Moduli **1**, e4 (2024) 1–19
doi:10.1112/mod.2024.3

Symmetric tensors on the intersection of two quadrics and Lagrangian fibration

Arnaud Beauville, Antoine Etesse, Andreas Höring, Jie Liu and Claire Voisin

Abstract

Let X be an n-dimensional (smooth) intersection of two quadrics, and let T^*X be its cotangent bundle. We show that the algebra of symmetric tensors on X is a polynomial algebra in n variables. The corresponding map $\Phi : T^*X \to \mathbb{C}^n$ is a Lagrangian fibration, which admits an explicit geometric description; its general fiber is a Zariski open subset of an abelian variety, which is a quotient of a hyperelliptic Jacobian by a 2-torsion subgroup. In dimension 3, Φ is the Hitchin fibration of the moduli space of rank 2 bundles with fixed determinant on a curve of genus 2.

Contents

1 Introduction **2**
1.1 Comments . . . 2
1.2 Strategy . . . 3
1.3 Notations . . . 3
2 The case n=3 **3**
3 Definition of Φ **4**
4 Fibers of φ **6**
4.1 Odd-dimensional intersection of 2 quadrics . . . 7
4.2 An auxiliary construction . . . 7
4.3 Proof of Proposition 4.1. . . . 8
5 Fibres of Φ **9**
5.1 Results . . . 9
5.2 Proof of the theorem: lemmas . . . 10
5.3 Proof of Theorem 5.1 . . . 11

Received 14 May 2024, accepted 15 May 2024.
2020 Mathematics Subject Classification 70H06 (Primary), 14J45 (Secondary)
Keywords: symmetric tensors, quadrics, Lagrangian fibration, completely integrable systems, Hitchin fibration.
© The Author(s), 2024. Published by Cambridge University Press on behalf of the Foundation Composition Mathematica, in partnership with the London Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3

6 Proof of Proposition 5.5 **11**
7 Symmetric tensors: second approach **13**
7.1 The cotangent bundle of a smooth quadric 13
7.2 Explicit description of symmetric tensors 14
7.3 The double cover 15
7.4 The divisor of s_0 16
7.5 Proof of part (a) of the theorem 17
References **18**

1. Introduction

Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a smooth n-dimensional complete intersection of two quadrics, with $n \geq 2$, and let T^*X be its cotangent bundle. The $\mathbb{C}$-algebra $H^0(T^*X, \mathscr{O}_{T^*X})$ is canonically isomorphic to the algebra of symmetric tensors $H^0(X, \mathsf{S}^\bullet T_X)$. Recall that T^*X carries a canonical symplectic structure. Our main result is the following theorem:

Theorem 1.1.
(a) The vector space $W := H^0(X, \mathsf{S}^2 T_X)$ *has dimension* n*, and the natural map* $\mathsf{S}^\bullet W \to H^0(X, \mathsf{S}^\bullet T_X)$ *is an isomorphism.*
(b) The corresponding map, $\Phi : T^*X \to W^* \cong \mathbb{C}^n$*, is a Lagrangian fibration.*
(c) When X *is general, the general fiber of* Φ *is of the form* $A \smallsetminus Z$*, where* A *is an abelian variety and* codim $Z \geq 2$.

We will give a precise geometric description of the map Φ and of the abelian variety A in Sections 4 and 5.

1.1 Comments

(1) For $n = 2$, (a) follows from Theorem 5.1 in [DOL19], while (b) and (c) are proved in [KL22]. The proof is based on the isomorphism $T_X \cong \Omega^1_X(1)$. The theorem also follows from the fact that X is a moduli space for parabolic rank 2 bundles on $\mathbb{P}^1$ [Cas15], so $\Phi : T^*X \to \mathbb{C}^2$ is identified to the *Hitchin fibration* (see [BHK10]).

For $n = 3$, X is isomorphic to the moduli space of vector bundles of rank 2 and fixed determinant of odd degree [New68]; again, the theorem follows from the properties of the Hitchin fibration (see Section 2). It would be interesting to have a modular interpretation of Φ for $n \geq 4$. Note that the Hitchin map for G-bundles is homogeneous quadratic only when G is SL(2) or a product of copies of SL(2), so this limits the possibilities of using it.

(2) The map Φ is an example of an algebraically completely integrable system; see Remark 5.1. There is an abundant literature on such systems; see, for instance, [A96].

A classical example, the geodesic flow on an ellipsoid, is discussed in detail in [K80]. The corresponding Lagrangian fibration takes place on the cotangent bundle of *one* quadric; it is not related to our Φ. However, some of the tools we use in Sections 4 and 5, in particular the variety $\mathscr{X}$ and the family of planes $\mathscr{F}$, appear already in [K80] (with a different purpose).

(3) Such a situation is rather exceptional: Most varieties do not admit nonzero symmetric tensors (for instance, hypersurfaces of degree ≥ 3 [HLS22]); when they do, even for varieties as simple

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

as quadrics, the algebra of symmetric tensors is fairly complicated (see, for instance, [BLi24]). We do not have a conceptual explanation for the particularly simple behavior in our case.

(4) For $n = 2$ or 3, the generality assumption on X in (c) is unnecessary. It seems likely that this is the case for all n, but our method does not allow us to make that conclusion.

1.2 Strategy

We will first treat the case $n = 3$, which is independent of the rest of this article (Section 2). For the general case, we will develop two different approaches. In the first one we exhibit a natural n-dimensional subspace $W \subset H^0(X, \mathsf{S}^2 T_X)$, from which we deduce a map $\Phi : T^*X \to W^* \cong \mathbb{C}^n$ (Section 3). We then show that Φ has the required properties, which implies (a), (b) and (c) for general X (5.1). In the second approach (Section 7), we directly prove (a) for all smooth X , by realizing X as a double covering of a quadric.

1.3 Notations

Throughout this article, X will be a smooth complete intersection of two quadrics in $\mathbb{P}^{n+2}$, with $n \geq 2$. We denote by T^*X its cotangent bundle and by $\mathbb{P}T^*X$ its projectivisation in the geometric sense (not in the Grothendieck sense). If V is a vector space, we denote by $\mathbb{P}(V)$ the associated projective space $V \smallsetminus \{0\}/\mathbb{C}^*$ parametrising 1-dimensional subspaces of V.

2. The case n=3

In this section we show how our general results can be obtained in the case $n = 3$ by interpreting X as a moduli space.

As in Section 4.1 below, we associate to X a genus 2 curve C such that the variety of lines in X is isomorphic to JC. Let us fix a line bundle N on C of degree 1; then X is isomorphic to the moduli space $\mathscr{M}$ of rank 2 stable vector bundles on C with determinant N [New68]. The cotangent bundle $T^*\mathscr{M}$ is naturally identified with the moduli space of *Higgs bundles*; that is, pairs (E, u) with $E \in \mathscr{M}$ and $u : E \to E \otimes K_C$ a homomorphism with $\operatorname{Tr} u = 0$. The *Hitchin map* $\Phi : T^*\mathscr{M} \to H^0(K_C^2)$ associates to a pair (E, u) the section $\det u$ of K_C^2. It is a Lagrangian fibration [Hit87].

Let $\omega \in H^0(K_C^2)$. *We assume in what follows that ω vanishes at 4 distinct points.* Let C_ω be the curve in the cotangent bundle T^*C defined by $z^2 = \omega$. The projection $\pi : C_\omega \to C$ is a double covering branched along $\operatorname{div}(\omega)$, and C_ω is a smooth curve of genus 5. Let P be the Prym variety associated to π, that is, the kernel of the norm map $\operatorname{Nm} : JC_\omega \to JC$; it is a 3-dimensional abelian variety.

Proposition 2.1. *The fibre $\Phi^{-1}(\omega)$ is isomorphic to the complement of a curve in P.*

Proof. Recall that the map $L \mapsto \pi_* L$ establishes a bijective correspondence between line bundles on C_ω and rank 2 vector bundles E on C endowed with a homomorphism $u : E \to E \otimes K_C$ such that $u^2 = \omega \cdot \mathrm{Id}_E$ or, equivalently, $\operatorname{Tr} u = 0$ and $\det u = \omega$ (see, for instance, [BNR89]). To get (E, u) in $\Phi^{-1}(\omega)$, we have to impose $\det E = N$ and E stable. Since $\det \pi_* L = \operatorname{Nm}(L) \otimes K_C^{-1}$, the first condition means that L belongs to the translate $P_N := \operatorname{Nm}^{-1}(K_C \otimes N)$ of P.

Then the vector bundle $\pi_* L$ is unstable if and only if it contains an invertible subsheaf M of degree 1; this is equivalent to saying that there is a nonzero map $\pi^* M \to L$; that is, $L = \pi^* M(p)$ for some point $p \in C_\omega$. The condition $L \in P_N$ means that $M^2(\pi(p)) = K_C \otimes N$, so

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

M is determined by p up to the 2-torsion of JC. Thus the locus of line bundles $L \in P_N$ such that $\pi_* L$ is unstable is a curve. $\square$

Let $\rho : C \to \mathbb{P}^1$ be the canonical double covering, with $B \subset \mathbb{P}^1$ its branch locus. Since the homomorphism $\mathsf{S}^2 H^0(K_C) \to H^0(K_C^2)$ is surjective, the divisor of ω is of the form $\rho^*(p+q)$, for some $p, q \in \mathbb{P}^1$; by assumption, we have $p \neq q$ and $p, q \notin B$.

Proposition 2.2. *Let Γ be the double covering of $\mathbb{P}^1$ branched along $B \cup \{p, q\}$. There is an exact sequence*

$$0 \to \mathbb{Z}/2 \to J\Gamma \to P \to 0 .$$

Proof. Let $\chi : \mathbb{P}^1 \to \mathbb{P}^1$ be the double covering branched along $\{p, q\}$. Since $\operatorname{div}(\omega) = \rho^*(p+q)$, there is a cartesian diagram of double coverings

$$\begin{tikzcd} C_\omega \arrow[r, "\xi"] \arrow[d, "\pi"'] & \mathbb{P}^1 \arrow[d, "\chi"] \\ C \arrow[r, "\rho"'] & \mathbb{P}^1 \end{tikzcd}$$

which gives rise to two commuting involutions σ, τ of C_ω, exchanging the two sheets of π and ξ, respectively. The field of rational functions on C_ω is

$$\mathbb{C}(x, y, z) \quad \text{with} \quad y^2 = f(x), z^2 = g(x),$$

where f and g are polynomials with $\operatorname{div} f = B$ and $\operatorname{div} g = \{p, q\}$. Then σ and τ change the sign of y and z, respectively.

The involution $\sigma\tau$ is fixed-point free, so the quotient $\Gamma := C_\omega / \langle \sigma\tau \rangle$ has genus 3; its field of functions is $\mathbb{C}(x, w)$, with $w = yz$ and $w^2 = f(x)g(x)$. We have again a cartesian square

$$\begin{tikzcd} C_\omega \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \Gamma \arrow[d, "\psi"] \\ C \arrow[r, "\rho"'] & \mathbb{P}^1. \end{tikzcd}$$

Let $\alpha \in J\Gamma$. We have $\operatorname{Nm}_\pi \varphi^* \alpha = \rho^* \operatorname{Nm}_\psi \alpha = 0$; hence, φ^* maps $J\Gamma$ into $P \subset JC_\omega$. Since φ is étale, we have $\operatorname{Ker} \varphi^* = \mathbb{Z}/2$; since $\dim J\Gamma = \dim P = 3$, φ^* is surjective. $\square$

3. Definition of Φ

Let Y be a smooth degree d hypersurface in $\mathbb{P}^N$, defined by an equation $f = 0$. Recall that one associates to f a section h_f of $\mathsf{S}^2 \Omega^1_Y(d)$, the *hessian* or *second fundamental form* of f [GH79]: at a point y of Y, the intersection of Y with the tangent hyperplane H to Y at y is a hypersurface in H singular at y, and $h_f(y)$ is the degree 2 term in the Taylor expansion of $f_{|H}$ at y.

Now let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of r hypersurfaces of degree d; let

$$V \subset H^0(\mathbb{P}^{n+r}, \mathscr{O}_{\mathbb{P}}(d))$$

be the r-dimensional subspace of degree d polynomials vanishing on X. By restricting h_f, for $f \in V$, to X, we get a linear map

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3

$$V \otimes \mathscr{O}_X \longrightarrow \mathsf{S}^2\Omega^1_X(d),$$

which gives at each point $x \in X$ a linear space of quadratic forms on the tangent space $T_x(X)$. Note that when $d=2$, the corresponding quadrics in $\mathbb{P}(T_x(X))$ can be viewed geometrically as follows: The projective space $\mathbb{P}(T_x(X))$ can be identified with the space of lines in $\mathbb{P}^{n+r}$ passing through x and tangent to X; then for each $q \in V$, the quadric defined by $h_q(x)$ parameterises the lines passing through x and contained in the quadric $\{q=0\}$.

Now we want to consider the 'inverse' of the quadratic form $h_f(x)$ on $T_x(X)$; that is, the form on $T^*_x(X)$ given in coordinates by the cofactor matrix. Intrinsically, each $f \in V$ gives a twisted symmetric morphism

$$h_f : T_X \longrightarrow \Omega^1_X(d),$$

which induces a twisted symmetric morphism on $(n-1)$-th exterior powers, namely,

$$\wedge^{n-1} h_f : \bigwedge^{n-1} T_X \longrightarrow \bigwedge^{n-1} \Omega^1_X((n-1)d)\,.$$

We now observe that $K_X = \mathscr{O}_X(-n-1-r+dr)$; hence

$$\bigwedge^{n-1} T_X \cong \Omega^1_X(n+1-r(d-1)) \text{ and } \quad \bigwedge^{n-1} \Omega^1_X \cong T_X(-n-1+r(d-1))\,,$$

so $\wedge^{n-1} h_f$ induces a symmetric morphism from $\Omega^1_X(n+1-r(d-1))$ to $T_X((n-1)d-n-1+r(d-1))$, hence provides a section

$$\wedge^{n-1} h_f \in H^0(X, \mathsf{S}^2 T_X(d(n+2r-1)-2(n+r+1))).$$

Being locally given by the cofactor matrix, $\wedge^{n-1} h_f$ is homogeneous of degree $n-1$ in f. Hence, we have constructed a linear map

$$\alpha : \mathsf{S}^{n-1} V \longrightarrow H^0(X, \mathsf{S}^2 T_X(d(n+2r-1)-2(n+r+1))) \quad \text{such that} \quad \alpha(f^{n-1}) = \wedge^{n-1} h_f\,.$$

From now on, we restrict to the case $d=2$, $r=2$, so X is the complete intersection of two quadrics, and the previous construction gives a linear map

$$\alpha : \mathsf{S}^{n-1} V \longrightarrow H^0(X, \mathsf{S}^2 T_X)\,.$$

Using the canonical isomorphism $H^0(T^*X, \mathscr{O}_{T^*X}) = H^0(X, \mathsf{S}^\bullet T_X)$, we deduce from α a morphism

$$\Phi : T^*X \longrightarrow \mathsf{S}^{n-1} V^* \cong \mathbb{C}^n\,.$$

We have $\Phi(\lambda v) = \lambda^2 \Phi(v)$ for $v \in T^*X$, $\lambda \in \mathbb{C}$, so Φ induces a rational map

$$\varphi : \mathbb{P}T^*X \dashrightarrow \mathbb{P}^{n-1},$$

whose indeterminacy locus Z is the image of $\Phi^{-1}(0)$.

Proposition 3.1.

(1) *α is injective.*
(2) *Φ is surjective.*
(3) *The image of Z by the structure map $p : \mathbb{P}T^*X \to X$ is a proper subvariety of X.*

Proof. Let x be a general point of X. We claim that the base locus in $\mathbb{P}(T_x(X))$ of the pencil of quadratic forms $\{h_q(x)\}_{q\in V}$ is smooth. Indeed, this locus can be viewed as the variety F_x of lines in X passing through x. Let F be the Fano variety of lines contained in X, and let

$$G \subset F \times X = \{(\ell, y) \mid y \in \ell\}\,.$$

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

Then F and therefore G are smooth [Reid72, Theorem 2.6], hence F_x, which is the fibre above x of the projection $G \to X$, is smooth since x is general. It follows that in an appropriate system of coordinates $(k_1, \ldots, k_n)$ of $T_x(X)$, the forms $\{h_q(x)\}$ can be written as

$$t\sum k_i^2 + \sum \alpha_i k_i^2 \quad \text{with } \alpha_i \text{ distinct in } \mathbb{C},\ t \in \mathbb{C}\,.$$

Then $\wedge^{n-1}h_q(x)$ is given by the diagonal matrix with entries $\beta_i := \prod_{j\neq i}(t+\alpha_j)$ $(i=1,\ldots,n)$. These polynomials in t are linearly independent; hence, they generate the space of quadratic forms on $T_x^*(X)$, which are diagonal in the basis (k_i). This linear system has dimension n, so α is injective; it has no base point, so φ induces a finite, surjective morphism $\mathbb{P}(T_x^*(X)) \to \mathbb{P}^{n-1}$. Thus, Φ is surjective, and $Z \cap \mathbb{P}(T_x^*(X)) = \varnothing$, which gives (2) and (3). ☐

We want to give a geometric construction of the rational map $\varphi : \mathbb{P}T^*X \dashrightarrow \mathbb{P}^{n-1}$. A point of $\mathbb{P}T^*X$ is a pair (x,H), where $x \in X$ and H is a hyperplane in $T_x(X)$. Restricting the pencil $\{h_q(x)\}_{q\in V}$ to H gives a pencil of quadrics on H, which for general(x,H) contains $n-1$ singular quadrics $q_1, \ldots, q_{n-1}$. The subset $\{q_1, \ldots, q_{n-1}\}$ of $\mathbb{P}(V)$ corresponds to a point $\varphi_{x,H}$ of $\mathbb{P}(\mathsf{S}^{n-1}V^*)$; namely, the hyperplane in $\mathsf{S}^{n-1}V$ spanned by $q_1^{n-1}, \ldots, q_{n-1}^{n-1}$.

Proposition 3.2. $\varphi(x,H) = \varphi_{x,H}$.

Proof. We can assume that x is general. We have seen that the restriction of φ to $\mathbb{P}(T_x^*X)$ is the morphism given by the linear system of quadratic forms $W \cong \mathsf{S}^{n-1}V$ spanned by the forms $\wedge^{n-1}h_q(x)$, for $q \in V$; in other words, φ maps the point H of $\mathbb{P}(T_x^*(X))$ to the hyperplane of forms in W vanishing at H.

On the other hand, $\varphi_{x,H}$ is the hyperplane of $\mathsf{S}^{n-1}V$ spanned by the q^{n-1} for those $q \in V$ such that $h_q(x)_{|H}$ is singular; this condition is equivalent to saying that the form $\wedge^{n-1}h_q(x)$ on T_x^*X vanishes at H. Therefore, $\varphi_{x,H}$ is spanned by quadratic forms vanishing at H, hence coincides with $\varphi(x,H)$. ☐

Corollary 3.3. $\operatorname{codim} Z \geq 2$.

Proof. Suppose that Z contains a component Z_0 of codimension 1; since $p(Z) \neq X$, we have $Z_0 = p^{-1}(p(Z_0))$. We claim that this is impossible; in fact, Z cannot contain a fibre $p^{-1}(x)$. Indeed, its doing this would mean that for $q \in V$, the form $h_q(x)$ is singular along all hyperplanes $H \subset T_x(X)$; that is, $h_q(x)$ has rank $\leq n-2$. But the rank of $h_q(x)$ is the rank of the restriction of q to the projective tangent subspace to X at x. Restricting a quadratic form to a hyperplane lowers its rank by up to two. Since a general q in V has rank $n+3$, its restriction to a codimension 2 subspace has rank $\geq n-1$. ☐

4. Fibers of φ

In an appropriate system of coordinates $(x_0, \ldots, x_{n+2})$, our variety X is defined by the equations $q_1 = q_2 = 0$, with

$$q_1 = \sum x_i^2 \quad q_2 = \sum \mu_i x_i^2 \qquad , \text{with } \mu_i \in \mathbb{C} \text{ distinct.}$$

Let $\Pi = \mathbb{P}(V)$ $(\cong \mathbb{P}^1)$ be the pencil of quadrics containing X. We choose a coordinate t on Π so that the quadrics of Π are given by $tq_1 - q_2 = 0$. Then the singular quadrics of Π correspond to the points $\mu_0, \ldots, \mu_{n+2}$.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3

The goal of this section is to describe the general fibre of the rational map $\varphi : \mathbb{P}T^*X \dashrightarrow \mathsf{S}^{n-1}\Pi$ ($\cong \mathbb{P}^{n-1}$). For $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathsf{S}^{n-1}\Pi$, let $C_{\mu,\lambda}$ denote the hyperelliptic curve $y^2 = \prod(t-\mu_i)\prod(t-\lambda_j)$, of genus n. We will then prove the following:

Proposition 4.1. *For λ general in $\mathsf{S}^{n-1}\Pi$, the fibre $\varphi^{-1}(\lambda)$ is birational to the quotient of the Jacobian $JC_{\mu,\lambda}$ by the group $\Gamma := \{\pm 1_{JC}\} \times \Gamma^+$, where $\Gamma^+ \cong (\mathbb{Z}/2Z)^{n-2}$ is a group of translations by 2-torsion elements.*

4.1 Odd-dimensional intersection of 2 quadrics

We briefly recall here the results of Reid's thesis ([Reid72]; see also [DR76]). Let $Y \subset \mathbb{P}^{2g+1}$ be a smooth intersection of 2 quadrics, and let Ξ ($\cong \mathbb{P}^1$) be the pencil of quadrics containing Y. Let $\Sigma \subset \Xi$ be the subset of $2g+2$ points corresponding to singular quadrics, and let C be the double covering of Ξ branched along Σ; this is a hyperelliptic curve of genus g. The intermediate Jacobian JY of Y is isomorphic to JC (as principally polarized abelian varieties). The variety F of $(g-1)$-planes contained in Y is also isomorphic to JC, but this isomorphism is not canonical.

In an appropriate system of coordinates, the equations of Y are of the form

$$\sum x_i^2 = \sum \alpha_i x_i^2 = 0, \qquad \text{with } \alpha_i \in \mathbb{C} \text{ distinct;}$$

then $\Sigma = \{\alpha_1, \dots, \alpha_{2g+2}\}$. The group $\Gamma := (\mathbb{Z}/2\mathbb{Z})^{2g+1}$ acts on Y (hence also on F) by changing the signs of the coordinates. Let $\Gamma^+ \subset \Gamma$ be the subgroup of elements that change an even number of coordinates. Choose an element $\gamma \in \Gamma \smallsetminus \Gamma^+$; there is an isomorphism $F \xrightarrow{\sim} JC$ such that γ corresponds to (-1_{JC}). Then the image of Γ^+ in $\mathrm{Aut}(JC)$ is the group T_2 of translations by 2-torsion elements of JC, and the image of Γ is $T_2 \times \{\pm 1_{JC}\}$ [DR76, Lemma 4.5].

4.2 An auxiliary construction

We consider the projective space $\mathbb{P}^{2n+1}$ equipped with the system of homogeneous coordinates

$$x_0, \dots, x_{n+2};\ y_1, \dots, y_{n-1}$$

and the affine space $\mathbb{A}^{n-1}$ equipped with the affine coordinates $\lambda_1, \dots, \lambda_{n-1}$. Let

$$\mathscr{X} \subset \mathbb{P}^{2n+1} \times \mathbb{A}^{n-1}$$

be the complete intersection of the two quadrics with equations

$$Q_1 = Q_2 = 0 \quad \text{with} \quad Q_1 = \sum_{i=0}^{n+2} x_i^2 + \sum_{j=1}^{n-1} y_j^2 \quad , \quad Q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 + \sum_{j=1}^{n-1} \lambda_j y_j^2 \,.$$

The second projection, $\mathscr{X} \to \mathbb{A}^{n-1}$, gives a family of complete intersections of two quadrics $\mathscr{X}_\lambda$ of dimension $2n-1$ parameterised by $\mathbb{A}^{n-1}$. Note that X is the intersection of $\mathscr{X}$ with the subspace $\mathbb{P}^{n+2} \subset \mathbb{P}^{2n+1}$ defined by $y_1 = \ldots = y_{n-1} = 0$.

Let $p : \mathscr{F} \to \mathbb{A}^{n-1}$ be the family of $(n-1)$-planes contained in the $\mathscr{X}_\lambda$; that is

$$\mathscr{F} = \{(P, \lambda) \mid \lambda \in \mathbb{A}^{n-1},\ P\ (n-1)\text{-plane} \subset \mathscr{X}_\lambda\} \,.$$

For λ general, the fibre $\mathscr{F}_\lambda$ is isomorphic to the Jacobian of the hyperelliptic curve $C_{\mu,\lambda}$ (4.1).

Let (P, λ) be a general point of $\mathscr{F}$. Then $P \cap \mathbb{P}^{n+2}$ is a point x of X. Let $\pi : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{n+2}$ be the projection $(x_i, y_j) \mapsto (x_i)$. Since the π_* differentials of Q_i and q_i coincide at x, the differential π_* maps $T_x(P) \subset T_x(\mathscr{X})$ into $T_x(X)$. Since P is general, $\pi_* T_x(P)$ is a hyperplane in $T_x(X)$; this

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3

will follow from the proof of Proposition 4.2, (1) below, where we explicitly construct pairs (P,λ) with this property.

Therefore, we have a rational map

$$\psi : \mathscr{F} \dashrightarrow \mathbb{P}T^*X \quad (P,\lambda) \mapsto (x = P\cap \mathbb{P}^{n+2}\,,\ \pi_* T_x(P))\,.$$

The symmetric group $\mathfrak{S}_{n-1}$ acts on $\mathbb{P}^{2n+1}$ by permuting the y_j and acts on the group $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ by changing their signs; this gives an action of the semi-direct product $G := (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_{n-1}$. We make G act on $\mathbb{A}^{n-1}$ through its quotient $\mathfrak{S}_{n-1}$, by permutation of the λ_i. This induces an action of G on $\mathscr{X}$ and therefore on $\mathscr{F}$, which is compatible via p with the action on the base. The map ψ is invariant under this action; hence, it factors through the quotient $\mathscr{F}/G$. By passing to the quotient, we get a map $p^\sharp : \mathscr{F}/G \to \mathbb{A}^{n-1}/\mathfrak{S}_{n-1}$.

Proposition 4.2. (1) *ψ induces a birational map $\psi^\sharp : \mathscr{F}/G \dashrightarrow \mathbb{P}T^*X$.*

(2) *There is a commutative diagram*

$$\begin{tikzcd} \mathscr{F}/G \arrow[r, dashed, "\psi^\sharp"] \arrow[d, "p^\sharp"'] & \mathbb{P}T^*X \arrow[d, dashed, "\varphi"] \\ \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} \arrow[r, "\sim", "\sigma"'] & \mathbb{A}^{n-1} \subset \mathbb{P}^{n-1} \end{tikzcd}$$

where $p^\sharp$ is deduced from p and where *σ is the isomorphism given by symmetric functions.*

Proof. (1) Let $(x,H) \in \mathbb{P}T^*X$; we want to describe the pairs (P,λ) such that $P\cap \mathbb{P}^{n+2} = \{x\}$ and $\pi_*T_x(P) = H$. The latter condition says that via the decomposition

$$T_x(\mathbb{P}^{2n+1}) = T_x(\mathbb{P}^{n+2}) \oplus \operatorname{Ker}\pi_*\,,$$

$T_x(P)$ identifies with the graph of a linear map

$$\alpha : H \to \operatorname{Ker}\pi_*\,.$$

Using the basis $(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{n-1}})$ of $\operatorname{Ker}\pi_*$, we have $\alpha = (\alpha_1,\dots,\alpha_{n-1})$, where the α_i are linear forms on H. The condition $P\subset \mathscr{X}_\lambda$ implies that the hessians $h_{Q_1}(x)$ and $h_{Q_2}(x)$ vanish on $T_x(P)$, which gives

$$h_{q_1}(x)_{|H} = -\sum_i \alpha_i^2 \quad h_{q_2}(x)_{|H} = -\sum_i \lambda_i\alpha_i^2\,. \tag{1}$$

This is a simultaneous diagonalisation of the quadratic forms $h_{q_1}(x)_{|H}$ and $h_{q_2}(x)_{|H}$; when they are in general position, this determines the λ_i up to permutation and the α_i up to sign and permutation, which proves (1).

(2) Let $(P,\lambda)\in\mathscr{F}$, and let $(x,H) := \psi(P,\lambda)$. According to Proposition 3.2, $\varphi(x,H)$ is given by the $(n-1)$-uple of quadrics $q\in\Pi$ such that the form $h_q(x)_{|H}$ is singular. Using $(\alpha_1,\dots,\alpha_{n-1})$ as coordinates on H, we see from (1) that this $(n-1)$-uple is given by $(\lambda_1,\dots,\lambda_{n-1})$, which proves (2). ☐

4.3 Proof of Proposition 4.1.

Let λ be a general element of $\mathbb{A}^{n-1}$. Let us denote by Γ the subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ of G. From Proposition 4.2 and the cartesian diagram

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

$$\begin{tikzcd} \mathscr{F}/\Gamma \arrow[r] \arrow[d, "p"] & \mathscr{F}/G \arrow[d, "p^\sharp"] \\ \mathbb{A}^{n-1} \arrow[r] & \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} \end{tikzcd}$$

we see that the fibre $\varphi^{-1}(\lambda)$ is birational to the quotient $\mathscr{F}_\lambda/\Gamma$. By (4.1) there is an isomorphism $\mathscr{F}_\lambda \xrightarrow{\sim} JC_{\mu,\lambda}$ such that Γ acts on $JC_{\mu,\lambda}$ as $\{\pm 1_J\} \times \Gamma^+$, where Γ^+ is a group of translations by 2-torsion elements. This proves the proposition.

5. Fibres of Φ

5.1 Results

We keep the settings of the previous section. Recall that our parameter λ lives in $\mathbb{A}^{n-1} \subset \mathsf{S}^{n-1}\Pi \cong \mathbb{P}^{n-1}$. For λ in $\mathbb{A}^{n-1}$, we denote by $\tilde{\lambda}$ a lift of λ in $\mathbb{C}^n$ for the quotient map $\mathbb{C}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$.

Theorem 5.1. *Assume that X is general. For $\lambda \in \mathbb{A}^{n-1}$ general, the fibre $\Phi^{-1}(\tilde{\lambda})$ is isomorphic to $A \smallsetminus Z$, where:*

- *A is the abelian variety quotient of $JC_{\mu,\lambda}$ by a 2-torsion subgroup, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-2}$;*
- *Z is a closed subvariety of codimension ≥ 2 in A.*

Corollary 5.2. *For every smooth complete intersection of two quadrics $X \subset \mathbb{P}^{n+2}$, the fibration $\Phi : T^*X \to \mathbb{C}^n$ is Lagrangian.*

Proof. Assume first that X is general. The symplectic form on T^*X is $d\eta$, where η is the Liouville form. By Theorem 5.1 and Hartogs' principle, the pull-back of η to a general fibre of Φ is the restriction of a 1-form on an abelian variety, hence is closed. This implies the result.

Let $p : \mathscr{X} \to B$ be a complete family of smooth intersection of two quadrics in $\mathbb{P}^{n+2}$. The constructions of §3 can be globalised over B: we have a rank 2 vector bundle $\mathscr{V}$ over B whose fibre at a point $b \in B$ is the space of quadratic forms vanishing on $\mathscr{X}_b$. We get a homomorphism $\mathsf{S}^{n-1}\mathscr{V} \to p_* T_{\mathscr{X}/B}$, which thus gives rise to a morphism $\boldsymbol{\Phi} : T^*(\mathscr{X}/B) \to \mathsf{S}^{n-1}\mathscr{V}^*$ over B which induces over each point $b \in B$ our map Φ. There is a natural Liouville form $\boldsymbol{\eta}$ on $T^*(\mathscr{X}/B)$: Since $d\boldsymbol{\eta}$ vanishes on a general fibre of $\boldsymbol{\Phi}$, it vanishes on all fibres. $\square$

Corollary 5.3. *Assume that X is general. The multiplication map $\mathsf{S}^\bullet H^0(X, \mathsf{S}^2 T_X) \to H^0(X, \mathsf{S}^\bullet T_X)$ is an isomorphism.*

(We will give in Section 7 a proof that is valid with no generality assumption.)

Proof. Theorem 5.1 implies that every function on a general fibre of Φ is constant; hence, the pull-back $\Phi^* : H^0(\mathbb{C}^n, \mathscr{O}_{\mathbb{C}^n}) \to H^0(T^*X, \mathscr{O}_{T^*X})$ is an isomorphism. The right-hand space is canonically isomorphic to $H^0(X, \mathsf{S}^\bullet T_X)$; hence, we get an algebra isomorphism $\mathbb{C}[t_1, \ldots, t_n] \xrightarrow{\sim} H^0(X, \mathsf{S}^\bullet T_X)$. By construction, the t_i are mapped to elements of $H^0(X, \mathsf{S}^2 T_X)$, so the Corollary follows. $\square$

Remark 5.4. Let $V_1, \ldots, V_n$ be the Hamiltonian vector fields on T^*X that are associated to the components of Φ. For λ general in $\mathbb{C}^n$, let us identify $\Phi^{-1}(\lambda)$ to $A \smallsetminus Z$ as in the theorem. Then by Hartogs' principle the V_i linearise on A; that is, they extend to a basis of $H^0(A, T_A)$. In principle, this allows to write explicit solutions of the Hamilton equations for Φ_i in terms of theta functions.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

5.2 Proof of the theorem: lemmas

We fix a general point $\lambda \in \mathbb{A}^{n-1}$. We denote by $\mathscr{F}^{\mathrm{o}}$ the open subset of $\mathscr{F}$ where the rational map ψ is well-defined and denote by $\mathscr{F}_\lambda^{\mathrm{o}}$ its intersection with the fibre $\mathscr{F}_\lambda$. Since λ is general, the complement of $\mathscr{F}_\lambda^{\mathrm{o}}$ in $\mathscr{F}_\lambda$ has codimension ≥ 2. The rational map ψ induces a morphism $\psi^{\mathrm{o}}: \mathscr{F}^{\mathrm{o}} \to \mathbb{P}T^*X$; we denote by $\psi_\lambda^{\mathrm{o}}$ its restriction to $\mathscr{F}_\lambda^{\mathrm{o}}$. Let $Z \subset \mathbb{P}T^*X$ be the indeterminacy locus of φ (§ 3), and let $\mathscr{F}_\lambda^{\mathrm{bad}} := (\psi_\lambda^{\mathrm{o}})^{-1}(Z) \subset \mathscr{F}_\lambda^{\mathrm{o}}$.

Proposition 5.5. *$\mathscr{F}_\lambda^{\mathrm{bad}}$ has codimension ≥ 2 in $\mathscr{F}_\lambda$.*

We postpone the proof of Proposition 5.5 to the next section; here we show how it implies Theorem 5.1.

Let $0_X \subset T^*X$ be the zero section, and let $q: T^*X \smallsetminus 0_X \to \mathbb{P}T^*X$ be the quotient map. Let $\varphi^{\mathrm{o}}: \mathbb{P}T^*X \smallsetminus Z \to \mathbb{P}^{n-1}$ be the morphism induced by φ. We thus have $q(\Phi^{-1}(\tilde{\lambda})) = (\varphi^{\mathrm{o}})^{-1}(\lambda)$, and the restriction

$$q_\lambda: \Phi^{-1}(\tilde{\lambda}) \to (\varphi^{\mathrm{o}})^{-1}(\lambda)$$

is an étale double cover, with Galois involution ι induced by (-1_{T^*X}).

We put $\mathscr{F}_\lambda^{\mathrm{oo}} := \mathscr{F}_\lambda^{\mathrm{o}} \smallsetminus \mathscr{F}_\lambda^{\mathrm{bad}}$ and consider the restriction

$$\psi_\lambda^{\mathrm{o}}: \mathscr{F}_\lambda^{\mathrm{oo}} \to (\varphi^{\mathrm{o}})^{-1}(\lambda) \quad \text{of} \quad \psi^{\mathrm{o}}\,.$$

Lemma 5.6. *The fibre $\Phi^{-1}(\tilde{\lambda})$ is Lagrangian, and has a trivial tangent bundle.*

Proof. The étale double cover q_λ induces by fibred product an étale double cover

$$\pi: \widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}} \to \mathscr{F}_\lambda^{\mathrm{oo}}$$

such that $\psi_\lambda^{\mathrm{o}}$ lifts to a morphism $\tilde{\psi}_\lambda^{\mathrm{o}}: \widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}} \to \Phi^{-1}(\tilde{\lambda})$.

By Proposition 5.5, the complement of $\mathscr{F}_\lambda^{\mathrm{oo}}$ in $\mathscr{F}_\lambda$ has codimension ≥ 2, so π extends to an étale double cover $\widetilde{\mathscr{F}}_\lambda \to \mathscr{F}_\lambda$, where $\widetilde{\mathscr{F}}_\lambda$ is an abelian variety or the disjoint union of two abelian varieties. The morphism $\tilde{\psi}_\lambda^{\mathrm{o}}: \widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}} \to \Phi^{-1}(\tilde{\lambda})$ is generically of maximal rank. Again by Proposition 5.5, the holomorphic 1-forms on $\widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}}$ are closed; hence by pull-back, the same holds for the holomorphic 1-forms on $\Phi^{-1}(\tilde{\lambda})$. As in the proof of Corollary 5.2, this implies that $\Phi^{-1}(\tilde{\lambda})$ is Lagrangian. The second assertion is a basic property of Lagrangian fibres. □

Lemma 5.7 *The morphism $\psi_\lambda^{\mathrm{o}}$ lifts to a morphism $\tilde{\psi}_\lambda^{\mathrm{o}}: \mathscr{F}_\lambda^{\mathrm{oo}} \to \Phi^{-1}(\tilde{\lambda})$.*

Proof. It suffices to show that the double covering $\pi: \widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}} \to \mathscr{F}_\lambda^{\mathrm{oo}}$ splits.

Assuming the contrary, $\widetilde{\mathscr{F}}_\lambda$ is an abelian variety. By Lemma 5.6 $H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1)$ has dimension n. It follows that the pull-back $(\tilde{\psi}_\lambda^{\mathrm{o}})^*: H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1) \to H^0(\widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}}, \Omega^1)$ is bijective. Since the Galois involution of the double covering π acts trivially on holomorphic 1-forms, the same holds for the Galois involution ι of the double covering $q_\lambda: \Phi^{-1}(\tilde{\lambda}) \to (\varphi^{\mathrm{o}})^{-1}(\lambda)$.

Now we observe that the 1-forms on $\Phi^{-1}(\tilde{\lambda})$ are 'pure'; that is, they extend to any smooth projective compactification of $\Phi^{-1}(\tilde{\lambda})$. This follows from the fact that this holds after pull-back to $\widetilde{\mathscr{F}}_\lambda^{\mathrm{oo}}$. But the quotient $\Phi^{-1}(\tilde{\lambda})/\iota$ is isomorphic to a Zariski open subset of $\varphi^{-1}(\lambda)$, which, by Proposition 4.1, has no nonzero holomorphic 1-forms, so any Zariski open subset has no nonzero closed pure holomorphic 1-forms. This contradiction proves the lemma. □

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

5.3 Proof of Theorem 5.1

Lemma 5.7 gives a factorisation,

$$\psi^{\mathrm{o}}_{\lambda} : \mathscr{F}^{\mathrm{oo}}_{\lambda} \xrightarrow{\tilde{\psi}^{\mathrm{o}}_{\lambda}} \Phi^{-1}(\tilde{\lambda}) \xrightarrow{q_{\lambda}} (\varphi^{\mathrm{o}})^{-1}(\lambda) \,.$$

By Proposition 4.1, $\psi^{\mathrm{o}}_{\lambda}$ induces a birational morphism,

$$\psi^{\mathrm{o}}_{\lambda,\Gamma} : \mathscr{F}^{\mathrm{oo}}_{\lambda}/\Gamma \longrightarrow (\varphi^{\mathrm{o}})^{-1}(\lambda)$$

it follows that for some subgroup $\Gamma' \subset \Gamma$ of index 2, the morphism $\tilde{\psi}^{\mathrm{o}}_{\lambda} : \mathscr{F}^{\mathrm{oo}}_{\lambda} \to \Phi^{-1}(\tilde{\lambda})$ factors through a birational morphism,

$$\tilde{\psi}^{\mathrm{o}}_{\lambda,\Gamma'} : \mathscr{F}^{\mathrm{oo}}_{\lambda}/\Gamma' \longrightarrow \Phi^{-1}(\tilde{\lambda}) \,.$$

By Lemma 5.6, the cotangent bundle of $\Phi^{-1}(\tilde{\lambda})$ is trivial. Therefore, the cotangent bundle of $\mathscr{F}^{\mathrm{oo}}_{\lambda}/\Gamma'$ is generically generated by its global sections. This implies that Γ' acts trivially on holomorphic 1-forms and, hence, is the subgroup Γ^{+} of Γ generated by translations, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-2}$; thus $\mathscr{F}_{\lambda}/\Gamma'$ is an abelian variety A.

To simplify notation, we write $A^{\mathrm{o}} := \mathscr{F}^{\mathrm{oo}}_{\lambda}/\Gamma'$ and $u := \tilde{\psi}^{\mathrm{o}}_{\lambda,\Gamma'}$. The rational map $u^{-1} : \Phi^{-1}(\tilde{\lambda}) \dashrightarrow A$ is everywhere defined (e.g. [BL92, Theorem 4.9.4]), so we have two morphisms

$$A^{\mathrm{o}} \xrightarrow{u} \Phi^{-1}(\tilde{\lambda}) \xrightarrow{u^{-1}} A$$

whose composition is the inclusion $A^{\mathrm{o}} \hookrightarrow A$. Since the tangent bundles of A and $\Phi^{-1}(\tilde{\lambda})$ are trivial, the determinant of $Tu : T_{A^{\mathrm{o}}} \to u^{*}T_{\Phi^{-1}(\tilde{\lambda})}$ is a function on A^{o}, hence constant by Proposition 5.5. Therefore, u is étale and birational, hence an open embedding. This implies that every function on $\Phi^{-1}(\tilde{\lambda})$ is constant (because its restriction to A^{o} is constant). Then the previous argument shows that u^{-1} is also an open embedding, hence $\Phi^{-1}(\tilde{\lambda})$ is isomorphic to an open subset of A containing A^{o}. This proves the theorem.

6. Proof of Proposition 5.5

We keep the notations of Section 4.2. Recall that we have coordinates $(x_0, \ldots, x_{n+2}; y_1, \ldots, y_{n-1})$ on $\mathbb{P}^{2n+1}$ and subspaces $\mathbb{P}^{n+2}$ and $\mathbb{P}^{n-2}$ in $\mathbb{P}^{2n+1}$ defined by $y=0$ and $x=0$.

Let $q_1(x) = q_2(x) = 0$ be the equations defining X in $\mathbb{P}^{n+2}$, and let R be the vector space of quadratic forms in $y = (y_1, \ldots, y_{n-1})$. We define an extended family $\mathscr{X}^{e} \subset \mathbb{P}^{2n+1} \times R^2$ as

$$\mathscr{X}^{e} = \{((x,y);(r_1,r_2)) \in \mathbb{P}^{2n+1} \times R^2 \mid q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0\} \,.$$

The fibre $\mathscr{X}^{e}_{r}$ at a point $r = (r_1, r_2)$ of R^2 is the intersection in $\mathbb{P}^{2n+1}$ of the two quadrics $q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0$. Let $\mathbb{G}$ be the Grassmannian of $(n-1)$-planes in $\mathbb{P}^{2n+1}$; we define as before

$$\mathscr{F}^{e} := \{(P,r) \in \mathbb{G} \times R^2 \mid P \subset \mathscr{X}^{e}_{r}\}$$

and the extended rational map $\psi^{e} : \mathscr{F}^{e} \dashrightarrow \mathbb{P}T^{*}X$, which maps a general $P \subset \mathscr{X}^{e}_{r}$ to the pair (x, H), with $\{x\} = P \cap \mathbb{P}^{n+2}$ and $H = \pi_{*}T_{x}(P)$.

We observe that a general pair $r = (r_1, r_2)$ of R^2 is simultaneously diagonalisable, so the restriction of ψ^{e} to $\mathscr{F}^{e}_{r}$ coincides, for an appropriate choice of the coordinates (y_i), with the map ψ_{λ} that we want to study.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

Proposition 6.1. *Assume that X is general.*

1. *Let $\Gamma \subset \mathscr{F}^e$ be the locus of points (P, r) such that either* $\dim P \cap \mathbb{P}^{n+2} > 0$, *or $P \cap \mathbb{P}^{n-2} \neq \varnothing$. Then Γ has codimension ≥ 2 in $\mathscr{F}^e$.*

2. *There exists no divisor in $\mathscr{F}^e \smallsetminus \Gamma$ that dominates R^2 and that is mapped to the base locus $Z \subset \mathbb{P}T^*X$ by ψ_e.*

We claim that this implies Proposition 5.5. Indeed, as just explained above, it suffices to prove the analogue of Proposition 5.5 for ψ^e. Next, it is clear that the indeterminacy locus of ψ^e is contained in Γ, so ψ^e is well-defined on $\mathcal{F}^e \smallsetminus \Gamma$. By Proposition 6.1, (1), it now suffices to prove the analogue of Proposition 5.5 for the restriction of ψ^e to $\mathcal{F}^e \smallsetminus \Gamma$. This is exactly the statement of Proposition 6.1, (2).

Proof of Proposition 6.1: (1) Let $\mathcal{Q}$ be the vector space of quadratic forms on $\mathbb{P}^{2n+1}$ of the form $q(x) + r(y)$ for some quadratic forms q and r. For each pair of integers (k, l) with $k \geq 0$, $l \geq -1$, let $\mathbb{G}_{k,l}$ be the locally closed subvariety of $(n-1)$-planes $P \in \mathbb{G}$ such that

$$\dim(P \cap \mathbb{P}^{n+2}) = k \qquad \dim(P \cap \mathbb{P}^{n-2}) = l\,.$$

(We put, by convention, $l = -1$ if $P \cap \mathbb{P}^{n-2} = \varnothing$.) Let

$$\mathscr{F}^{\mathcal{Q}} := \{(P, (Q_1, Q_2)) \in \mathbb{G} \times \mathcal{Q}^2 \mid Q_{1|P} = Q_{2|P} = 0\}$$

and

$$\mathscr{F}^{\mathcal{Q}}_{k,l} := \mathscr{F}^{\mathcal{Q}} \cap (\mathbb{G}_{k,l} \times \mathcal{Q}^2)\,.$$

The general fibre of the projection $\mathscr{F}^{\mathcal{Q}} \to \mathcal{Q}^2$ is an abelian variety, and we recover $\mathscr{F}^e$ by restricting $\mathscr{F}^{\mathcal{Q}}$ to pairs of quadratic forms of the form $(q_1(x) + r_1(y), q_2(x) + r_2(y))$, with $(q_1(x), q_2(x))$ fixed. Because we assume X general, the pair $(q_1(x), q_2(x))$ is general in $\mathcal{Q}^2$. It thus suffices to prove the result for the larger family $\mathscr{F}^{\mathcal{Q}}$; that is, to show that $\mathscr{F}^{\mathcal{Q}}_{k,l}$ has codimension ≥ 2 in $\mathscr{F}^{\mathcal{Q}}$.

This is done by a dimension count. For $P \in \mathbb{G}$, let φ_P be the restriction map $\mathcal{Q} \to H^0(P, \mathscr{O}_P(2))$. The fibre of the projection $\mathscr{F}^{\mathcal{Q}} \to \mathbb{G}$ is the vector space $(\mathrm{Ker}\varphi_P)^{\oplus 2}$. For P general, φ_P is surjective: This is the case, for instance, if P is contained in the $(n+2)$-plane in $\mathbb{P}^{2n+1}$ defined by $y_i = x_i$ $(i = 1, \dots, n-1)$. However, φ_P is not surjective for $P \in \mathbb{G}_{k,l}$ because the forms $r(y)_{|P}$ are singular along $P \cap \mathbb{P}^{n+2}$ and the forms $q(x)_{|P}$ are singular along $P \cap \mathbb{P}^{n-2}$; this implies that the subspaces $P \cap \mathbb{P}^{n+2}$ and $P \cap \mathbb{P}^{n-2}$ are apolar for all forms in $\mathrm{Im}\varphi_P$. Therefore, the corank of φ_P is $\geq (k+1)(l+1)$, and there is equality when P is contained in the subspace defined by $x_0 = \ldots = x_{n+1-k} = y_1 = \ldots = y_{n-2-l} = 0$, hence for P general in $\mathbb{G}_{k,l}$. Thus our assertion follows from:

$$\begin{aligned}
\operatorname{codim}(\mathscr{F}^{\mathcal{Q}}_{k,l}, \mathscr{F}^{\mathcal{Q}}) &= \operatorname{codim}(\mathbb{G}_{k,l}, \mathbb{G}) - 2(k+1)(l+1) \\
&= k(k+1) + (l+1)(l+4) - 2(k+1)(l+1) \\
&= (k-l)(k-l-1) + 2(l+1) \\
&\geq 2 \quad \text{if} \quad k \geq 1 \quad \text{or} \quad l \geq 0\,.
\end{aligned}$$

(2) The base locus $Z \subset \mathbb{P}T^*X$ has codimension ≥ 2 (Corollary 3.3). Note that ψ^e is well-defined in $\mathscr{F}^e \smallsetminus \Gamma$. If $\mathscr{D}$ is a codimension 1 subvariety in $\mathscr{F}^e \smallsetminus \Gamma$, with $\psi^e(\mathscr{D}) \subset Z$, the map ψ^e does not have maximal rank along $\mathscr{D}$. This contradicts the following lemma:

Lemma 6.2. *ψ^e has maximal rank on $\mathscr{F}^e \smallsetminus \Gamma$.*

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

Proof. Let (x,H) be a point of $\mathbb{P}T^*X$; we view H as a hyperplane in the projective tangent space to x at X. The fibre of $\psi^e:\mathscr{F}^e\smallsetminus\Gamma\to\mathbb{P}T^*X$ at (x,H) is the locus

$$(\psi^e)^{-1}(x,H)=\{(P,r_1,r_2)\in\mathbb{G}\times R^2 \mid P\cap\mathbb{P}^{n+2}=\{x\}\,,\ P\cap\mathbb{P}^{n-2}=\varnothing\,,\ \pi(P)=H, \tag{2}$$

$$(q_i+r_i)_{|P}=0 \quad (i=1,2)\}\,. \tag{3}$$

The equations (2) define a smooth, locally closed subvariety $\mathbb{G}_{x,H}$ of $\mathbb{G}$. Let $P\in\mathbb{G}_{x,H}$, and let $\chi_P:R\to H^0(P,\mathscr{O}_P(2))$ be the restriction map. We will show below that the image of χ_P is the space of quadratic forms on P that are singular at x. Since the forms $q_{i|P}$ are singular at x, this implies that the solutions of (3) form an affine space over $(\operatorname{Ker}\chi_P)^{\oplus 2}$. Therefore, $(\psi^e)^{-1}(x,H)$ admits an affine fibration over $\mathbb{G}_{x,H}$, hence is smooth.

Clearly the quadrics in $\operatorname{Im}\chi_P$ are singular at x. To prove the opposite inclusion, choose the coordinates (x_i) so that $x=(1,0,\ldots,0)$. Since $P\cap\mathbb{P}^{n+2}=\{x\}$, there exist linear forms $\ell_1,\ldots,\ell_{n+2}$ in the y_j so that P is defined by $x_i=\ell_i(y)$ for $i=1,\ldots,n+2$. Then a quadratic form on $\mathbb{P}^{2n+1}$ singular at x can be written as a form in $x_1,\ldots,x_{n+2};y_1,\ldots,y_{n-1}$; hence, its restriction to P is in $\operatorname{Im}\chi_P$. This proves the lemma and, hence, also the proposition. □

7. Symmetric tensors: second approach

7.1 The cotangent bundle of a smooth quadric

We consider a smooth quadric $Q\subset\mathbb{P}^{n+1}$ defined by an equation $q=0$. Its cotangent bundle $\mathbb{P}T^*Q$ parameterises pairs (x,P) with $x\in Q$ and P a $(n-1)$-plane tangent to Q at x. Thus, we get a morphism γ from $\mathbb{P}T^*Q$ to the Grassmannian $\mathbb{G}$ of $(n-1)$-planes in $\mathbb{P}^{n+1}$, which is the morphism defined by the linear system $|\mathscr{O}_{\mathbb{P}T^*Q}(1)|$. It is birational onto its image, but contracts the subvariety $\mathscr{C}\subset\mathbb{P}T^*Q$ that consists of pairs (x,P), such that P is tangent to Q along a line $\ell\subset Q$ with $x\in\ell$; then $\gamma^{-1}(P)$ consists of the pairs (x,P) with $x\in\ell$.

Let $h_q\in H^0(Q,\mathsf{S}^2\Omega^1_Q(2))$ be the hessian form of q (§3). Choosing coordinates (x_i) such that $q(x)=\sum x_i^2$, we have $h_q=\sum(dx_i)^2$ (note that this is, up to a scalar, the unique element of $H^0(Q,\mathsf{S}^2\Omega^1_Q(2))$ invariant under $\operatorname{Aut}(Q)$). Then $h_q(x)$ is non-degenerate at each point x of Q, so h_q induces an isomorphism $\Omega^1_Q(1)\xrightarrow{\sim}T_Q(-1)$, hence also $\mathsf{S}^2\Omega^1_Q(2)\xrightarrow{\sim}\mathsf{S}^2T_Q(-2)$. The image in $H^0(Q,\mathsf{S}^2T_Q(-2))$ of h_q by this isomorphism is $h'_q=\sum\partial_j^2$. We will view h'_q as an element of $H^0(\mathbb{P}T^*Q,\mathscr{O}_{\mathbb{P}T^*Q}(2)\otimes p^*\mathscr{O}_Q(-2))$, where $p:\mathbb{P}T^*Q\to Q$ is the projection.

Proposition 7.1. *The divisor of h'_q is $\mathscr{C}$. The projection $p_{|\mathscr{C}}:\mathscr{C}\to Q$ is a smooth quadric fibration, and $\mathscr{C}$ is a prime divisor for $n\geq 3$.*

Proof. Let $x\in Q$; the hyperplane tangent to Q at x cuts out a cone over the smooth quadric $Q_x\subset\mathbb{P}(T_x(Q))$ defined by $h_q(x)=0$ (Section 3). The isomorphism $T_x(Q)\xrightarrow{\sim}T^*_x(Q)$ given by $h_q(x)$ carries Q_x into the dual quadric Q^*_x in $\mathbb{P}(T^*_x(Q))$. On the other hand, a point $y\in p^{-1}(x)$ corresponds to a hyperplane $H_y\subset\mathbb{P}(T_x(Q))$, and y belongs to $\mathscr{C}$ if and only if H_y is tangent to Q_x; that is, if $y\in Q^*_x$. This proves the equality $\mathscr{C}=\operatorname{div}(h'_q)$ and thus, that the fiber of $p_{|\mathscr{C}}:\mathscr{C}\to Q$ at x is Q_x, which is smooth and connected if $n\geq 3$. □

Remark 7.2 The variety $\mathscr{C}$ is an example of a total dual VMRT [HLS22]. For the proof of the theorem, we will combine this tool with the birational transformation of $\mathbb{P}T^*X$ defined by a double cover. (Compare with [AH23]).

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3

We will have to consider the following situation: Let Q' be another quadric in $\mathbb{P}^{n+1}$, such that the intersection $B := Q \cap Q'$ is a smooth hypersurface in Q. The surjection $T_Q \to N_{B/Q}$ gives a section of $\mathbb{P}T^*Q$ over B, hence an embedding $s : B \hookrightarrow \mathbb{P}T^*Q$.

Lemma 7.3. *The image $s(B)$ is not contained in $\mathscr{C}$.*

Proof. Let $x \in B$. The point $s(x)$ in $\mathbb{P}(T_x^*(Q))$ corresponds to the hyperplane image of $T_x(B)$ in $T_x(Q)$; we must therefore show that this hyperplane is not tangent to the quadric $Q_x := h_q(x)$. In terms of projective space, this means that the projective tangent space to Q' at x is not tangent, at a smooth point y, to the cone cut out on Q by the projective tangent space to Q' at x.

Suppose that this is the case, with $y = (y_0, \ldots, y_{n+1})$. We can assume that Q' is defined by $\sum \alpha_i x_i^2 = 0$, with $\alpha_i \in \mathbb{C}$ distinct. Then the (projective) tangent space to Q' at x, given by $\sum (\alpha_i x_i)\xi_i = 0$, must coincide with the tangent space to Q at y, given by $\sum y_i \xi_i = 0$. This implies $y = (\alpha_0 x_0, \ldots, \alpha_{n+1} x_{n+1})$. Thus the point x must satisfy

$$\sum x_i^2 = \sum \alpha_i x_i^2 = \sum \alpha_i^2 x_i^2 = 0 \, .$$

If these relations hold for all x in B, the quadric $\sum \alpha_i^2 x_i^2 = 0$ must belong to the pencil spanned by Q and Q'. This means that there exist scalars λ, μ, ν such that

$$\lambda \alpha_i^2 + \mu \alpha_i + \nu = 0 \quad \text{for all } i \, ,$$

which is impossible since the α_i are distinct. Therefore, there exists $x \in B$ such that $s(x) \notin \mathscr{C}$. ☐

7.2 Explicit description of symmetric tensors

We keep the notation of the previous sections: $X \subset \mathbb{P} = \mathbb{P}^{n+2}$ is defined by $q_1 = q_2 = 0$, and with

$$q_1 = \sum_{i=0}^{n+2} x_i^2, \qquad q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 \qquad \text{with } \mu_i \in \mathbb{C} \text{ distinct.}$$

We put $\partial_i := \dfrac{\partial}{\partial x_i}\cdot$ We have an exact sequence

$$0 \to T_X \to T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} \mathscr{O}_X(2)^2 \to 0 \, ,$$

where dq_i maps the restriction of a vector field V on $\mathbb{P}$ to $V \cdot q_i$. This gives the exact sequence of symmetric tensors

$$0 \to \mathsf{S}^2 T_X \to \mathsf{S}^2 T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} T_{\mathbb{P}|X}(2)^2 \, , \tag{4}$$

where $dq_i(V_1 V_2) = (V_1 \cdot q_i) V_2 + (V_2 \cdot q_i) V_1$ for V_1, V_2 in $H^0(X, T_{\mathbb{P}|X})$.

Proposition 7.4. *The quadratic vector fields $s_i := \sum_{j \neq i} \dfrac{(x_i \partial_j - x_j \partial_i)^2}{\mu_j - \mu_i}$ in $H^0(X, \mathsf{S}^2 T_{\mathbb{P}|X})$ belong to the image of $H^0(X, \mathsf{S}^2 T_X)$.*

Proof. According to the exact sequence (4), we have to prove $dq_1(s_i) = dq_2(s_i) = 0$.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

We have $(x_i\partial_j - x_j\partial_i)\cdot q_1 = 0$; hence, $dq_1(s_i)=0$ and $dq_2(x_i\partial_j - x_j\partial_i, x_i\partial_j - x_j\partial_i) = 4(\mu_j - \mu_i)x_ix_j(x_i\partial_j - x_j\partial_i)$. Hence, using $\sum x_j\partial_j = 0$ and $q_{1|X}=0$,

$$dq_2(s_i) = 4x_i^2\sum_{j\neq i} x_j\partial_j - 4x_i(\sum_{j\neq i} x_j^2)\partial_i = 0\,, \quad \text{which proves the proposition.}$$

□

In the rest of this article, we will consider the s_i to be elements of $H^0(X, \mathsf{S}^2T_X)$.

7.3 The double cover

Let $p_0 : \mathbb{P}^{n+2} \dashrightarrow \mathbb{P}^{n+1}$ be the projection $(x_0,\dots,x_{n+2}) \mapsto (x_1,\dots,x_{n+2})$. The image $p_0(X)$ is the smooth quadric Q in $\mathbb{P}^{n+1}$ defined by

$$\sum_{i=1}^{n+2}(\mu_i-\mu_0)x_i^2 = 0\,.$$

The restriction $\pi : X \to Q$ of p_0 is a double covering that is branched along the subvariety $B \subset Q$ defined by

$$\sum_{i=1}^{n+2} x_i^2 = \sum_{i=1}^{n+2}\mu_i x_i^2 = 0\,.$$

It is a smooth complete intersection of 2 quadrics in $\mathbb{P}^{n+1}$. The ramification locus $R\subset X$ of π (isomorphic to B) is the hyperplane section $x_0 = 0$ of X.

The tangent map of $\pi : X \to Q$ gives a morphism,

$$\tau : T_X \to \pi^*T_Q,$$

which is an isomorphism outside of R. Consider the normal exact sequence

$$0 \to T_R \to T_{X|R} \to N_{R/X} \to 0\,.$$

The involution $\iota : (x_0,\dots,x_{n+2}) \mapsto (-x_0, x_1,\dots,x_{n+2})$ acts on $T_{X|R}$; this splits the exact sequence, giving a decomposition

$$T_{X|R} = T_R \oplus N_{R/X}$$

into eigenspaces for the eigenvalues $+1$ and -1. Let $\rho : T_{X|R} \to T_R$ be the projection on the first summand. We deduce from ρ a sequence of homomorphisms

$$h^k : H^0(X, \mathsf{S}^kT_X) \longrightarrow H^0(X, \mathsf{S}^kT_{X|R}) \xrightarrow{\mathsf{S}^k\rho} H^0(R, \mathsf{S}^kT_R)\,.$$

Since $\iota_*\partial_0 = -\partial_0$ and $\iota_*\partial_j = \partial_j$ for $j>0$, we have

$$h^2(s_0) = 0 \quad\text{and}\quad h^2(s_i) = \sum_{\substack{j>0\\ j\neq i}} \frac{(x_i\partial_j - x_j\partial_i)^2}{\mu_j-\mu_i} \quad \text{for } i>0 \tag{5}$$

in other words, h^2 maps $s_1,\dots,s_{n+2}$ to the elements $\hat{s}_1,\dots,\hat{s}_{n+2}$ of $H^0(R, \mathsf{S}^2T_R)$ constructed in Proposition 7.2.1 applied to R.

Let $\pi^*\mathbb{P}T^*Q$ be the pull-back under π of the projective bundle $\mathbb{P}T^*Q \to Q$. The homomorphism $\tau : T_X \to \pi^*T_Q$ gives rise to the birational map $g : \pi^*\mathbb{P}T^*Q \dashrightarrow \mathbb{P}T^*X$. Following the geometric description of the tangent map as an elementary transformation of vector bundles in the sense of Maruyama in [Mar72] and [Mar73, Corollary 1.1.1], one has a commutative diagram

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

$$
\begin{tikzcd}
 & \Gamma \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\pi^*\mathbb{P}T^*Q \arrow[rr, dashed, "g"] \arrow[dr, "p"'] & & \mathbb{P}T^*X \arrow[dl, "q"] \\
 & X &
\end{tikzcd}
\tag{6}
$$

where p and q are the canonical projections; $\nu:\Gamma\to\mathbb{P}T^*X$ is the blow-up along the subspace $\mathbb{P}T^*R\subset\mathbb{P}T^*X$ defined by the projection ρ; and $\mu:\Gamma\to\pi^*\mathbb{P}T^*Q$ is the blow-up of the image B' of the embedding $B\hookrightarrow\pi^*\mathbb{P}T^*Q$ deduced from the surjective homomorphism $\pi^*T_Q\to\pi^*N_{B/X}$.

Let E_μ be the exceptional divisor of μ. By [Mar73, Theorem 1.1], there is an isomorphism

$$\mu^*\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)\otimes\mathscr{O}_\Gamma(-E_\mu)\cong\nu^*\mathscr{O}_{\mathbb{P}T^*X}(1) \tag{7}$$

as well as the equality

$$\nu_*E_\mu=q^*R\,. \tag{8}$$

7.4 The divisor of s_0

We now consider the divisor $\mathscr{C}\subset\mathbb{P}T^*Q$ defined in (7.1) and the cartesian diagram

$$
\begin{tikzcd}
\pi^*\mathbb{P}T^*Q \arrow[r, "\pi'"] \arrow[d] & \mathbb{P}T^*Q \arrow[d] \\
X \arrow[r, "\pi"] & Q\,.
\end{tikzcd}
$$

Set $\mathscr{C}':={\pi'}^{-1}(\mathscr{C})$. The projection $\mathscr{C}'\to X$ is again a smooth quadric fibration, so $\mathscr{C}'$ is smooth and connected for $n\geq 3$.

Recall that we have defined the element $s_0:=\displaystyle\sum_{j=1}^{n+2}\frac{(x_0\partial_j-x_j\partial_0)^2}{\mu_j-\mu_0}\in H^0(X,\mathsf{S}^2T_X)$ (7.2). We will now view s_0 as an element of $H^0(\mathbb{P}T^*X,\mathscr{O}(2))$.

Proposition 7.5 *Assume $n\geq 3$. We have $g_*\mathscr{C}'=\operatorname{div}(s_0)$.*

Proof. We first show that $g_*\mathscr{C}'\in|\mathscr{O}_{\mathbb{P}T^*X}(2)|$. By Proposition 7.1 we have $\mathscr{C}'\in|\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(2)\otimes p^*\mathscr{O}_X(-2)|$. Using (7), (8) and the projection formula, we get the linear equivalences

$$\nu_*\mu^*\mathscr{C}'\sim 2\nu_*\mu^*(c_1(\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)-p^*R))\sim 2(c_1(\mathscr{O}_{\mathbb{P}T^*X}(1))+q^*R)-2q^*R=c_1(\mathscr{O}_{\mathbb{P}T^*X}(2))\,.$$

Thus, it is enough to prove that $\nu_*\mu^*\mathscr{C}'$ is irreducible. Since $\mathscr{C}'$ is irreducible and μ is the blow-up along $B'\subset\pi^*\mathbb{P}T^*Q$, it suffices to show that B' is not contained in $\mathscr{C}'$. If this is the case, then we have $\pi'(B')\subset\pi'(\mathscr{C}')=\mathscr{C}$. But $\pi'(B')=s(B)$, where $s:B\hookrightarrow\mathbb{P}T^*Q$ is the embedding defined by the surjective homomorphism $T_Q\to N_{B/Q}$. Then the result follows from Lemma 7.3.

Since $g_*\mathscr{C}'$ and $\operatorname{div}(s_0)$ are linearly equivalent effective divisors and $g_*\mathscr{C}'$ is irreducible, it suffices to show that their restrictions to $\mathbb{P}T^*_x(X)$ coincide at a general point $x\in X$.

Fix a point $x=(x_0,\ldots,x_{n+2})\in X\smallsetminus R$ so that $x_0\neq 0$. Then the tangent map $T\pi(x):T_x(X)\to T_{\pi(x)}(Q)$ is an isomorphism; in diagram (6), the maps μ,ν and g restricted over the fibres at x are all isomorphisms. Let us show that $\mathscr{C}'$ and $T\pi(\operatorname{div}(s_0))$ define the same quadric in $\mathbb{P}(T_{\pi(x)}(Q))$.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

Note that $\mathscr{C}' \cap \mathbb{P}(T^*_x(X)) = \mathscr{C} \cap \mathbb{P}(T^*_{\pi(x)}(Q))$ is the quadric defined by the element h'_q of (7.1). In the coordinates (z_i) defined by $z_i = (\mu_i - \mu_0)^{1/2} x_i$, the equation of Q is $\sum_{j=1}^{n+2} z_j^2 = 0$, so

$$h'_q = \sum_{j=1}^{n+2} (\frac{\partial}{\partial z_j})^2 = \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0} \,.$$

On the other hand, since $\pi(x_0, \ldots, x_{n+2}) = (x_1, \ldots, x_{n+2})$, we have $T\pi(\partial_0) = 0$ and $T\pi(\partial_j) = \partial_j$ for $j > 0$; hence,

$$T\pi(s_0) = x_0^2 \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0} \,.$$

Since $x_0 \neq 0$, this proves the proposition. □

7.5 Proof of part (a) of the theorem

Suppose now that $n \geq 3$. Consider the double cover $\pi : X \to Q$ and the ramification divisor $R \subset X$. The restriction maps h^k defined in Section 7.3 yield a homomorphism of graded $\mathbb{C}$-algebras

$$h : S(X) := H^0(X, \mathsf{S}^\bullet T_X) \longrightarrow H^0(R, \mathsf{S}^\bullet T_R) =: S(R).$$

Proposition 7.6 *The kernel $\mathscr{I}$ of h is the ideal generated by s_0.*

Proof. Since $\mathscr{I}$ is a homogeneous ideal, it suffices to prove that every homogeneous element $s \in \mathscr{I}$ can be written as $s = s' s_0$ for some element $s' \in S(X)$.

Choose an element $s \in \mathscr{I}$ of degree k. This element corresponds to an effective Cartier divisor G in the linear system $|\mathscr{O}_{\mathbb{P}T^*X}(k)|$. Recall the commutative diagram (6):

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\pi^*\mathbb{P}T^*Q \arrow[rr, dashed, "g"] \arrow[dr, "p"'] & & \mathbb{P}T^*X \arrow[dl, "q"] \\
 & X &
\end{tikzcd}$$

Choose $\hat{G} := \mu_* \nu^* G \subset \pi^*\mathbb{P}T^*Q$. By (7), $\hat{G}$ belongs to the linear system $|\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(k)|$.

Here is the key observation: Since $s \in \mathscr{I}$, the divisor $\hat{G} \subset \pi^*\mathbb{P}T^*Q$ contains p^*R. Indeed, since $(\pi^*T_Q)_{|R}$ is invariant under ι, the homomorphism $\tau_{|R}$ factors as

$$\tau_{|R} : T_{X|R} \xrightarrow{\rho} T_R \longrightarrow (\pi^*T_Q)_{|R} \,.$$

Therefore, we have a commutative diagram,

$$\begin{tikzcd}
H^0(X, \mathsf{S}^k T_X) \arrow[r, "h^k"] \arrow[d, "\mathsf{S}^k\tau"'] & H^0(R, \mathsf{S}^k T_R) \arrow[d] \\
H^0(X, \mathsf{S}^k \pi^* T_Q) \arrow[r] & H^0(R, \mathsf{S}^k (\pi^*T_Q)_{|R})
\end{tikzcd}$$

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

and $\mathsf{S}^k\tau(s)$ vanishes on R. But $\hat{G}$ is the divisor of $\mathsf{S}^k\tau(s)$, viewed as a section of $\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(k)$; hence, $\hat{G}$ contains p^*R.

Now we want to show that the divisor $\mathscr{C}' \subset \pi^*\mathbb{P}T^*Q$ is a component of $\hat{G} - p^*R$. Recall from (7.1) that $\mathscr{C}$ is the union of the lines ℓ that are contracted by the morphism $\gamma : \mathbb{P}T^*Q \to \mathbb{G}$ and that $c_1(\mathscr{O}_{\mathbb{P}T^*Q}(1)) \cdot \ell = 0$. Thus the curves $\ell' := \pi'^*\ell$ cover $\mathscr{C}'$ and satisfy $c_1(\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)) \cdot \ell' = 0$. On the other hand, the divisor $R \subset X$ is a hyperplane section, so $p^*R \cdot \ell' = R \cdot p_*\ell' > 0$. Therefore,

$$(\hat{G} - p^*R) \cdot \ell' < 0\,,$$

so $\mathscr{C}'$ is a component of $\hat{G}$. Thus, $g_*\mathscr{C}'$ is a component of G. Since $g_*\mathscr{C}' = \operatorname{div}(s_0)$ by Proposition 7.5, this proves the proposition. $\square$

The following proposition implies part (a) of our main theorem:

Proposition 7.7. *Assume $n \geq 2$. For any choice of indices $0 \leq i_1 < \ldots < i_n \leq n+2$, the homomorphism $\mathbb{C}[t_1, \ldots, t_n] \to S(X)$, which maps t_j to s_{i_j}, with $\deg(t_i) = 2$, is an isomorphism of graded $\mathbb{C}$-algebras.*

Proof. We argue by induction on n. The statement for $n = 2$ follows from [DOL19, Theorem 5.1], except for the fact that any two of the s_i generate $H^0(X, \mathsf{S}^2T_X)$. Up to the permuting of the coordinates, it suffices to prove that s_0 and s_1 are linearly independent. But $h^2 : H^0(X, \mathsf{S}^2T_X) \to H^0(R, \mathsf{S}^2T_R)$ maps s_0 to zero and maps s_i (for $i > 0$) to the corresponding elements $\hat{s}_i$ of $H^0(R, \mathsf{S}^2T_R)$; this implies our assertion.

Assume $n \geq 3$. By the induction hypothesis, the homomorphism $\mathbb{C}[t_1, \ldots, t_{n-1}] \to S(R)$, which maps t_i to $\hat{s}_i$, is an isomorphism of graded $\mathbb{C}$-algebras (with $\deg(t_i) = 2$). It follows that h is surjective and that $(s_0, \ldots, s_{n-1})$ form a basis of $H^0(X, \mathsf{S}^2T_X)$ and generate the $\mathbb{C}$-algebra $S(X)$. Thus we have a surjective homomorphism $u : \mathbb{C}[t_0, \ldots, t_{n-1}] \to S(X)$, with $u(t_i) = s_i$.

In particular, the Krull dimension of $S(X)$ is at most n. On the other hand, the ring $S(X)$ is a domain, and s_0 is neither zero nor a unit. Thus, by Krull's Hauptidealsatz, the Krull dimension of $S(X)$ is equal to n; hence, u is an isomorphism. By permutation of the coordinates, we get the same result for any choice of n elements in $\{s_0, \ldots, s_{n+2}\}$. This proves the proposition. $\square$

Conflicts of Interest

None.

Financial Support

J. Liu is supported by the National Key Research and Development Program of China (No. 2021YFA1002300), the NSFC grants (No. 12001521 and No. 12288201) and the CAS Project for Young Scientists in Basic Research (No. YSBR-033).

Journal Information

Moduli is published as a joint venture of the Foundation Compositio Mathematica and the London Mathematical Society. As not-for-profit organisations, the Foundation and Society reinvest 100% of any surplus generated from their publications back into mathematics through their charitable activities.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms.
https://doi.org/10.1112/mod.2024.3

References

[A96] M. Audin, *Spinning tops. A course on integrable systems*, Cambridge Studies in Advanced Mathematics, vol. **51** (CUP, Cambridge, 1996).

[AH23] F. Anella and A. Höring, *The cotangent bundle of a K3 surface of degree two*, Épijournal Géom. Algébr. (2023). Special volume in honour of Claire Voisin, Art. 3.

[BHK10] I. Biswas, Y. I. Holla and C. Kumar, *On moduli spaces of parabolic vector bundles of rank 2 over* $\mathbb{CP}^1$, Michigan Math. J. **59** (2010), 467–475.

[BL92] C. Birkenhake and H. Lange, *Complex abelian varieties*, Grundlehren der mathematischen Wissenschaften, vol. **302** (Springer-Verlag, Berlin, 1992).

[BLi24] A. Beauville and J. Liu, *The algebra of symmetric tensors on smooth projective varieties*, Sci. China Math., to appear (2024).

[BNR89] M. Beauville, S. Narasimhan and S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math. **398** (1989), 169–179.

[Cas15] C. Casagrande, *Rank* 2 *quasiparabolic vector bundles on* $\mathbb{P}^1$ *and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. **280** (2015), 981–988.

[DOL19] B. De Oliveira and C. Langdon, Twisted symmetric differentials and the quadric algebra of subvarieties of $\mathbb{P}^N$ *of low codimension*, Eur. J. Math. **5** (2019), 454–475.

[DR76] U. Desale and S. Ramanan, *Classification of vector bundles of rank* 2 *on hyperelliptic curves*, Invent. Math. **38** (1976/77), 161–185.

[GH79] P. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. École Norm. Sup. **12** (1979), 355–452.

[Hit87] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987) 91–114.

[HLS22] A. Höring, J. Liu and F. Shao, *Examples of Fano manifolds with non-pseudoeffective tangent bundle*, J. Lond. Math. Soc. **106** (2022), 27–59.

[K80] H. Knörrer, *Geodesics on the ellipsoid*, Invent. Math. **59** (1980), 119–143.

[KL22] H. Kim and Y. Lee, *Lagrangian fibration structure on the cotangent bundle of a del Pezzo surface of degree* \$4\$, Preprint arXiv:2210.01317.

[Mar72] M. Maruyama, *On a family of algebraic vector bundles*, Dissertation Kyoto University, 1972, https://doi.org/10.14989/doctor.r2072.

[Mar73] M. Maruyama, *On a family of algebraic vector bundles*, in Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki (Kinokuniya, Tokyo, 1973), 95–146.

[New68] P. Newstead, *Stable bundles of rank* 2 *and odd degree over a curve of genus* 2, Topology **7** (1968), 205–215.

[Reid72] M. Reid, *The complete intersection of two or more quadrics*. PhD thesis (Cambridge University, 1972).

Arnaud Beauville arnaud.beauville@univ-cotedazur.fr
Université Côte d'Azur, CNRS – Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice cedex 2, France.

Antoine Etesse antoine.etesse@ens-lyon.fr
ENS de Lyon, UMPA, CNRS UMR 5669, 46 allée d'Italie, 69364 Lyon Cedex 07, France.

Andreas Höring Andreas.Hoering@univ-cotedazur.fr
Université Côte d'Azur, CNRS – Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice cedex 2, France.

Jie Liu jliu@amss.ac.cn
Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.

Claire Voisin claire.voisin@imj-prg.fr
Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France.

Downloaded from https://www.cambridge.org/core. IP address: 18.218.251.71, on 11 Jan 2025 at 00:45:54, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1112/mod.2024.3