, β, μ) is any probability space, $T : X \to X$ is any measurable, measure preserving transformation of X and B is a T invariant set in β in the sense that $T^{-1}B = B$. Then for any sequence of natural numbers $\mathbf{k} = (k_n)_{n=1}^{\infty}$ and any natural number M

$$\frac{1}{M} \sum_{n=1}^{M} \mu(B \cap T^{-k_n} B) = \mu(B),$$

for each $M \geqslant 1$. Hence if

$$\lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} \mu(B \cap (T^{k_n})^{-1} B) = \mu(B)^2. \tag{1}$$

then

$$\mu(B) = \mu(B)^2.$$

Therefore $\mu(B)$ is either zero or one. Now set $X = [0,1)$, μ is equal to the Lebesgue measure, $Tx = \langle x + 1/2 \rangle$ and $B = [0,(1/4)) \cup [(1/2),(3/4))$. Then T preserves μ and B is obviously T invariant while $\mu(B) = 1/2$. This is a contradiction. ∎

Recall that a measurable, measure preserving map $T : X \to X$ of a probability space (X, β, μ) is ergodic if given any B in β with $T^{-1}B = B$ then $\mu(B)$ is either zero or one. Plainly our example above works because T is not ergodic. It is unclear to the author whether Theorem A remains true with $\geqslant$ replaced by $=$ under the additional assumption that T is ergodic.

We need to establish some standard notation. We say that a statement is true μ almost everywhere if the subset of X on which it is true has full μ measure. We also say that two functions f and g are equivalent if $f - g = 0$ μ almost everywhere. Let $\|f\| = \left(\int_X |f|^2 d\mu\right)^{1/2}$ and let $L^2 = L^2(X, \beta, \mu)$ denote the space of equivalence classes for μ measurable functions such that the norm $\|f\|$ is finite. Given a sequence of functions $(f_N)_{N=1}^{\infty}$ defined on a probability space (X, β, μ) we say that $(f_N)_{N=1}^{\infty}$ converges to a function g defined on (X, β, μ) in L^2 norm if $\lim_{N\to\infty} \|f_N - g\| = 0$. We say that $(f_N)_{N=1}^{\infty}$ converges almost everywhere to g if $\mu\Big(\big\{x \in X : \lim_{N\to\infty} f_N(x) = g(x)\big\}\Big) = 1$. In [2] it is shown that if $\mathbf{k} = (k_n)_{n\geqslant 1}$ is Hartman uniformly distributed on $\mathbf{Z}$ then this is equivalent to the statement that if $f \in L^2(X, \beta, \mu)$ and $T : X \to X$ is measurable and measure preserving, then

$$\text{(2)} \qquad \lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f\left(T^{k_n} x\right) - \mathbf{E}(f \mid \mathcal{I}) \right\|.$$

Here $\mathbf{E}(f \mid \mathcal{I})$ is the projection of f onto the subspace of L^2 of T invariant functions. Suppose that instead of being Hartman uniform distributed on $\mathbf{Z}$ the sequence $\mathbf{k} = (k_n)_{n\geqslant 1}$ has the property that if $f \in L^2(X, \beta, \mu)$ and $T : X \to X$ is measurable and measure preserving then

$$\text{(3)} \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(T^{k_n} x\right) = \mathbf{E}(f \mid \mathcal{I}),$$

almost everywhere with respect to μ. Note that hypothesis (2) says that if f is in $L^2(X, \beta, \mu)$ then the sequence $\left(1/N \sum_{n=1}^{N} f\left(T^{k_n} x\right)\right)_{N=1}^{\infty}$ converges to $\mathbf{E}(f \mid \mathcal{I})(x)$ in L^2 norm and (3) says the convergence is almost everwhere. In general convergence

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almost everywhere does not necessarily imply convergence in norm, nor vice-versa. The hypothesis (3) does however imply the hypothesis (2). See [**4**, Lemma 4] for a proof of this. Also, in the case where $G = \mathbf{Z}$, to prove (3) we need a much more refined condition on $\mathbf{k}$ than (1). See [**3**] for details of this. A number of families of sequences of natural numbers for which (3) is true can also be found in [**3**]. In [**2**] it is shown that in the presence of (3) the ergodicity of T is equivalent to

$$\lim_{M\to\infty} \frac{1}{M}\sum_{n=1}^{M} \mu(A\cap T^{-k_n}B) = \mu(B)\mu(A), \tag{4}$$

for each pair $A, B \in \beta$. This of course implies (1) for the particular transformation T. In the absence of (3) condition (4) still implies the ergodicity of T. For ergodic T, property (1) for $\mathbf{k} = (k_n)_{n\geqslant 1}$, is not obviously implied by (2) but is implied by (3). It is however possible to establish (1) for particular $\mathbf{k} = (k_n)_{n\geqslant 1}$ for ergodic T without recourse to (3) as the following Theorem demonstates. This suggests that (1) for ergodic T is not equivalent to either (2) or (3).

THEOREM C. *For a sequence of integers $\mathbf{k} = (k_i)_{i=1}^{\infty}$ suppose that the system of neighbourhoods $A_n = [1,n]\cap\mathbf{k}$ $(n = 1,2,\cdots)$ satisfies*

$$|A_n \,\triangle\, (h + A_n)| = o(|A_n|),$$

for any h in $\mathbf{N}$, where $\triangle$ denotes the symmetric difference of two sets, and the set $h + A_n$ denotes $\{h + k : k \in A_n\}$. Then (3) follows if for each set A in the σ-algebra β we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mu(T^{-k_i}A\cap A) = \mu(A)^2. \tag{5}$$

PROOF: Recall that L^2 is a Hilbert space under the inner product $\langle f, g\rangle = \int_X f\bar{g}\,d\mu$, where $\bar{g}$ is the complex conjugate of g. Let $Uf(x) = f(Tx)$. Then $\|Uf\| = \|f\|$ because T is measure preserving.

In the special case where for A in β we set $a = \chi_A$ (the characteristic function of A) the hypothesis (5) may be rewritten

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \langle U^{k_i}a, a\rangle = \langle a, 1\rangle\langle 1, a\rangle. \tag{5$'$}$$

By taking linear combinations of characteristic functions, this statement is also seen to remain true for simple functions a. For arbitrary L^2 functions f given $\varepsilon > 0$ we can find simple functions a such that $\|f-a\|_2 \leqslant \varepsilon$. Further by (5′) we can find a natural number $n = n(\varepsilon)$ such that if $N \geqslant n(\varepsilon)$ then

$$\left|\frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}a, a\rangle - \langle a,1\rangle\langle 1,a\rangle\right| \leqslant \varepsilon.$$

Thus, also if $N \geqslant n(\varepsilon)$ then

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}f, f\rangle - \langle f,1\rangle\langle 1,f\rangle\right| &\leqslant \left|\frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}f, f\rangle - \frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}a, f\rangle\right| \\ &\quad + \left|\frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}a, f\rangle - \frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}a, f\rangle\right| \\ &\quad + \left|\sum_{i=1}^{N}\langle U^{k_i}a, a\rangle - \langle a,1\rangle\langle 1,a\rangle\right| \\ &\quad + \left|\langle a,1\rangle\langle 1,a\rangle - \langle f,1\rangle\langle 1,a\rangle\right| \\ &\quad + \left|\langle f,1\rangle\langle 1,a\rangle - \langle f,1\rangle\langle 1,a\rangle\right| \\ &\leqslant \frac{1}{N}\sum_{i=1}^{N}\left|\langle U^{k_i}(f-a), f\rangle\right| \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\left|\langle U^{k_i}a, f-a\rangle\right| \\ &\quad + \varepsilon + \left|\langle f-a,1\rangle\langle 1,a\rangle\right| + \left|\langle f,1\rangle\langle 1,f-a\rangle\right|.\end{aligned}$$

Using the fact that $\|f\| = \langle f,f\rangle^{1/2}$ and Cauchy's inequality this is

$$\leqslant \varepsilon\|f\| + \varepsilon\big(\|f\|+\varepsilon\big) + \varepsilon + \varepsilon\big(\|f\|+\varepsilon\big) + \varepsilon\|f\|.$$

Thus we have shown that if $f \in L^2$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}f, f\rangle = \langle f,1\rangle\langle 1,f\rangle \tag{6}$$

Now $L^2 = H \oplus H^{\perp}$ where $H = \bigcap_{n=1}^{\infty} U^n L^2(X,\beta,\mu)$ and note that if

$$V = L^2 \ominus UL^2 = \left\{f \in L^2(X,\beta,\mu) : f \perp L^2\big(X, T^{-1}\beta, \mu\big)\right\}$$

where $T^{-1}\beta$ is the σ algebra generated by $\{A = T^{-1}B : B \in \beta\}$, then

$$H^{\perp} = \bigoplus_{n=0}^{\infty} U^n V.$$

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Since the spaces are mutually orthogonal it is clear that for $f \in U^iV$ and $g \in U^jV$ with $i \neq j$ different we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i}f, g\rangle = 0. \tag{7}$$

By taking linear combinations of such f and g and using approximation arguments in the $L^2(X, \beta, \mu)$ norm, we must have (7) for all f and g in $H^\perp$. Of course for f in H and g in $H^\perp$ or vice versa then (7) is still true and we see that in order to prove Theorem C it suffices to show (7) assuming that $\int_X f(x)d\mu = 0$. This is the only point at which we need to use the hypothesis on the sequence of integers $\mathbf{k}$. We first note that $UH = H$ and therefore U is a unitary operator on H, that is, in particular it has an inverse there. Let $S(f) = \{U^n f : n \in \mathbf{Z}\}$, and let $Z(f)$ be the $\|.\|$ closure of the linear span of $S(f)$. Now by (6) for arbitrary positive integers ℓ

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\langle U^{k_i+\ell}f, U^\ell f\rangle = 0.$$

Also by the hypothesis on the sequence $\mathbf{k}$ we have

$$\sum_{i=1}^{N}\langle U^{k_i+\ell}f, U^\ell f\rangle = \sum_{i=1}^{N}\langle U^{k_i}f, U^\ell f\rangle + o(N).$$

Taking linear combinations of the $U^\ell f$ and then taking limits completes the proof. ∎

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