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A CLASS OF MELLIN MULTIPLIERS

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ABSTRACT. We examine a class of functions which can serve as Mellin multipliers in the setting of the spaces $F_{p,\mu}$ which we have used extensively in other papers. The conditions to be satisfied by such a multiplier h do not involve h' explicitly. This means that multipliers involving Γ-functions can be handled by means of the asymptotics of $\Gamma(z)$ alone, without the need to study $\psi = \Gamma'/\Gamma$, thereby saving effort in the case of complicated multipliers.

1. In [7], Rooney introduced a class $\mathcal{A}$ of Mellin multipliers such that each multiplier $h \in \mathcal{A}$ gives rise to a corresponding bounded linear mapping T from $L_{p,\mu}$ into $L_{p,\mu}$ for $1 < p < \infty$ and suitable complex numbers μ. In particular, the relation

$$\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s), \quad \operatorname{Re} s = -\operatorname{Re}\mu$$

holds for all $f \in L_{p,\mu} \cap L_{2,\mu}$, where $\mathcal{M}$ denotes the Mellin transform.

Recently we have been concerned with multipliers for continuous linear mappings from $F_{p,\mu}$ into $F_{p,\mu}$, where $F_{p,\mu}$ is a certain subspace of smooth functions in $L_{p,\mu}$. It was proved in [3, Theorem 3.3] that every multiplier which gives rise to a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$ does likewise for $F_{p,\mu}$, i.e. every $L_{p,\mu}$ multiplier is an $F_{p,\mu}$ multiplier. However, the class of $F_{p,\mu}$ multipliers is strictly larger.

The definition of Rooney's class $\mathcal{A}$ involves a condition on h', the derivative of h, and this condition can be tedious to verify if h is complicated. We have been particularly interested in multipliers involving products and/or quotients of gamma functions where the appropriate condition on h' can be checked via the asymptotics of Γ and $\psi = \Gamma'/\Gamma$. However, the calculations involving ψ are unnecessary. We shall obtain another criterion involving h, but not h', which will guarantee that h is an $F_{p,\mu}$ multiplier and which will be applicable in particular to our Γ-function multipliers. See [4], [5] and [6] for details of this application, along with the necessary background.

2. First let us establish the notation to be used. Throughout we shall assume that $1 < p < \infty$ and that μ is a suitable complex number.

DEFINITION 2.1.

(i) We denote by $L_{p,\mu}$ the set

$$L_{p,\mu} = \{f : \|f\|_{p,\mu} < \infty\} \tag{2.1}$$

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where

(2.2) $$\|f\|_{p,\mu} = \{ \int_0^\infty |x^{-\mu} f(x)|^p dx/x \}^{1/p}.$$

(ii) We denote by $F_{p,\mu}$ the set

(2.3) $$F_{p,\mu} = \{f \in C^\infty(0,\infty) : \delta^i f \in L_{p,\mu} \text{ for } i = 0,1,2,\ldots\}$$

where

(2.4) $$(\delta f)(x) = xf'(x).$$

For $i = 0,1,2,\ldots$ and $f \in F_{p,\mu}$, define $\nu_i^{p,\mu}(f)$ by

(2.5) $$\nu_i^{p,\mu}(f) = \|\delta^i f\|_{p,\mu}.$$

REMARK 2.2.

(i) The expression $\| \ \|_{p,\mu}$ in (2.2) defines a norm on $L_{p,\mu}$ and $(L_{p,\mu}, \| \ \|_{p,\mu})$ is a Banach space.

(ii) For each $i = 0,1,2,\ldots,$ the expression $\nu_i^{p,\mu}(f)$ defines a seminorm on $F_{p,\mu}$ and $\nu_0^{p,\mu}(f)$ defines a norm. The topology generated by the multinorm $\{\nu_i^{p,\mu}\}_{i=0}^\infty$ turns $F_{p,\mu}$ into a Fréchet space.

(iii) The seminorms $\{\nu_i^{p,\mu}\}_{i=0}^\infty$ are more convenient here than the equivalent family of seminorms $\{\gamma_i^{p,\mu}\}_{i=0}^\infty$ defined by

$$\gamma_i^{p,\mu}(f) = \|x^i D^i f(x)\|_{p,\mu}$$

which are used in [4] and elsewhere.

(iv) Although μ is assumed to be real in [7], we can allow μ to be complex without any difficulty. The spaces $(L_{p,\mu}, \| \ \|_{p,\mu})$ and $(L_{p,\mathrm{Re}\,\mu}, \| \ \|_{p,\mathrm{Re}\,\mu})$ are identical, so that there is no loss of generality in taking μ real when it is convenient.

(v) Our set $L_{p,\mu}$ corresponds to $L^p_{-\mu}$ in [7].

DEFINITION 2.3. For suitable functions f, we define $\mathcal{M}f$, the Mellin transform of f, by

(2.6) $$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

for suitable complex numbers s.

THEOREM 2.4. *For $1 < p \leq 2$ and $f \in L_{p,\mu}$, $\mathcal{M}f$ exists almost everywhere on the line*

(2.7) $$\mathrm{Re}\, s = -\mathrm{Re}\,\mu$$

the integral (2.6) being interpreted in terms of mean convergence.

PROOF. See [7] but note Remark 2.2(v). ∎

DEFINITION 2.5. The set $\mathcal{A}$ consists of all functions h for which there exist extended real numbers α and β (depending on h) with $\alpha < \beta$ such that

(i) $h(s)$ is analytic on the strip $\alpha < \mathrm{Re}\, s < \beta$

(ii) $h(s)$ is bounded on every closed substrip $\alpha' \leq \mathrm{Re}\, s \leq \beta'$ where $\alpha < \alpha' \leq \beta' < \beta$

(iii) for $\alpha < \mathrm{Re}\, s < \beta$, $|h'(s)| = 0(|\,\mathrm{Im}\, s|^{-1})$ as $|\,\mathrm{Im}\, s| \to \infty$.

THEOREM 2.6. *Every function $h \in \mathcal{A}$ is an $L_{p,\mu}$ multiplier. More precisely, for α, β as in Definition 2.5, there exists a linear operator T such that*

(i) T is a bounded linear operator from $L_{p,\mu}$ into $L_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$

(ii)

$$\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s) \text{ on the line } \operatorname{Re} s = -\operatorname{Re}\mu \tag{2.8}$$

whenever $f \in L_{p,\mu} \cap L_{2,\mu}$, $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [7, Theorem 1]. ■

THEOREM 2.7. *Every function $h \in \mathcal{A}$ is an $F_{p,\mu}$ multiplier. More precisely, for α, β as in Definition 2.5, there exists a linear operator T such that*

(i) T is a continuous linear operator from $F_{p,\mu}$ into $F_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$

(ii) (2.8) holds for $f \in F_{p,\mu} \cap F_{2,\mu}$ where $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [3, Theorem 3.3]. ■

REMARK 2.8.

(i) In the situation of Theorems 2.6 and 2.7 we shall call T a (Mellin) multiplier transform having h as its multiplier.

(ii) Functions other than those in $\mathcal{A}$ can act as multipliers of continuous operators, simple examples being 1 and $-s$ which correspond to the identity operator and δ, as in (2.4), the latter only being meaningful in $F_{p,\mu}$ rather than in $L_{p,\mu}$.

3. As indicated in § 1 we now introduce a class of functions which can serve as multipliers but which are characterised by conditions which do not involve a"growth" estimate of the derivative. It turns out that the growth estimate in Definition 2.5(iii) is a consequence of the alternative conditions, these being easier to check in certain cases.

DEFINITION 3.1. The set $\mathcal{B}$ consists of all functions h for which there exist real numbers α and β (depending on h) with $\alpha < \beta$ such that

(i) $h(s)$ is analytic on the strip $\alpha < \operatorname{Re} s < \beta$

(ii) $sh(s)$ is bounded on every closed substrip $\alpha' \leq \operatorname{Re} s \leq \beta'$ where $\alpha < \alpha' \leq \beta' < \beta$.

THEOREM 3.2. *$\mathcal{B}$ is a subset of $\mathcal{A}$.*

PROOF. We check the conditions of Definition 2.5. Condition (i) for $\mathcal{A}$ follows from condition (i) for $\mathcal{B}$. Also boundedness of $sh(s)$ on the strip $\alpha' \leq \operatorname{Re} s \leq \beta'$ guarantees boundedness of $h(s)$ on the same strip with $|h(s)| = 0(\operatorname{Im} s|^{-1})$ as $|\operatorname{Im} s| \to \infty$ within the strip. It remains to get a similar estimate for the derivative. For given α' and β', let

$$\epsilon = \frac{1}{2}\min(\beta - \beta', \alpha' - \alpha)$$

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$$M = \sup\{ |sh(s)| : \alpha' - \epsilon \leq \operatorname{Re} s \leq \beta' + \epsilon \}.$$

Note that $[\alpha' - \epsilon, \beta' + \epsilon] \subset (\alpha, \beta)$ so that M exists by Definition 3.1(ii). Let $\rho = \epsilon/2$. Then for $\alpha' \leq \operatorname{Re} s \leq \beta'$, we may write

$$sh'(s) = \frac{d}{ds}\big(sh(s)\big) - h(s) = \frac{1}{2\pi i}\int_{C_\rho} \frac{zh(z)}{(z-s)^2}\,dz - h(s)$$

where C_ρ denotes the circle with centre s and radius ρ, C_ρ lying entirely within the strip $\alpha' - \epsilon \leq \operatorname{Re} s \leq \beta' + \epsilon$ by choice of ρ. By a standard estimate, we obtain

$$|sh'(s)| \leq \frac{1}{2\pi}\frac{M}{\rho^2}\cdot 2\pi\rho + |h(s)| = \frac{M}{\rho} + |h(s)|.$$

As noted above, $|h(s)|$ is bounded on $\alpha' \leq \operatorname{Re} s \leq \beta'$ and hence so is $|sh'(s)|$. It now follows that $|h'(s)| = 0(|\operatorname{Im} s|^{-1})$ as $|\operatorname{Im} s| \longrightarrow \infty$ within the strip $\alpha' \leq \operatorname{Re} s \leq \beta'$. This verifies the third condition in Definition 2.5 and therefore completes the proof. ∎

The multiplier transforms corresponding to multipliers in $\mathcal{B}$ form a subset of those corresponding to those in $\mathcal{A}$. For instance we lose the identity transformation whose multiplier h, given by $h(s) \equiv 1$, belongs to $\mathcal{A}$ but not to $\mathcal{B}$. The transforms corresponding to multipliers in $\mathcal{B}$ can be characterised as convolution integral operators by virtue of the following result.

THEOREM 3.3. *Let $h \in \mathcal{B}$ and let α and β be as in Definition 3.1. Then there exists a function k such that*

(i) $k \in L_{1,\mu}$ for all μ satisfying $\alpha < -\operatorname{Re}\mu < \beta$

(ii) $(\mathcal{M}k)(s) = h(s)$ on the strip $\alpha < \operatorname{Re} s < \beta$.

The corresponding multiplier transform T is given by

$$(3.1) \qquad (Tf)(x) = (k * f)(x) \equiv \int_0^\infty k(x/t)f(t)\,dt/t \quad (f \in L_{p,\mu})$$

and is a bounded linear mapping from $L_{p,\mu}$ into itself for $1 < p < \infty, \alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [8, Theorem 2.35]. ∎

EXAMPLE 3.4. Let us review a familiar operator in the context of the class $\mathcal{B}$. Consider the multiplier

$$h(s) = \Gamma(\eta + s)/\Gamma(\eta + \gamma + s)$$

where η and γ are complex numbers with $\operatorname{Re}\gamma > 0$. h is analytic in the half-plane $\operatorname{Re} s > -\operatorname{Re}\eta$. Take $\alpha = -\operatorname{Re}\eta$, β to be any real number such that $\beta > \alpha$. For condition (ii) in Definition 3.1 we may make use of the formula [1, 1.18(6)]

$$(3.2) \qquad |\Gamma(x+iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\pi|y|/2} \text{ as } |y| \longrightarrow \infty.$$

Then if we write $\gamma = \gamma_1 + i\gamma_2$, $\eta = \eta_1 + i\eta_2$, $s = \sigma + i\tau$, take $-\eta_1 < \alpha' \leq \sigma \leq \beta' < \beta$ and note that (3.2) is uniform in x for x in a compact subset of $\mathbb{R}$, we get

$$(3.3) \qquad \left| s\frac{\Gamma(\eta+s)}{\Gamma(\eta+\gamma+s)}\right| \leq (C + |\tau|)\frac{|\eta_2+\tau|^{\eta_1+\sigma-1/2}}{|\eta_2+\gamma_2+\tau|^{\eta_1+\gamma_1+\sigma-1/2}} \times \exp[-\pi\{|\eta_2+\tau| - |\eta_2+\gamma_2+\tau|\}/2]$$

(C a constant) and for boundedness as $|\tau| \to \infty$ we require $1 - \gamma_1 \leq 0$ i.e. $\operatorname{Re}\gamma \geq 1$. In this case the corresponding multiplier transform is the Erdélyi-Kober operator $K_1^{\eta,\gamma}$ given by

$$(3.4) \qquad (K_1^{\eta,\gamma} f)(x) = [\Gamma(\gamma)]^{-1} x^{\eta} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\eta-\gamma} f(t)\, dt$$

which has the form (3.1) with the kernel

$$(3.5) \qquad k(t) = \begin{cases} [\Gamma(\gamma)]^{-1}(1-t)^{\gamma-1} t^{\eta} & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}.$$

We deduce that, for $\operatorname{Re}\gamma \geq 1$, $K_1^{\eta,\gamma}$ is a bounded linear mapping from $L_{p,\mu}$ into itself whenever $\operatorname{Re}\eta > \operatorname{Re}\mu$ (as $\beta > \alpha$ was arbitrary). However, it is well-known that the resulting operator remains bounded under the weaker condition $\operatorname{Re}\gamma > 0$ (and $\operatorname{Re}\eta > \operatorname{Re}\mu$ as before). Indeed we can check that the kernel k in (3.5) belongs to $L_{1,\mu}$ under these conditions. Thus the set $\mathcal{B}$ does not tell the whole story in the $L_{p,\mu}$ setting, i.e. $h \in \mathcal{B}$ is sufficient to guarantee a convolution integral operator but not necessary.

REMARK 3.5. At this stage the reader may wonder why we have introduced $\mathcal{B}$ at all. It is true that the multiplier h in Example 3.4 belongs to $\mathcal{A}$ under the weaker condition $\operatorname{Re}\gamma > 0$, as can be checked via the asymptotics of the function $\psi = \Gamma'/\Gamma$ given by [1, 1.18(7)]. However, although our class $\mathcal{B}$ may seem to be deficient in the $L_{p,\mu}$ setting, it comes into its own (suitably modified) in the $F_{p,\mu}$ setting. It is in that setting that we can obtain the most elegant theory for multipliers involving products and/or quotients of gamma functions. Accordingly, we shall proceed to $F_{p,\mu}$ for our subsequent discussions.

4. When we are working in $F_{p,\mu}$, polynomials are available to us as multipliers, with the polynomial $P(s) = \sum_{i=0}^{n} a_i s^i$ corresponding to the continuous linear operator $P(-\delta) = \sum_{i=0}^{n} a_i(-\delta)^i$. We shall exploit this to the full in defining our next class of multipliers.

DEFINITION 4.1. The set C consists of all functions h for which there exist extended real numbers α and β (depending on h) such that

(i) $h(s)$ is analytic on the strip $\alpha < \operatorname{Re} s < \beta$

(ii) for each α_0 and β_0 satisfying $\alpha < \alpha_0 < \beta_0 < \beta$, there exists a non-negative integer $N \equiv N(\alpha_0, \beta_0, h)$ such that

$$(4.1) \qquad \begin{cases} (\alpha_0 - s)^{-N} s h(s) \text{ is bounded on every closed substrip} \\ \alpha' \leq \operatorname{Re} s \leq \beta' \text{ where } \alpha_0 < \alpha' < \beta' < \beta_0. \end{cases}$$

REMARK 4.2.

(i) Condition (ii) in Definition 4.1 says that if we restrict attention to $\alpha_0 < \operatorname{Re} s < \beta_0$ we can find a non-negative integer N such that $(\alpha_0 - s)^{-N} h(s)$ defines a multiplier in $\mathcal{B}$. However if we change α_0 and β_0, we are allowed to change N in order to control the growth of h.

(ii) Instead of introducing the factor $(\alpha_0 - s)^{-N}$, we could equally well have introduced $(\beta_0 - s)^{-N}$, with the same effect.

Immediately we can prove

THEOREM 4.3. *Let $h \in \mathcal{C}$ and let α, β be as in Definition 4.1. Then there exists an operator T such that, for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$,*

(i) T is a continuous linear operator from $F_{p,\mu}$ into $F_{p,\mu}$

(ii) $\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s)$ on the line $\operatorname{Re} s = -\operatorname{Re}\mu$ for all $f \in F_{p,\mu} \cap F_{2,\mu}$.

PROOF. Choose α_0 and β_0 such that $\alpha < \alpha_0 < \beta_0 < \beta$. With N as in Definition 4.1(ii), let

$$h_0(s) = (\alpha_0 - s)^{-N} h(s) \quad (\alpha_0 < \operatorname{Re} s < \beta_0). \tag{4.2}$$

By Theorem 3.3 and Remark 4.2(i), there exists a function k_0 such that, for all μ satisfying $\alpha_0 < -\operatorname{Re}\mu < \beta_0$,

$$k_0 \in L_{1,\mu} \text{ and } (\mathcal{M}k)(s) = h_0(s) \text{ for } \operatorname{Re} s = -\operatorname{Re}\mu. \tag{4.3}$$

Let T_0 be the convolution integral operator generated by k_0 via (3.1) and let

$$T = (\alpha_0 + \delta)^N T_0 \tag{4.4}$$

where $\alpha_0 + \delta$ stands for $\alpha_0 I + \delta$, I being the identity operator on $F_{p,\mu}$. Under the stated conditions, T_0 is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$. Also, if $f \in F_{p,\mu}$, a standard result involving the Mellin convolution $*$ allows us to say that

$$\delta^i(T_0 f) = \delta^i(k_0 * f) = k_0 * \delta^i f = T_0(\delta^i f) \text{ for } i = 0, 1, 2, \ldots.$$

Hence T_0 defines a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ under the stated conditions and the same is therefore true of T. For appropriate p and μ and for $f \in F_{p,\mu} \cap F_{2,\mu}$,

$$\begin{aligned}\big(\mathcal{M}(Tf)\big)(s) &= (\alpha_0 - s)^N \big(\mathcal{M}(T_0 f)\big)(s) = (\alpha_0 - s)^N (\mathcal{M}k_0)(s)(\mathcal{M}f)(s) \\ &= (\alpha_0 - s)^N h_0(s)(\mathcal{M}f)(s) = h(s)(\mathcal{M}f)(s)\end{aligned}$$

where we have used successively (4.4), (4.3) and (4.2). Since the above argument applies to any strip $\alpha_0 < \operatorname{Re} s < \beta_0$ where $\alpha < \alpha_0 < \beta_0 < \beta$, we have constructed an operator T satisfying the requirements of the theorem. (That the versions of T coming from different substrips agree on the intersection of the substrips is proved by an argument similar to that in [7, Lemma 3.2].) This completes the proof. ∎

The conditions in Definition 4.1 led to the appearance of δ and choice of α_0 ensured that $\alpha_0 + \delta$ was an invertible operator. The operator δ itself is invertible on $F_{p,\mu}$ iff $\operatorname{Re}\mu \neq 0$, a condition which may or may not be satisfied throughout the range $\alpha < -\operatorname{Re}\mu < \beta$, depending on the values of α and β. Nevertheless, we can now obtain an equivalent characterisation of $\mathcal{C}$ which is easier to use in that there is no explicit mention of α_0 and β_0.

THEOREM 4.4. *A function h belongs to the class $\mathcal{C}$ if and only if there exist extended real numbers α and β (depending on h) such that*

(i) $h(s)$ is analytic on the strip $\alpha < \operatorname{Re} s < \beta$

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(ii) *for each closed substrip* $\alpha' \le \operatorname{Re} s \le \beta'$ *with* $\alpha < \alpha' \le \beta' < \beta$, *there exists a non-negative integer N such that h(s) is uniformly of order* $|s|^N$ *as* $|s| \longrightarrow \infty$, *in the sense that there exist constants M and K such that*

$$|s^{-N}h(s)| \le M \quad \forall s : \alpha' \le \operatorname{Re} s \le \beta' \text{ and } |s| > K. \tag{4.5}$$

PROOF. Let $h \in \mathcal{C}$. Then condition (i) of the theorem is satisfied and it remains to check (ii). With α', β' as stated in (ii), choose α_0 and $\beta_0 : \alpha < \alpha_0 < \alpha' \le \beta' < \beta_0 < \beta$. By Definition 4.1 (ii), there exists a positive integer N' such that $(\alpha_0 - s)^{-N'} sh(s)$ is bounded on the strip $\alpha' \le \operatorname{Re} s \le \beta'$ and from this (4.5) follows easily with $N = N' - 1$.

Conversely, let h satisfy the conditions of Theorem 4.4. We need only check that h satisfies Definition 4.1(ii). Given α and β, choose α_0 and $\beta_0 : \alpha < \alpha_0 < \beta_0 < \beta$ and consider the substrip $\alpha' \le \operatorname{Re} s \le \beta'$ where $\alpha_0 < \alpha' < \beta' < \beta_0$. The quantity $sh(s)/(\alpha_0 - s)^{N+1}$ is bounded in modulus for $\alpha' \le \operatorname{Re} s \le \beta'$ with N as in (4.5), since for such s satisfying $|s| > K$ we can use (4.5) and when $|s| \le K$, we use boundedness of a continuous function on a compact set. This leads to (4.1) with N replaced by $N + 1$ and the proof is complete. ∎

We shall use Theorem 4.4 to rehabilitate the Erdélyi-Kober operator we discussed in §3.

EXAMPLE 4.5. Consider again the function $h(s)$ in Example 3.4. Let

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re} z \ne 0, -1, -2, \ldots\}. \tag{4.6}$$

Then $h(s)$ is analytic in the region corresponding to $\eta + s \in \Omega$. Suppose that $\eta - \mu \in \Omega$. We can find a strip containing the line $\operatorname{Re} s = -\operatorname{Re} \mu$ where $h(s)$ is analytic. Calling this strip $\alpha < \operatorname{Re} s < \beta$, we see that on any closed substrip $\alpha' \le \operatorname{Re} s \le \beta'$ containing $\operatorname{Re} s = -\operatorname{Re} \mu$ in its interior, there is an estimate of the form (3.3) for $\Gamma(\eta + s)/\Gamma(\eta + \gamma + s)$ which shows that its modulus behaves like $|s|^{-\operatorname{Re}\gamma}$ as $|s| \longrightarrow \infty$ in this substrip. Accordingly we may simply choose any integer N such that $N > -\operatorname{Re}\gamma$ to see that h satisfies the conditions of Theorem 4.4 for any $\gamma \in \mathbb{C}$. By Theorems 4.3 and 4.4, h is the multiplier of an operator, called $K_1^{\eta,\gamma}$ as before, which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ provided only that $1 < p < \infty$ and $\eta - \mu \in \Omega$.

For this operator we can say more. The function $1/h$ has the same form as h with η and γ replaced by $\eta + \gamma$ and $-\gamma$ respectively. Thus $1/h$ will be the multiplier of $K_1^{\eta+\gamma,-\gamma}$ which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ provided only that $1 < p < \infty$ and $\eta + \gamma - \mu \in \Omega$. Combining our results,

$$\left\{\begin{array}{l}\text{if } 1 < p < \infty, \quad \eta - \mu \in \Omega \text{ and } \eta + \gamma - \mu \in \Omega \text{ then} \\ K_1^{\eta,\gamma} \text{ is a homeomorphism from } F_{p,\mu} \text{ onto } F_{p,\mu} \text{ with inverse } K_1^{\eta+\gamma,-\gamma}.\end{array}\right. \tag{4.7}$$

This is in accord with known results [2, Chapter 3], which also hold for $p = 1$ and $p = \infty$ (although our theory here has to be modified to handle these values of p.)

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REMARK 4.6.

(i) Other Erdélyi-Kober operators can be handled similarly. It is worth repeating the point that in Example 4.5 we have made use of formula (3.2) for the Γ-function but we did not require to use a corresponding result for $\psi = \Gamma'/\Gamma$. When the multiplier h consists of products and quotients of many Γ-functions, the saving in effort becomes well worthwhile, as illustrated in [6].

(ii) Statement (4.7) illustrates another point. In general, a multiplier $h \in \mathcal{C}$ will give rise to a homeomorphism on $F_{p,\mu}$ provided that $1/h$ also belongs to $\mathcal{C}$ and that corresponding strips overlap. We shall summarise the situation briefly in the following theorem.

THEOREM 4.7. *Let h be such that*

(i) $h \in \mathcal{C}$, *with numbers* α *and* β *as in Theorem 4.4*

(ii) $1/h \in \mathcal{C}$, *with corresponding numbers* α_1 *and* β_1

(iii) $S \equiv \{s : \alpha < \operatorname{Re} s < \beta\} \cap \{s : \alpha_1 < \operatorname{Re} s < \beta_1\}$ *is non-empty.*

Then for $1 < p < \infty$ *and* $-\mu \in S$, *h is the multiplier of a homeomorphism T from* $F_{p,\mu}$ *onto* $F_{p,\mu}$.

PROOF. This is almost immediate. ∎

5. In Theorem 3.3 we saw that multipliers in $\mathcal{B}$ gave rise to convolution integral operators, although there were convolution integral operators on $L_{p,\mu}$ which did not arise in this way, such as $K_1^{\eta,\gamma}$ for $0 < \operatorname{Re}\gamma < 1$ in Example 3.4. It turns out that we can give a precise characterisation of the continuous linear operators on $F_{p,\mu}$ which correspond to multipliers in the class $\mathcal{C}$. For this, we need one further simple piece of notation.

DEFINITION 5.1. For any $a > 0$, define the dilation operator λ_a on $F_{p,\mu}$ by

$$(\lambda_a f)(x) = f(ax) \quad (x > 0). \tag{5.1}$$

THEOREM 5.2. *A function h is in the class* $\mathcal{C}$, *with* α, β *as in Definition 4.1, if and only if it is the Mellin multiplier of a mapping T such that*

(i) T is a continuous linear mapping from $F_{p,\mu}$ *into* $F_{p,\mu}$ *for* $1 < p < \infty$ *and* $\alpha < -\operatorname{Re}\mu < \beta$

(ii) T commutes with λ_a *for all* $a > 0$.

PROOF. Certainly if $h \in \mathcal{C}$, (i) will follow from Theorem 4.3 and (ii) is easily checked since, under the appropriate conditions,

$$\big(\mathcal{M}(\lambda_a f)\big)(s) = a^{-s}(\mathcal{M}f)(s)$$

and a^{-s} will commute with $h(s)$. The reverse implication is more complicated and we omit details which can be found in [8, Theorem 3.24]. ∎

REMARK 5.3. Condition (ii) in Theorem 5.2 is the analogue for the Mellin transform of translation invariance for the Fourier transform. The theorem is an analogue of results for Fourier multipliers to be found in [9], and the proof uses techniques such as interpolation which are also found in [9].

This concludes our brief look at a class of multipliers which includes many of the Mellin multipliers which arise in common applications.

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