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VECTOR SUBSPACES OF THE SET OF NON-NORM-ATTAINING FUNCTIONALS

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This paper is dedicated to Professor Richard M. Aron

Abstract

An example is found of a nonreflexive Banach space X such that the union of $\{0\}$ and the set $X^* \setminus \mathsf{NA}(X)$ of non-norm-attaining functionals on X contains no two-dimensional subspace.

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1. Preliminaries

The concept of *lineability* appeared in the early nineties as an algebraic measure of the size of subsets of infinite-dimensional vector spaces (see [12]). When these vector spaces are Banach spaces, we can also talk about *spaceability* and *dense-lineability*.

DEFINITION 1 [12]. A subset M of a Banach space is said to be:

(1) *n-lineable* if $M \cup \{0\}$ contains an n-dimensional vector subspace;
(2) *lineable* if $M \cup \{0\}$ contains an infinite-dimensional vector subspace;
(3) *dense-lineable* if $M \cup \{0\}$ contains an infinite-dimensional dense vector subspace;
(4) *spaceable* if $M \cup \{0\}$ contains an infinite-dimensional closed vector subspace.

For a wider perspective of these new concepts, we refer the reader to [3–7, 11, 13], where it is proved that several pathological properties occur more often than one might expect in the sense described in the definitions above.

The paper [1] considers the problem of the lineability of the set $\mathsf{NA}(X)$ of norm-attaining functionals on an infinite-dimensional Banach space X. Some positive results are given in that paper. In this one, we shall consider the following two problems.

QUESTION 1 [3]. Let X be a nonreflexive Banach space. Is $X^* \setminus \mathsf{NA}(X)$ always lineable?

QUESTION 2 [3]. Let X be a nonreflexive Banach space. Can X always be equivalently renormed to make $X^* \setminus \mathsf{NA}(X)$ lineable?

In the following sections, we shall give a negative answer to Question 1, and an approach to a negative answer to Question 2. We shall now present some partial results relative to the previous questions that appear in [1].

THEOREM 1.1 [1]. *Let K be an infinite compact Hausdorff topological space. Then $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$ is lineable. If, in addition, K possesses a nontrivial convergent sequence, then $\mathcal{C}(K)^* \setminus \mathsf{NA}(\mathcal{C}(K))$ is spaceable.*

THEOREM 1.2 [1]. *Let (Ω, Σ, μ) be a σ-finite measure space with a countably-infinite number of disjoint measurable sets of positive measure. Then $\mathsf{L}_1(\mu)^* \setminus \mathsf{NA}(\mathsf{L}_1(\mu))$ is spaceable.*

We refer the reader to [10], since in much of this paper we shall make use of concepts and notations from the geometry of Banach spaces, such as exposed points and smoothness.

2. Sufficient conditions

In this section, we shall find some sufficient conditions to assure that the set $X^* \setminus \mathsf{NA}(X)$ is lineable or spaceable. We shall base all the results in this section upon the following remark.

REMARK 1. Let X be a smooth Banach space. If $x^* \in \mathsf{NA}(X) \cap \mathsf{S}_{X^*}$, then x^* is not only an extreme point of B_{X^*} but an (ω^*-strongly) exposed point.

We present the next proposition as a consequence of the previous remark. Note that $\exp(\mathsf{B}_X)$ denotes the set of exposed points of the unit ball B_X of a Banach space X.

PROPOSITION 2.1. *Let X be a smooth Banach space. If Y is a vector subspace of X^* such that $Y \cap \exp(\mathsf{B}_{X^*}) = \emptyset$, then $Y \subseteq X^* \setminus \mathsf{NA}(X) \cup \{0\}$.*

PROOF. Assume that there is $0 \neq y^* \in Y \cap \mathsf{NA}(X)$. Then, by Remark 1,

$$y^*/\|y^*\| \in \mathsf{NA}(X) \cap \mathsf{S}_{X^*} \subseteq \exp(\mathsf{B}_{X^*}).$$

Therefore, $y^*/\|y^*\| \in Y \cap \exp(\mathsf{B}_{X^*})$, which is a contradiction. □

For the moment, we shall focus our attention on spaces of continuous functions. We shall begin by presenting the following result, which can be found in [8].

LEMMA 2.2 [8]. *Let K be a compact Hausdorff topological space and X a rotund Banach space. Then*

$$\operatorname{ext}(\mathsf{B}_{\mathcal{C}(K,X)}) = \{f \in \mathcal{C}(K, X) \mid \|f(t)\| = 1 \textit{ for all } t \in K\}.$$

In order for the the next theorem to make sense, we clarify that a nontrivial compact Hausdorff topological space is a compact Hausdorff topological space with more than one point.

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THEOREM 2.3. *Let K be a nontrivial compact Hausdorff topological space and X a nonzero rotund Banach space. If $\mathcal{C}(K, X)$ is infinite-dimensional, then $\mathcal{C}(K, X) \setminus \mathrm{ext}(\mathsf{B}_{\mathcal{C}(K,X)})$ is spaceable.*

PROOF. First of all, note that $\mathcal{C}(K, X)$ is infinite-dimensional only when K is infinite or X is infinite-dimensional. Therefore, we must distinguish these two cases. Let us fix an arbitrary $s \in K$, and consider the continuous linear operator

$$\begin{aligned} \delta_s : \mathcal{C}(K, X) &\longrightarrow X, \\ f &\longmapsto \delta_s(f) = f(s). \end{aligned}$$

Observe that it suffices to prove that $\ker(\delta_s)$ is infinite-dimensional. Indeed, $\ker(\delta_s)$ is closed, and according to Lemma 2.2, $\ker(\delta_s) \subseteq \mathcal{C}(K, X) \setminus \mathrm{ext}(\mathsf{B}_{\mathcal{C}(K,X)})$. Finally, in order to prove that $\ker(\delta_s)$ is infinite-dimensional, we shall consider the two cases mentioned above.

(1) Assume that K is infinite. Choose an infinite sequence $(t_n)_{n\in\mathbb{N}} \subseteq K \setminus \{s\}$. By Urysohn's lemma, for every $n \geq 0$ there exists $f_n \in \mathcal{C}(K)$ such that $f(s) = f(t_n) = 0$ and $f(t_{n+1}) = 1$, where $t_0 = s$. Now, choose any $x \in X \setminus \{0\}$. The family $\{f_n x \mid n \geq 0\}$ is linearly independent and contained in $\ker(\delta_s)$.

(2) Assume that X is infinite-dimensional. Since K contains more than one point, again by applying Urysohn's lemma, we deduce the existence of a function $f \in \mathcal{C}(K) \setminus \{0\}$ such that $f(s) = 0$. Now, choose an infinite linearly-independent family $\{x_n \mid n \in \mathbb{N}\} \subset X$. The family $\{f x_n \mid n \in \mathbb{N}\}$ is linearly independent and contained in $\ker(\delta_s)$. □

The previous theorem allows us to state and prove the following sufficient condition to assure the spaceability of the set $X^* \setminus \mathsf{NA}(X)$.

THEOREM 2.4. *Let X be a smooth Banach space. Let K be a nontrivial compact Hausdorff topological space and Y a nonzero rotund Banach space such that $\mathcal{C}(K, Y)$ is infinite-dimensional. Assume that X^* contains an isometric copy of $\mathcal{C}(K, Y)$. Then $X^* \setminus \mathsf{NA}(X)$ is spaceable.*

PROOF. In accordance with Theorem 2.3, $\mathcal{C}(K, Y) \setminus \mathrm{ext}(\mathsf{B}_{\mathcal{C}(K,Y)})$ is spaceable. So, let W be an infinite-dimensional closed vector space contained in $\mathcal{C}(K, Y) \setminus \mathrm{ext}(\mathsf{B}_{\mathcal{C}(K,Y)})$. Since $\exp(\mathsf{B}_{\mathcal{C}(K,Y)}) \subseteq \mathrm{ext}(\mathsf{B}_{\mathcal{C}(K,Y)})$ (see [10]), we deduce by Proposition 2.1 that $W \subseteq X^* \setminus \mathsf{NA}(X) \cup \{0\}$, and the result holds. □

To finish this section, we shall focus on spaces of integrable functions. In order for the previous theorem to make sense, we want to recall that a σ-finite measure space (Ω, Σ, μ) is said to be *nontrivial* if there exist at least two disjoint measurable sets of positive measure.

THEOREM 2.5. *Let (Ω, Σ, μ) be a nontrivial σ-finite measure space and X a nonzero Asplund Banach space. If $\mathsf{L}_1(\mu, X)$ is infinite-dimensional, then the set $\mathsf{L}_1(\mu, X) \setminus \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)})$ is lineable.*

PROOF. First of all, notice that $\mathsf{L}_1(\mu, X)$ is infinite-dimensional only when (Ω, Σ, μ) has a countably-infinite number of disjoint measurable sets of positive measure or X is infinite-dimensional. Therefore, we shall have to distinguish these two cases.

(1) Assume that (Ω, Σ, μ) has a countably-infinite number of disjoint measurable sets of positive measure. Choose $\{A_n \mid n \in \mathbb{N}\}$ to be an infinite family of disjoint measurable sets of positive measure. Let us fix an element $x \in \mathsf{S}_X$. We will show that

$$\operatorname{span}\{(\chi_{A_1} + \chi_{A_2})x, (\chi_{A_3} + \chi_{A_4})x, \ldots \mid n \in \mathbb{N}\} \subseteq \mathsf{L}_1(\mu, X) \setminus \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)}).$$

Let $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$, not all zero, and set

$$g := \lambda_1(\chi_{A_1} + \chi_{A_2})x + \cdots + \lambda_k(\chi_{A_{2k-1}} + \chi_{A_{2k}})x \in \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)}).$$

Let $f \in \mathsf{S}_{\mathsf{L}_\infty(\mu,X^*)}$ attain its norm only at g. Then

$$\begin{aligned}
1 &= \int_\Omega f(t)\,(g(t))\,d\mu(t) \\
&= \int_\Omega f(t)\,(\lambda_1\chi_{A_1}(t)x)\,d\mu(t) + \int_\Omega f(t)\,(\lambda_1\chi_{A_2}(t)x)\,d\mu(t) \\
&\quad + \cdots + \int_\Omega f(t)\,(\lambda_k\chi_{A_{2k-1}}(t)x)\,d\mu(t) + \int_\Omega f(t)\,(\lambda_k\chi_{A_{2k}}(t)x)\,d\mu(t) \\
&\le |\lambda_1|\mu(A_1) + |\lambda_1|\mu(A_2) + \cdots + |\lambda_k|\mu(A_{2k-1}) + |\lambda_k|\mu(A_{2k}) \\
&= \|g\|_1 \\
&= 1.
\end{aligned}$$

Therefore, for every $i \in \{1, \ldots, k\}$,

$$\int_\Omega f(t)\,(\lambda_i\chi_{A_{2i-1}}(t)x)\,d\mu(t) = |\lambda_i|\mu(A_{2i-1}),$$

and

$$\int_\Omega f(t)\,(\lambda_i\chi_{A_{2i}}(t)x)\,d\mu(t) = |\lambda_i|\mu(A_{2i}),$$

which means that f attains its norm at

$$\frac{\lambda_i\chi_{A_{2i-1}}x}{|\lambda_i|\mu(A_{2i-1})} \quad \text{and} \quad \frac{\lambda_i\chi_{A_{2i}}x}{|\lambda_i|\mu(A_{2i})},$$

for those $\lambda_i \neq 0$. This is a contradiction.

(2) Assume that X is infinite-dimensional. Choose $\{x_n \mid n \in \mathbb{N}\} \subset \mathsf{S}_X$ to be an infinite family of linearly-independent elements. Since (Ω, Σ, μ) is nontrivial, there exist at least two disjoint measurable sets A and B of positive measure. We will show that

$$\operatorname{span}\{(\chi_A + \chi_B)x_n \mid n \in \mathbb{N}\} \subseteq \mathsf{L}_1(\mu, X) \setminus \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)}).$$

Let $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$, not all zero, and set

$$g := \lambda_1(\chi_A + \chi_B)x_1 + \cdots + \lambda_k(\chi_A + \chi_B)x_k \in \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)}).$$

Let $f \in \mathsf{S}_{\mathsf{L}_\infty(\mu,X^*)}$ attain its norm only at g. Then

$$\begin{aligned}
1 &= \int_\Omega f(t)\,(g(t))\,d\mu(t) \\
&= \int_\Omega f(t)\,(\lambda_1\chi_A(t)x_1)\,d\mu(t) + \int_\Omega f(t)\,(\lambda_1\chi_B(t)x_1)\,d\mu(t) \\
&\quad + \cdots + \int_\Omega f(t)\,(\lambda_k\chi_A(t)x_k)\,d\mu(t) + \int_\Omega f(t)\,(\lambda_k\chi_B(t)x_k)\,d\mu(t) \\
&\le |\lambda_1|\mu(A) + |\lambda_1|\mu(B) + \cdots + |\lambda_k|\mu(A) + |\lambda_k|\mu(B) \\
&= \|g\|_1 \\
&= 1.
\end{aligned}$$

Therefore, for every $i \in \{1, \ldots, k\}$,

$$\int_\Omega f(t)\,(\lambda_i\chi_A(t)x_i)\,d\mu(t) = |\lambda_i|\mu(A),$$

and

$$\int_\Omega f(t)\,(\lambda_i\chi_B(t)x_i)\,d\mu(t) = |\lambda_i|\mu(B),$$

which means that f attains its norm at

$$\frac{\lambda_i\chi_A x_i}{|\lambda_i|\mu(A)} \quad \text{and} \quad \frac{\lambda_i\chi_B x_i}{|\lambda_i|\mu(B)},$$

for those $\lambda_i \neq 0$. This is a contradiction. □

The previous theorem allows us to state and prove a sufficient condition to assure the lineability of the set $X^* \setminus \mathsf{NA}(X)$.

THEOREM 2.6. *Let X be a smooth Banach space. Let (Ω, Σ, μ) be a nontrivial σ-finite measure space and Y a nonzero Asplund Banach space such that $\mathsf{L}_1(\mu, Y)$ is infinite-dimensional. Assume that X^* contains an isometric copy of $\mathsf{L}_1(\mu, Y)$. Then $X^* \setminus \mathsf{NA}(X)$ is lineable.*

PROOF. In accordance with Theorem 2.5, $\mathsf{L}_1(\mu, X) \setminus \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)})$ is lineable. So, let W be an infinite-dimensional vector space contained in $\mathsf{L}_1(\mu, X) \setminus \exp(\mathsf{B}_{\mathsf{L}_1(\mu,X)})$. By Proposition 2.1, we have that $W \subseteq X^* \setminus \mathsf{NA}(X) \cup \{0\}$, and the result holds. □

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3. A counterexample

In this section, we shall take care of both Questions 1 and 2. In the first place, we shall present a (negative) solution to Question 1. We shall begin with the next theorem, which is a sufficient condition to assure that the set $X^* \setminus \mathsf{NA}(X)$ is not even 2-lineable.

THEOREM 3.1. *Let X be a Banach space such that X is a maximal subspace of X^{**}. For $n \in \mathbb{N}$, let X^n be the nth dual of X. Then $X^{n+1} \setminus \mathsf{NA}(X^n)$ is not even 2-lineable. If, in addition, X is isometrically isomorphic to its bidual X^{**}, then $X^* \setminus \mathsf{NA}(X)$ is not even 2-lineable either.*

PROOF. In the first place, assume that Y is a vector subspace contained in $X^{**} \setminus \mathsf{NA}(X^*) \cup \{0\}$. Since $X \subset \mathsf{NA}(X^*)$, we deduce that $X \cap Y = \{0\}$. The maximality of X implies that Y has dimension at most one. In the second place, observe that, since X is a maximal subspace of X^{**}, X^n is a maximal subspace of X^{n+2} for all $n \in \mathbb{N}$. Finally, if X is isometrically isomorphic to its bidual X^{**}, then $X^* \setminus \mathsf{NA}(X)$ is not 2-lineable because $X^{***} \setminus \mathsf{NA}(X^{**})$ is not so. □

In order to provide a negative answer to Question 1, we have to find an example of a Banach space satisfying the hypothesis of Theorem 3.1. It is well known that the James space $\mathcal{J}$ is one such. By virtue of [14, 15] we have the following result.

THEOREM 3.2 [15]. *The real vector space*

$$\mathcal{J} := \{(\alpha_n)_{n\in\mathbb{N}} \in c_0 \mid \|(\alpha_n)_{n\in\mathbb{N}}\|_a < \infty\},$$

endowed with the norm

$$\|(\alpha_n)_{n\in\mathbb{N}}\|_a := 2^{-1/2} \sup_{m\geq 2,\, p_1<\cdots<p_m} \left(\sum_{n=1}^{m-1} (\alpha_{p_n} - \alpha_{p_{n+1}})^2 + (\alpha_{p_m} - \alpha_{p_1})^2 \right)^{1/2},$$

is a real Banach space satisfying the following conditions:

(i) *the space $\mathcal{J}$ is of codimension 1 in its bidual $\mathcal{J}^{**}$;*
(ii) *the space $\mathcal{J}$ is isometrically isomorphic to its bidual $\mathcal{J}^{**}$.*

Now we are able to answer Question 1 negatively.

EXAMPLE 1. In accordance with Theorem 3.1, the James space and all its duals answer Question 1 negatively.

To finish this paper, we shall present an approach to a negative solution to Question 2. In concrete terms, we shall answer negatively the next question by following a similar process to the above.

QUESTION 3 [3]. Let X be a nonreflexive dual Banach space. Can X always be equivalently dually renormed to make $X^* \setminus \mathsf{NA}(X)$ lineable?

Before discussing the solution to the previous question, let us note that, as indicated in the next results, not every equivalent norm on a dual Banach space is a dual norm (see, for instance, [9, p. 27]). On this topic, in [2] the following result is shown.

THEOREM 3.3 [2]. *Let X be a real Banach space. If $x^* \in X^*$ is an L^2-summand vector, then $x^* \in \mathsf{NA}(X)$.*

The previous theorem reveals another way to prove that there are always equivalent norms on nonreflexive dual Banach spaces that are not dual norms (see, again, [2]).

COROLLARY 3.4 [2]. *Let X be a nonreflexive real Banach space. Consider $x^* \in \mathsf{S}_{X^*} \setminus \mathsf{NA}(X)$ and $x^{**} \in \mathsf{S}_{X^{**}}$ such that $x^{**}(x^*) = 1$. Then the equivalent norm on X^* given by*

$$\|y^*\| = \sqrt{\|m\|^2 + \|\delta x^*\|^2}, \qquad y^* = m + \delta x^*, \qquad m \in \ker(x^{**}),\ \delta \in \mathbb{R},$$

is not a dual norm.

Observe that, because of what has previously been discussed, a negative answer to Question 3 does not necessarily answer Question 2 negatively.

THEOREM 3.5. *Let X be a Banach space such that X is a maximal subspace of X^{**}. For every $n \in \mathbb{N}$, the nth dual X^n of X cannot be equivalently dually renormed to make $X^{n+1} \setminus \mathsf{NA}(X^n)$ 2-lineable.*

PROOF. Obviously, it suffices to show that the result holds for X^*. If $\|\cdot\|$ is an equivalent dual norm on X^*, then there exists an equivalent norm $|\cdot|$ on X such that $|\cdot|^* = \|\cdot\|$. Now, it is sufficient to apply Theorem 3.1 to $(X, |\cdot|)$. □

EXAMPLE 2. In accordance with Theorem 3.5, all the duals of the James space answer Question 3 negatively.

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References

[1] M. D. Acosta, A. Aizpuru, R. M. Aron and F. J. García-Pacheco, 'Functionals that do not attain their norm', *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), 407–418.
[2] A. Aizpuru and F. J. García-Pacheco, 'A note on L^2-summand vectors in dual spaces', *Glasgow Math. J.* (2009), to appear.
[3] R. M. Aron, V. I. Gurariy and J. B. Seoane-Sepúlveda, 'Lineability and spaceability of sets of functions on $\mathbb{R}$', *Proc. Amer. Math. Soc.* **133** (2005), 795–803.
[4] R. M. Aron, D. Pérez-García and J. B. Seoane-Sepúlveda, 'Algebrability of the set of non-convergent Fourier series', *Studia Math.* **175** (2006), 83–90.
[5] F. Bayart, 'Topological and algebraic genericity of divergence and of universality', *Studia Math.* **167** (2005), 161–181.
[6] F. Bayart, 'Linearity of sets of strange functions', *Michigan Math. J.* **53** (2005), 291–303.
[7] F. Bayart and L. Quarta, 'Algebras in sets of queer functions', *Israel J. Math* **158** (2007), 285–296.
[8] V. I. Bogachev, J. F. Mena-Jurado and J. C. Navarro-Pascual, 'Extreme points in spaces of continuous functions', *Proc. Amer. Math. Soc.* **123** (1995), 1061–1067.

https://doi.org/10.1017/S0004972708000348 Published online by Cambridge University Press

[9] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64 (John Wiley & Sons, New York, 1993).

[10] J. Diestel, *Geometry of Banach Spaces–Selected Topics*, Lecture Notes in Mathematics, 485 (Springer, Berlin, 1975).

[11] F. J. García-Pacheco, N. Palmberg and J. B. Seoane-Sepúlveda, 'Lineability and algebrability of pathological phenomena in analysis', *J. Math. Anal. Appl.* **326** (2007), 929–939.

[12] V. I. Gurariy, 'Linear spaces composed of everywhere non-differentiable functions', *C. R. Acad. Bulgare Sci.* **44** (1991), 13–16.

[13] V. I. Gurariy and L. Quarta, 'On lineability of sets of continuous functions', *J. Math. Anal. Appl.* **294** (2004), 62–72.

[14] R. C. James, 'Bases and reflexivity of Banach spaces', *Ann. Math.* **52** (1950), 518–527.

[15] R. C. James, 'A non-reflexive Banach space which is isomorphic to its second conjugate', *Proc. Natl. Acad. Sci. USA* **37** (1951), 174–177.

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