Bull. Aust. Math. Soc. **102** (2020), 374–386
doi:10.1017/S0004972720000106

APPLICATIONS OF SYSTEMS OF QUADRATIC FORMS TO GENERALISED QUADRATIC FORMS

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(Received 25 December 2019; accepted 4 January 2020; first published online 13 February 2020)

Abstract

A system of quadratic forms is associated to every generalised quadratic form over a division algebra with involution of the first kind in characteristic two. It is shown that this system determines the isotropy behaviour and the isometry class of generalised quadratic forms. An application of this construction to the Witt index of generalised quadratic forms is also given.

2010 *Mathematics subject classification*: primary 11E04; secondary 11E39, 12F10.

Keywords and phrases: generalised quadratic form, system of quadratic forms, Witt index, division algebra with involution.

1. Introduction

By a theorem of Jacobson [2], the theory of hermitian forms over a quaternion algebra (or a quadratic extension) with the canonical involution in characteristic different from two, may be reduced to that of quadratic forms. The main idea of [2] is to associate to every hermitian space (V, h) on these algebras with involution the quadratic form $v \mapsto h(v, v)$. This construction determines the isotropy and the isometry class of the hermitian forms. Jacobson's construction was generalised in [8] to arbitrary characteristic.

Associating some quadratic form over the base field to a hermitian form of arbitrary index to give information about it, is a very difficult problem and seems to be far out of reach in general. Considering this, one can instead use some generalisations of quadratic forms to study hermitian forms. In [5], a system of quadratic forms was associated to every hermitian form over a division algebra with involution of the first kind, which controls the isotropy and metabolicity of these forms and determines their isometry classes. This system, which is a natural extension of Jacobson's construction, was also used to study the behaviour of hermitian forms under finite field extensions of odd degree and quadratic separable extensions (see [5, Section 5] and [6]).

The theory of hermitian forms over fields of characteristic two divides into two distinct theories: that of hermitian forms and that of generalised quadratic forms. The

This research is partially supported by the University of Kashan under Grant No. 890193/1.

relationship between these two types of objects is essentially identical to that between bilinear and quadratic forms. The aim of this work is to establish a construction similar to [5] for generalised quadratic forms in characteristic two. Let (D, θ) be a division algebra with involution of the first kind over a field F of characteristic two and let ρ be a generalised quadratic form over (D, θ). For every basis $\mathcal{B}$ of the quotient $D/\mathrm{Symd}(D, \theta)$, we associate a system of quadratic forms $q_{\rho,\mathcal{B}}$ to ρ. It is shown that this system can be used to determine the isotropy and hyperbolicity of generalised quadratic forms, as well as their isometry classes (see Propositions 4.3 and 4.8 and Theorem 6.4). It is also shown in Theorem 5.4 that there is a natural relation between the Witt indices of the form ρ and the system $q_{\rho,\mathcal{B}}$. These results are based on a strengthening of a key result from [6], about the Witt index of systems of quadratic forms in arbitrary characteristic, and is of independent interest (see Proposition 3.1).

It should be mentioned that some ideas used in this work are of course parallel to those given in [6]. However, this work includes a number of improvements. It also reveals some new aspects, as it covers totally singular generalised quadratic forms. Although totally singular forms are in some sense simpler than nonsingular forms, their study, even over fields, has its own limitations, as various standard tools are unavailable for these forms.

2. Preliminaries

Let V be a finite-dimensional vector space over a field F. A *quadratic form* on V is a map $q : V \to F$ satisfying $q(\alpha u + \beta v) = \alpha^2 q(u) + \beta^2 q(v) + \alpha\beta \mathfrak{b}_q(u, v)$ for all $u, v \in V$ and $\alpha, \beta \in F$, where $\mathfrak{b}_q : V \times V \to F$ is a symmetric bilinear form. The form $\mathfrak{b}_q$ is called the *polar form* of q. For a subspace W of V we use the notation

$$W^{\perp_{\mathfrak{b}_q}} = \{v \in V \mid \mathfrak{b}_q(v, w) = 0 \text{ for all } w \in W\}.$$

We also set $\mathrm{rad}(\mathfrak{b}_q) = V^{\perp_{\mathfrak{b}_q}}$ and $\mathrm{rad}(q) = \{v \in \mathrm{rad}(\mathfrak{b}_q) \mid q(v) = 0\}$. The form q is called *nonsingular* if $\mathrm{rad}(\mathfrak{b}_q) = \{0\}$, *regular* if $\mathrm{rad}(q) = \{0\}$ and *totally singular* if $\mathfrak{b}_q$ is trivial.

A *system of quadratic forms* on the space V is an m-tuple $q = (q_1, \ldots, q_m)$, where every $q_i : V \to F$ is a quadratic form. The system q may be identified with a quadratic map $q : V \to F^m$. This map induces a bilinear map $\mathfrak{b}_q : V \times V \to F^m$ which is given by $\mathfrak{b}_q(u, v) = q(u + v) - q(u) - q(v)$. If W is a subspace of V, the *orthogonal complement* of W with respect to $\mathfrak{b}_q$ is defined as

$$W^{\perp_{\mathfrak{b}_q}} = \{v \in V \mid \mathfrak{b}_q(v, w) = 0 \text{ for all } w \in W\}.$$

The *radicals* of $\mathfrak{b}_q$ and q are $\mathrm{rad}(\mathfrak{b}_q) = V^{\perp_{\mathfrak{b}_q}}$ and $\mathrm{rad}(q) = \{v \in \mathrm{rad}(\mathfrak{b}_q) \mid q(v) = 0\}$. We say that q is *nonsingular* if $\mathrm{rad}(\mathfrak{b}_q) = \{0\}$, *regular* if $\mathrm{rad}(q) = \{0\}$ and *totally singular* if $\mathfrak{b}_q$ is trivial. Clearly, q is totally singular if and only if every q_i is totally singular. We also say that q is *strongly nonsingular* if q_i is nonsingular for some $1 \leqslant i \leqslant m$, and *totally nonsingular* if every q_i is nonsingular.

A system of quadratic forms q on V is called *isotropic* if there exists a nonzero vector $v \in V$ such that $q(v) = 0$. Such a vector v is called an *isotropic* vector of q.

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A subspace S of V is called *totally isotropic* if every vector $v \in S$ is isotropic, that is, $q|_S = 0$. We say that (V, q) (or the system q itself) is *metabolic* if there exists a totally isotropic subspace L of V such that $\dim_F L \geqslant \frac{1}{2}\dim_F V$. Such a subspace L is called a *Lagrangian* of (V, q). An *isometry* between two systems of quadratic forms (V, q) and (V', q') is an isomorphism of vector spaces $f : V \to V'$ such that $q'(f(v)) = q(v)$ for all $v \in V$.

Let $q : V \to F^m$ and $q' : V' \to F^m$ be two systems of quadratic forms. The *orthogonal sum* of q and q' is a system $q \perp q' : V \oplus V' \to F^m$ defined by

$$(q \perp q')((v, v')) = q(v) + q'(v') \quad \text{for } v \in V \text{ and } v' \in V.$$

Every system of quadratic forms $q : V \to F^m$ has a decomposition $q \simeq q_{\mathrm{ts}} \perp q_{\mathrm{ns}}$, where $q_{\mathrm{ts}} = q|_{\mathrm{rad}(\mathfrak{b}_q)}$ is totally singular and q_{ns} is nonsingular. We refer the reader to [7, Chapter 9] for basic concepts and facts regarding systems of quadratic forms.

Let A be a central simple algebra over a field F. An *involution* on A is an antiautomorphism σ of A of period two. An involution on A is said to be of *the first kind* if it restricts to the identity map on F. The sets of *symmetric* and *symmetrised* elements of an algebra with involution (A, σ) are defined as

$$\mathrm{Sym}(A, \sigma) = \{x \in A \mid \sigma(x) = x\}, \quad \mathrm{Symd}(A, \sigma) = \{x + \sigma(x) \mid x \in A\}.$$

We refer the reader to [4] for a reference on algebras with involution.

Let (D, θ) be a division algebra with involution over F and let V be a right vector space over D. A *hermitian form* on V is a bi-additive map $h : V \times V \to D$ for which

(i) $h(ud, vd') = \theta(d)h(u, v)d'$ for all $u, v \in V$ and $d, d' \in D$, and
(ii) $h(v, u) = \theta(h(u, v))$ for all $u, v \in V$.

It follows immediately that $h(v, v) \in \mathrm{Sym}(D, \theta)$ for all $v \in V$. For a D-subspace W of V we use the notation

$$W^{\perp_h} = \{v \in V \mid h(v, w) = 0 \text{ for all } w \in W\}.$$

Let (D, θ) be a division algebra with involution of the first kind over a field F of characteristic two. Let V be a finite-dimensional right vector space over D. A map $\rho : V \to D/\mathrm{Symd}(D, \theta)$ is called a *generalised quadratic form* over (D, θ) if $\rho(v\alpha) = \theta(\alpha)\rho(v)\alpha$ for all $v \in V$ and $\alpha \in D$, and

$$\rho(u + v) - \rho(u) - \rho(v) = h_\rho(u, v) + \mathrm{Symd}(D, \theta) \quad \text{for all } u, v \in V,$$

where $h_\rho : V \times V \to D$ is a hermitian form. The pair (V, ρ) is called a *generalised quadratic space* over (D, θ). By [1, (1.1)], ρ uniquely determines h_ρ. Also, for every $v \in V$,

$$h_\rho(v, v) = d + \theta(d), \tag{2.1}$$

where $d \in D$ is any representative of the class $\rho(v) \in D/\mathrm{Symd}(D, \theta)$. Note that if $(D, \theta) = (F, \mathrm{id})$ then $D/\mathrm{Symd}(D, \theta) = F$, ρ is a quadratic form over F and h_ρ is the polar form of ρ.

Let (V, ρ) be a generalised quadratic space over (D, θ). A basis $\{v_1, \ldots, v_n\}$ of V over D is called an *orthogonal basis* of (V, ρ) if $h_\rho(v_i, v_j) = 0$ for all $i \neq j$. In view of [3, Chapter I, (6.2.4)], if $D \neq F$ then every generalised quadratic form ρ over (D, θ) has an orthogonal basis. An *isometry* between two generalised quadratic spaces (V, ρ) and (V', ρ') over (D, θ) is an isomorphism of right vector spaces $f : V \to V'$ satisfying $\rho'(f(v)) = \rho(v)$ for all $v \in V$.

Let (V, ρ) be a generalised quadratic space over (D, θ). Set $\mathrm{rad}(h_\rho) = V^{\perp_{h_\rho}}$ and $\mathrm{rad}(\rho) = \{v \in \mathrm{rad}(h_\rho) \mid \rho(v) = 0\}$. We say that ρ is *nonsingular* if $\mathrm{rad}(h_\rho) = \{0\}$, *regular* if $\mathrm{rad}(\rho) = \{0\}$ and *totally singular* if h_ρ is trivial. Let W be a D-subspace of V with $V \simeq \mathrm{rad}(h_\rho) \oplus W$. Then $\rho \simeq \rho_{\mathrm{ts}} \perp \rho_{\mathrm{ns}}$, where $\rho_{\mathrm{ts}} = \rho|_{\mathrm{rad}(h_\rho)}$ is totally singular and $\rho|_{\mathrm{ns}} = \rho|_W$ is nonsingular. Similarly, every generalised quadratic form ρ has a decomposition $\rho \simeq \rho_0 \perp \rho_{\mathrm{re}}$, where $\rho_0 = \rho|_{\mathrm{rad}(\rho)}$ is trivial and ρ_{re} is regular.

A generalised quadratic form ρ on V is called *isotropic* if there exists a nonzero vector $v \in V$ such that $\rho(v) = 0$. Such a vector v is called an *isotropic* vector of ρ. The form ρ is called *anisotropic* if it is not isotropic. A nonsingular generalised quadratic space (V, ρ) is called *hyperbolic* if there exists a D-subspace L of V with $\dim_D L = \frac{1}{2} \dim_D V$ such that $\rho|_L$ is trivial. It is easy to see that every regular generalised quadratic form ρ decomposes as $\rho \simeq \rho_{\mathrm{an}} \perp \rho_{\mathrm{hyp}}$, where ρ_{an} is anisotropic and ρ_{hyp} is hyperbolic (see [3, Chapter I, (6.5.1)]).

3. The Witt index of systems of quadratic forms

Let F be a field of arbitrary characteristic and let (V, q) be a system of quadratic forms over F. We call the maximal dimension of totally isotropic subspaces of V the *Witt index* of q and we denote it by $w(q)$. Note that (V, q) is metabolic if and only if $w(q) \geq \frac{1}{2} \dim_F V$. Also, if q is strongly nonsingular and metabolic then $w(q) = \frac{1}{2} \dim_F V$.

The proof of the following result is similar to that of [5, (2.3)].

Proposition 3.1. *Let $(V, q) \simeq (U, \psi) \perp (W, \varphi)$ be an isometry of systems of quadratic forms over a field F of arbitrary characteristic. Suppose that φ is strongly nonsingular. If φ is metabolic, then $w(q) = w(\psi) + w(\varphi)$.*

Proof. Clearly, $w(q) \geqslant w(\psi) + w(\varphi)$, hence it suffices to prove that $w(q) \leqslant w(\psi) + w(\varphi)$. We may identify U and W with subspaces of V such that $V = U + W$ and $U \cap W = \{0\}$. Therefore, for every $v \in V$, there exist unique elements $u \in U$ and $w \in W$ such that $v = u + w$ and $q(v) = \psi(u) + \varphi(w)$.

Set $n = w(q)$ and $s = w(\varphi) = \frac{1}{2} \dim_F W$. Let L be a totally isotropic subspace of (V, q) with $\dim_F L = n$. Let $\pi_1 : L \to U$ and $\pi_2 : L \to W$ be the restrictions to L of natural projections $V \to U$ and $V \to W$. Then $v = \pi_1(v) + \pi_2(v)$ for every $v \in L$, which implies that $q(v) = \psi(\pi_1(v)) + \varphi(\pi_2(v))$. Since $q|_L$ is trivial, one concludes that

$$\psi(\pi_1(v)) = -\varphi(\pi_2(v)) \quad \text{and} \quad \mathfrak{b}_\psi(\pi_1(v), \pi_1(v')) = -\mathfrak{b}_\varphi(\pi_2(v), \pi_2(v')), \tag{3.1}$$

for all $v, v' \in L$.

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Set $W_1 := L \cap W = \ker(\pi_1)$ and $W' := \mathrm{Im}(\pi_2) \subseteq W$. Note that $W_1 \subseteq W'$. Let L' be a Lagrangian of (W, φ). Then $\dim_F L' = s$. Set $r = \dim_F W'$ and $t = \dim_F(W' \cap L') - \dim_F(W_1 \cap L')$. We claim that

$$w(\psi) \geqslant t + n - r. \tag{3.2}$$

Choose a basis $\mathcal{B}$ of $W_1 \cap L'$ and extend it to a basis of $W' \cap L'$, by adding some vectors $w_1, \ldots, w_t \in (W' \cap L') \setminus (W_1 \cap L')$. Since $w_i \in W'$, there exists $v_i \in L$ such that $w_i = \pi_2(v_i)$, $i = 1, \ldots, t$. Set $u_i = \pi_1(v_i) \in U$, $i = 1, \ldots, t$, and let U' be the subspace of U spanned by $\{u_1, \ldots, u_t\}$. We prove that $(L \cap U) + U'$ is a totally isotropic subspace of (U, ψ). Clearly, $L \cap U$ is a totally isotropic subspace of (U, ψ). Moreover, as $w_1, \ldots, w_t \in L'$, using (3.1), one concludes that

$$\psi(u_i) = \psi(\pi_1(v_i)) = -\varphi(\pi_2(v_i)) = -\varphi(w_i) = 0,$$

for $i = 1, \ldots, t$. Hence, U' is also a totally isotropic subspace of (U, ψ). Now, if $u \in L \cap U$ then $\pi_1(u) = u$ and $\pi_2(u) = 0$, so (3.1) implies that

$$\mathfrak{b}_\psi(u_i, u) = \mathfrak{b}_\psi(\pi_1(v_i), \pi_1(u)) = -\mathfrak{b}_\varphi(\pi_2(v_i), \pi_2(u)) = 0,$$

for $i = 1, \ldots, t$. It follows that $(L \cap U) + U'$ is a totally isotropic subspace of (U, ψ). To prove (3.2), we show that $\dim_F((L \cap U) + U') = t + n - r$. Note that $L \cap U = \ker(\pi_2)$, so

$$\dim_F(L \cap U) = \dim_F L - \dim_F W' = n - r.$$

Hence, it is enough to prove that

$$\dim_F U' = t \quad \text{and} \quad (L \cap U) \cap U' = \{0\}. \tag{3.3}$$

To prove (3.3), it suffices to show that if $\sum_{i=1}^t \alpha_i u_i \in L \cap U$ for some $\alpha_1, \ldots, \alpha_t \in F$, then $\alpha_1 = \cdots = \alpha_t = 0$. Since $v_i \in L$ for $i = 1, \ldots, t$, the condition $\sum_{i=1}^t \alpha_i u_i \in L \cap U$ implies that

$$\sum_{i=1}^t \alpha_i w_i = \sum_{i=1}^t \alpha_i v_i - \sum_{i=1}^t \alpha_i u_i \in L.$$

It follows that $\sum_{i=1}^t \alpha_i w_i \in L \cap W = W_1$. Since $w_1, \ldots, w_t \in (W' \cap L') \setminus (W_1 \cap L')$ and $\mathcal{B} \cup \{w_1, \ldots, w_t\}$ is a basis of $W' \cap L'$, one concludes that $\sum_{i=1}^t \alpha_i w_i = 0$, hence $\alpha_1 = \cdots = \alpha_t = 0$. This completes the proof of (3.3), and therefore (3.2).

We now prove that $W' \subseteq W_1^{\perp_{\mathfrak{b}_\varphi}}$. Let $w \in W_1$ and $w' \in W'$. Write $w' = \pi_2(v)$ for some $v \in L$. Using (3.1), one concludes that

$$\mathfrak{b}_\varphi(w, w') = \mathfrak{b}_\varphi(\pi_2(w), \pi_2(v)) = -\mathfrak{b}_\psi(\pi_1(w), \pi_1(v)) = -\mathfrak{b}_\psi(0, \pi_1(v)) = 0.$$

Hence, $W' \subseteq W_1^{\perp_{\mathfrak{b}_\varphi}}$, as claimed.

Set $X = W_1 \cap L'$ and $l = \dim_F X$. Then $\dim_F W' \cap L' = l + t$. The inclusion $X \subseteq W_1$ implies that $W_1^{\perp_{\mathfrak{b}_\varphi}} \subseteq X^{\perp_{\mathfrak{b}_\varphi}}$, so

$$W' \subseteq X^{\perp_{\mathfrak{b}_\varphi}}.$$

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Also, the inclusion $X \subseteq L'$ yields

$$L' = L'^{\perp_{\mathfrak{b}_\varphi}} \subseteq X^{\perp_{\mathfrak{b}_\varphi}}.$$

Since φ is strongly nonsingular, we have $\dim_F X^{\perp_{\mathfrak{b}_\varphi}} \leqslant 2s - l$. Now, W' is an r-dimensional subspace of $X^{\perp_{\mathfrak{b}_\varphi}}$ and L' is an s-dimensional subspace of $X^{\perp_{\mathfrak{b}_\varphi}}$ with $\dim_F W' \cap L' = l + t$. Hence, $r + s - l - t \leqslant 2s - l$, which implies that $r - t \leqslant s$. Using (3.2), one concludes that $w(\psi) \geqslant n - s = w(q) - w(\varphi)$, that is, $w(q) \leqslant w(\psi) + w(\varphi)$. □

Using Proposition 3.1, we can strengthen [5, (2.3)] as follows.

Corollary 3.2. *Let $(V, q) \simeq (U, \psi) \perp (W, \varphi)$ be an isometry of systems of quadratic forms over a field F of arbitrary characteristic. Suppose that φ is strongly nonsingular. If q and φ are metabolic, then so is ψ.*

Proof. The hypotheses imply that $w(q) \geqslant \frac{1}{2}\dim_F V$ and $w(\varphi) = \frac{1}{2}\dim_F W$. It follows from Proposition 3.1 that $w(\psi) = w(q) - w(\varphi) \geqslant \frac{1}{2}\dim_F U$. Hence, ψ is metabolic. □

4. The construction

From now on, unless stated otherwise, all fields are implicitly supposed to be of characteristic two.

We fix (D, θ) as a finite-dimensional division algebra with involution of the first kind over a field F. For $d \in D$ we denote the element $d + \mathrm{Symd}(D, \theta)$ in the quotient of vector spaces $D/\mathrm{Symd}(D, \theta)$ by $\overline{d}$. We also denote $D/\mathrm{Symd}(D, \theta)$ by $\overline{D}$.

The following result is easily verified. (Compare [5, (3.1)] which gives the analogous statement for hermitian forms.)

Lemma 4.1. *Let (V, ρ) be a generalised quadratic space over (D, θ) and let $\pi : \overline{D} \to F$ be a nonzero F-linear map. Considering V as a vector space over F, the map $q_{\rho,\pi} : V \to F$ given by $q_{\rho,\pi}(v) = \pi(\rho(v))$ is a quadratic form over F. Moreover, the polar form of $q_{\rho,\pi}$ is given by $\mathfrak{b}_{q_{\rho,\pi}}(u, v) = \pi(\overline{h_\rho(u, v)})$.*

Let (V, ρ) be a generalised quadratic space over (D, θ). Fix a basis

$$\mathcal{B} = \{u_1, \ldots, u_m\},$$

of $\overline{D}$ over F and let $\{\pi_1, \ldots, \pi_m\}$ denote its dual basis of $\mathrm{Hom}(\overline{D}, F)$. For $i = 1, \ldots, m$, define the map $q^{u_i}_{\rho,\mathcal{B}} : V \to F$ by

$$q^{u_i}_{\rho,\mathcal{B}}(v) = \pi_i(\rho(v)).$$

In view of Lemma 4.1, considering V as a vector space over F, every $q^{u_i}_{\rho,\mathcal{B}}$ is a quadratic form with the polar form $\mathfrak{b}_{q^{u_i}_{\rho,\mathcal{B}}}(u, v) = \pi_i(\overline{h_\rho(u, v)})$. We also have

$$\rho(v) = \sum_{i=1}^{m} q^{u_i}_{\rho,\mathcal{B}}(v) u_i,$$

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for all $v \in V$. Let

$$q_{\rho,\mathcal{B}} = (q^{u_1}_{\rho,\mathcal{B}}, \ldots, q^{u_m}_{\rho,\mathcal{B}}).$$

Then $q_{\rho,\mathcal{B}}$ is a system of quadratic forms. Clearly, if (V, ρ') is another generalised quadratic space over (D, θ) then $q_{\rho\perp\rho',\mathcal{B}} \simeq q_{\rho,\mathcal{B}} \perp q_{\rho',\mathcal{B}}$. We will denote $q^{u_i}_{\rho,\mathcal{B}}$ and $q_{\rho,\mathcal{B}}$ by q_i and q_ρ respectively if the basis $\mathcal{B}$ is clear from the context.

Note that if $(D, \theta) = (F, \mathrm{id})$, that is, ρ is a quadratic form over F, then for every basis $\mathcal{B}$ of $\overline{D} = F$, the form $q_{\rho,\mathcal{B}}$ is a scalar multiple of ρ. More precisely, taking $\mathcal{B} = \{\alpha\}$ for some $\alpha \in F^\times$, one has $q_{\rho,\mathcal{B}} = \alpha^{-1} \cdot \rho$. In particular, if $\mathcal{B} = \{1\}$ then $q_{\rho,\mathcal{B}} = \rho$.

Lemma 4.2. *Let (V, ρ) and (V', ρ') be two generalised quadratic spaces over (D, θ). If $(V, \rho) \simeq (V', \rho')$ then $q_\rho \simeq q_{\rho'}$.*

Proof. Let $f : (V, \rho) \simeq (V', \rho')$ be an isometry. Then f is an isomorphism of vector spaces over F satisfying

$$q^{u_i}_{\rho',\mathcal{B}}(f(v)) = \pi_i(\rho'(f(v))) = \pi_i(\rho(v)) = q^{u_i}_{\rho,\mathcal{B}}(v),$$

for all $v \in V$ and $i = 1, \ldots, m$. Hence, $q_{\rho'}(f(v)) = q_\rho(v)$ for all $v \in V$, proving the result. □

The following result is evident.

Proposition 4.3. *Let (V, ρ) be a generalised quadratic space over (D, θ) and let $v \in V$. Then $\rho(v) = 0$ if and only if $q_\rho(v) = 0$. In particular, ρ is isotropic if and only if q_ρ is isotropic.*

Corollary 4.4. *Let (V, ρ) be a generalised quadratic space over (D, θ). If $q_\rho(v) = 0$ for some $v \in V$ then $q_\rho(vd) = 0$ for all $d \in D$.*

Proof. If $q_\rho(v) = 0$ then $\rho(v) = 0$ by Proposition 4.3. Hence, $\rho(vd) = 0$ for all $d \in D$. Again, Proposition 4.3 implies that $q_\rho(vd) = 0$ for all $d \in D$. □

Lemma 4.5. *Let (V, ρ) be a generalised quadratic space over (D, θ) and let $v \in V$.*

(i) *If $v \in \mathrm{rad}(h_\rho)$ then $v \in \mathrm{rad}(\mathfrak{b}_{q_\rho})$, that is, $v \in \mathrm{rad}(\mathfrak{b}_{q_i})$ for all $i = 1, \ldots, m$.*

(ii) *If $v \notin \mathrm{rad}(h_\rho)$ then $v \notin \mathrm{rad}(\mathfrak{b}_{q_i})$ for all $i = 1, \ldots, m$; in particular, $v \notin \mathrm{rad}(\mathfrak{b}_{q_\rho})$.*

Proof. Suppose that $v \in \mathrm{rad}(h_\rho)$. Then $h_\rho(v, w) = 0$ for all $w \in V$. It follows that $\mathfrak{b}_{q_i}(v, w) = \pi_i(\overline{h_\rho(v, w)}) = 0$ for all $w \in V$ and $i = 1, \ldots, m$, that is, $v \in \mathrm{rad}(\mathfrak{b}_{q_\rho})$. This proves (i). To prove (ii), suppose that $h_\rho(v, w) \neq 0$ for some $w \in V$. Fix an index i with $1 \leqslant i \leqslant m$. Set $d = h_\rho(v, w)^{-1} u'_i$, where $u'_i \in D$ is a representative of $u_i \in \overline{D}$. Then $\mathfrak{b}_{q_i}(v, wd) = \pi_i(\overline{h_\rho(v, w)d}) = \pi_i(u_i) = 1 \neq 0$, hence $v \notin \mathrm{rad}(\mathfrak{b}_{q_i})$. □

Corollary 4.6. *Let (V, ρ) be a generalised quadratic space over (D, θ). Then*

(i) $\mathrm{rad}(h_\rho) = \mathrm{rad}(\mathfrak{b}_{q_\rho}) = \mathrm{rad}(\mathfrak{b}_{q_i})$ *for all $i = 1, \ldots, m$;*

(ii) $\mathrm{rad}(\rho) = \mathrm{rad}(q_\rho)$.

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Proof. Part (i) follows from Lemma 4.5. Part (ii) follows from (i) and Proposition 4.3. □

Corollary 4.7. *Let (V, ρ) be a generalised quadratic space over (D, θ). Then*

(i) *ρ is nonsingular if and only if q_ρ is totally nonsingular;*
(ii) *ρ is regular (respectively, totally singular) if and only if q_ρ is regular (respectively, totally singular).*

Proof. The result follows from Corollary 4.6. □

Proposition 4.8. *Let (V, ρ) be a nonsingular generalised quadratic space over (D, θ). Then ρ is hyperbolic if and only if q_ρ is metabolic.*

Proof. The proof is very similar to that of [5, (4.2)]. If ρ is hyperbolic then there is a subspace $L \subseteq V$ with $\rho|_L = 0$ such that $\dim_D L = \frac{1}{2}\dim_D V$. Hence, $q_\rho|_L = 0$ and $\dim_F L = \frac{1}{2}\dim_F V$, that is, q_ρ is metabolic.

To prove the converse, write $\rho \simeq \rho_{\mathrm{an}} \perp \rho_{\mathrm{hyp}}$, where ρ_{an} is anisotropic and ρ_{hyp} is hyperbolic. Then $q_\rho \simeq q_{\rho_{\mathrm{an}}} \perp q_{\rho_{\mathrm{hyp}}}$ by Lemma 4.2. As already observed, the system $q_{\rho_{\mathrm{hyp}}}$ is metabolic. Also, by Corollary 4.7, $q_{\rho_{\mathrm{hyp}}}$ is totally nonsingular. Hence, using Corollary 3.2, one concludes that $q_{\rho_{\mathrm{an}}}$ is metabolic. In view of Proposition 4.3, this is impossible, unless ρ_{an} is trivial. □

We conclude this section with an application of the system q_ρ to C_i-fields, suggested by the referee. Recall that a field K (of arbitrary characteristic) is called a *C_i-field* if every homogeneous polynomial of degree d over K in more than d^i variables has a nontrivial zero.

The next result follows from a theorem of Lang and Nagata (see [9, (15.8)]).

Theorem 4.9. *Let K be a C_i-field and let (V, q) be a system of r quadratic forms over K. If $\dim_K V > r2^i$ then q is isotropic.*

Let n be the degree of D over F. By [4, (2.6)], $\dim_F \mathrm{Symd}(D, \theta) = n(n-1)/2$. Hence, the quotient $\overline{D}$ has dimension $m = n(n+1)/2$ over F.

Proposition 4.10. *Suppose that F is a C_i-field. Let (V, ρ) be a generalised quadratic space over (D, θ) and let $s = \dim_D V$. If $s > 2^{i-1}(n+1)/n$ then ρ is isotropic.*

Proof. In view of Proposition 4.3, it suffices to show that the system of m quadratic forms (V, q_ρ) is isotropic. Observe that $\dim_F V = n^2 s$. If $s > 2^{i-1}(n+1)/n$ then $\dim_F V > n(n+1)2^{i-1} = m2^i$. Hence, the result follows from Theorem 4.9. □

5. Comparison of Witt indices

The aim of this section is to find a relation between the Witt indices of a generalised quadratic form ρ and its associated system of quadratic forms q_ρ. We start with the following simple observation.

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Remark 5.1. Let (V,ρ) be a generalised quadratic space over (D,θ) and $\rho_{\mathrm{ts}} = \rho|_{\mathrm{rad}(h_\rho)}$. Then $\mathrm{rad}(\rho)$ is the unique maximal totally isotropic D-subspace of ρ_{ts}. Moreover, every maximal totally isotropic D-subspace of (V,ρ) contains $\mathrm{rad}(\rho)$. This follows from the fact that for every totally isotropic D-subspace R of (V,ρ) and for every $v \in \mathrm{rad}(\rho)$, the D-subspace $vD + R$ is also totally isotropic. Similarly, if (V,q) is a system of quadratic forms over F, then $\mathrm{rad}(q)$ is the unique maximal totally isotropic subspace of $q_{\mathrm{ts}} := q|_{\mathrm{rad}(\mathfrak{b}_q)}$ and every maximal totally isotropic subspace of (V,q) contains $\mathrm{rad}(q)$.

Definition 5.2. Let (V,ρ) be a generalised quadratic space over (D,θ). We call the maximal dimension (over D) of totally isotropic D-subspaces of (V,ρ) the *Witt index* of (V,ρ) and we denote it by $w(\rho)$.

Note that if ρ is nonsingular then $w(\rho) \leqslant \frac{1}{2}\dim_D V$. Also, a nonsingular generalised quadratic space (V,ρ) is hyperbolic if and only if $w(\rho) = \frac{1}{2}\dim_D V$.

Lemma 5.3. *Let (V,ρ) be a generalised quadratic space over (D,θ). Then there exists a decomposition $\rho \simeq \rho_{\mathrm{ts}} \perp \rho_{\mathrm{ns}}$ such that ρ_{ts} is totally singular, ρ_{ns} is nonsingular and $w(\rho) = w(\rho_{\mathrm{ts}}) + w(\rho_{\mathrm{ns}})$.*

Proof. Set $W = \mathrm{rad}(h_\rho)$ and let W' be any D-subspace of V with $V \simeq W \oplus W'$. Then $\rho \simeq \rho_{\mathrm{ts}} \perp \rho|_{W'}$, where $\rho_{\mathrm{ts}} := \rho|_W$ is totally singular and $\rho|_{W'}$ is nonsingular. Write $R = \mathrm{rad}(\rho)$, $r = \dim_D R$ and $n = w(\rho)$. Let L be a totally isotropic D-subspace of (V,ρ) with $\dim_D L = n$. By Remark 5.1, $w(\rho_{\mathrm{ts}}) = r$ and $R \subseteq L$. Let $\{v_1,\ldots,v_r\}$ be a basis of R over D and extend it to a D-basis $\{v_1,\ldots,v_n\}$ of L. Let S_1 be the D-subspace of V spanned by $v_{r+1},\ldots,v_n$ and let S_2 be any D-subspace of V satisfying $V \simeq W \oplus S_1 \oplus S_2$. Set $S = S_1 \oplus S_2$. Then S is a D-subspace of V with $V \simeq W \oplus S$. Hence, $\rho \simeq \rho_{\mathrm{ts}} \perp \rho|_S$ and $\rho|_S$ is nonsingular. Note that S_1 is a totally isotropic subspace of $(S,\rho|_S)$, hence $w(\rho|_S) \geqslant n - r = w(\rho) - w(\rho_{\mathrm{ts}})$. On the other hand, the isometry $\rho \simeq \rho_{\mathrm{ts}} \perp \rho|_S$ shows that $w(\rho) \geqslant w(\rho_{\mathrm{ts}}) + w(\rho|_S)$. Hence, $w(\rho|_S) = n - r$, proving the claim. □

We are now ready to prove the following natural result.

Theorem 5.4. *Let (V,ρ) be a generalised quadratic space over (D,θ). Then we have $w(q_\rho) = \dim_F D \cdot w(\rho)$.*

Proof. By Lemma 5.3, there exists a decomposition $\rho \simeq \rho_{\mathrm{ts}} \perp \rho_{\mathrm{ns}}$ such that $\rho_{\mathrm{ts}} := \rho_{\mathrm{rad}(h_\rho)}$ is totally singular, ρ_{ns} is nonsingular and $w(\rho) = w(\rho_{\mathrm{ts}}) + w(\rho_{\mathrm{ns}})$. Set $k = \dim_F D$, $n = w(\rho)$ and $r = w(\rho_{\mathrm{ts}})$. We want to show that $w(q_\rho) = kn$. Note that $w(\rho_{\mathrm{ns}}) = n - r$, hence one can write $\rho_{\mathrm{ns}} = \rho_{\mathrm{hyp}} \perp \rho_{\mathrm{an}}$, where ρ_{hyp} is hyperbolic of dimension $2(n-r)$ over D and ρ_{an} is anisotropic. Then

$$q_\rho \simeq q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{hyp}}} \perp q_{\rho_{\mathrm{an}}}. \tag{5.1}$$

By Corollary 4.7 and Proposition 4.8, q_{hyp} is totally nonsingular and metabolic. Hence, in view of Proposition 3.1, the isometry (5.1) implies that

$$w(q_\rho) = w(q_{\rho_{\mathrm{hyp}}}) + w(q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}}).$$

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Since $q_{\rho_{\mathrm{hyp}}}$ is totally nonsingular and metabolic, one has $w(q_{\rho_{\mathrm{hyp}}}) = k(n-r)$ by dimension count. Hence, it suffices to show that $w(q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}}) = kr$.

Let $R = \mathrm{rad}(\rho)$. Then $\mathrm{rad}(q_\rho) = R$ by Corollary 4.6. As observed in Remark 5.1, R is the unique maximal totally isotropic D-subspace of ρ_{ts} and also the unique maximal totally isotropic F-subspace of $q_{\rho_{\mathrm{ts}}}$. In particular, $\dim_D R = w(\rho_{\mathrm{ts}}) = r$. Let L be a totally isotropic subspace of $q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}}$ with $\dim_F L = w(q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}})$. Then $R \subseteq L$ by Remark 5.1. According to Corollary 4.4, for every $v \in L$, vD is a totally isotropic subspace of $q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}} \simeq q_{\rho_{\mathrm{ts}} \perp \rho_{\mathrm{an}}}$. Hence, $R + vD$ is a totally isotropic subspace of $q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}}$, which implies that it is a totally isotropic subspace of $\rho_{\mathrm{ts}} \perp \rho_{\mathrm{an}}$, thanks to Proposition 4.3. However, we have $w(\rho_{\mathrm{ts}} \perp \rho_{\mathrm{an}}) = r$. Since $R + vD$ is a D-subspace of $\rho_{\mathrm{ts}} \perp \rho_{\mathrm{an}}$, one concludes that $v \in R$. Thus $R = L$. It follows that $w(q_{\rho_{\mathrm{ts}}} \perp q_{\rho_{\mathrm{an}}}) = \dim_F R = kr$, proving the statement. □

6. A classification theorem

In this section we show that the system q_ρ can be used to classify generalised quadratic forms.

LEMMA 6.1. *Let (V, ρ) be a generalised quadratic space over (D, θ). If $D \neq F$ and $\rho(v) \in \mathrm{Sym}(D,\theta)/\mathrm{Symd}(D,\theta)$ for all $v \in V$ then ρ is totally singular.*

PROOF. From relation (2.1), $h_\rho(v, v) = 0$ for all $v \in V$. In view of [3, Chapter I, (6.2.3)], the assumption $D \neq F$ implies that h_ρ is trivial, so ρ is totally singular. □

PROPOSITION 6.2. *Let (V, ρ) and (V', ρ') be anisotropic generalised quadratic spaces over (D, θ). If $q_\rho \perp q_{\rho'}$ is metabolic then $\rho \simeq \rho'$.*

PROOF. Let L be a Lagrangian of $(V \perp V', q_\rho \perp q_{\rho'})$. By Proposition 4.3, q_ρ is anisotropic. Hence, $L \cap V = \{0\}$, that is, the projection $\pi' : L \to V'$ is injective. In particular, $\dim_F L \leqslant \dim_F V'$. Similarly, we have $\dim_F L \leqslant \dim_F V$. However, note that $\dim_F L \geqslant \frac{1}{2}(\dim_D V + \dim_D V')$, so

$$\dim_F V = \dim_F V' = \dim_F L.$$

It follows that the projections $\pi : L \to V$ and $\pi' : L \to V'$ are both isomorphisms. Now, if $D = F$ then $q_\rho \simeq \alpha \cdot \rho$ and $q_{\rho'} \simeq \alpha \cdot \rho'$ for some $\alpha \in F^\times$. We prove that the isomorphism $\pi' \circ \pi^{-1} : V \to V'$ is an isometry $\rho \simeq \rho'$. For $v \in V$, choose $v' \in V'$ such that $(v, v') = \pi^{-1}(v) \in L$. Since $(q_\rho \perp q_{\rho'})(v, v') = 0$, one concludes that $q_\rho(v) = q_{\rho'}(v')$. Hence,

$$\begin{aligned}\rho'(\pi' \circ \pi^{-1}(v)) &= \alpha^{-1} q_{\rho'}(\pi' \circ \pi^{-1}(v)) = \alpha^{-1} q_{\rho'}(\pi'(v, v')) \\ &= \alpha^{-1} q_{\rho'}(v') = \alpha^{-1} q_\rho(v) = \rho(v),\end{aligned}$$

so $\pi' \circ \pi^{-1}$ is an isometry.

Suppose that $D \neq F$. We claim that ρ is totally singular if and only if ρ' is totally singular. Suppose that ρ is totally singular and let $v_1', v_2' \in V'$. Choose $v_1, v_2 \in V$ such

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that $(v_1, v_1') \in L$ and $(v_2, v_2') \in L$. The equality $\mathfrak{b}_{q_\rho \perp q_{\rho'}}((v_1, v_1'), (v_2, v_2')) = 0$ implies that $\mathfrak{b}_{q_{\rho'}}(v_1', v_2') = \mathfrak{b}_{q_\rho}(v_1, v_2) = 0$. Hence,

$$h_{\rho'}(v_1', v_2') \in \mathrm{Symd}(D, \theta) \quad \text{for all } v_1', v_2' \in V'. \tag{6.1}$$

For $d \in D$, applying (6.1) to vectors $v_1', v_2'd \in V'$, one concludes that

$$h_{\rho'}(v_1', v_2')d = h_{\rho'}(v_1', v_2'd) \in \mathrm{Symd}(D, \theta) \quad \text{for every } v_1', v_2' \in V' \text{ and } d \in D.$$

However, $\mathrm{Symd}(D, \theta) \subsetneq D$, so $h_{\rho'}(v_1', v_2') = 0$ for all $v_1', v_2' \in V'$, that is, ρ' is totally singular. Similarly, ρ is totally singular if ρ' is totally singular.

We now consider two cases.

Case 1: ρ is totally singular. Hence, ρ' is also totally singular. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V over D. For $i = 1, \ldots, n$, choose $v_i' \in V'$ such that $(v_i, v_i') \in L$. As $(q_\rho \perp q_{\rho'})(v_i, v_i') = 0$, we have $q_\rho(v_i) = q_{\rho'}(v_i')$, hence $\rho(v_i) = \rho'(v_i')$ for $i = 1, \ldots, n$. It follows that

$$\rho\Big(\sum_{i=1}^{n} v_i d_i\Big) = \sum_{i=1}^{n} \theta(d_i)\rho(v_i)d_i = \sum_{i=1}^{n} \theta(d_i)\rho'(v_i')d_i = \rho'\Big(\sum_{i=1}^{n} v_i' d_i\Big), \tag{6.2}$$

for all $d_1, \ldots, d_n \in D$. We claim that $\mathcal{B}' = \{v_1', \ldots, v_n'\}$ is a basis of V' over D. Since $\dim_D V = \dim_D V'$, it suffices to show that $\mathcal{B}'$ is linearly independent over D. Suppose $\sum_{i=1}^n v_i' d_i = 0$ for some $d_1, \ldots, d_n \in D$. Then $\rho'(\sum_{i=1}^n v_i' d_i) = 0$, hence $\rho(\sum_{i=1}^n v_i d_i) = 0$ by (6.2). Since ρ is anisotropic, $\sum_{i=1}^n v_i d_i = 0$, hence $d_1 = \cdots = d_n = 0$, proving the claim. It is now clear from (6.2) that the D-linear map $f : V \to V'$ induced by $f(v_i) = v_i'$ for $i = 1, \ldots, n$, is an isometry $\rho \simeq \rho'$.

Case 2: ρ is not totally singular. As $D \neq F$, there exists $v \in V$ with $\rho(v) \notin \mathrm{Sym}(D, \theta)/\mathrm{Symd}(D, \theta)$ by Lemma 6.1, so $h_\rho(v, v) \neq 0$. Choose $v' \in V'$ such that $(v, v') \in L$. Then $q_{\rho'}(v') = q_\rho(v)$. Hence, $\rho'(v') = \rho(v)$, which implies that $h_{\rho'}(v', v') = h_\rho(v, v) \neq 0$. We use induction on $\dim_D V$. Define the map $f : vD \to v'D$ by $f(vd) = v'd$ for $d \in D$. Then f is an isometry $\rho|_{vD} \simeq \rho'|_{v'D}$. This proves the result in the case where $\dim_D V = 1$. Suppose now that $\dim_D V > 1$ and let $W = (vD)^{\perp_{h_\rho}} \subseteq V$ and $W' = (v'D)^{\perp_{h_{\rho'}}} \subseteq V'$. Since $h_\rho(v, v) = h_{\rho'}(v', v') \neq 0$, both $\rho|_{vD}$ and $\rho'|_{v'D}$ are nonsingular. It follows from [3, Chapter I, (3.6.2)] that $\rho \simeq \rho|_{vD} \perp \rho|_W$ and $\rho' \simeq \rho'|_{v'D} \perp \rho'|_{W'}$. So, $q_\rho \simeq q_{\rho|_{vD}} \perp q_{\rho|_W}$ and $q_{\rho'} \simeq q_{\rho'|_{v'D}} \perp q_{\rho'|_{W'}}$, which implies

$$q_\rho \perp q_{\rho'} \simeq (q_{\rho|_{vD}} \perp q_{\rho|_{v'D}}) \perp (q_{\rho|_W} \perp q_{\rho|_{W'}}).$$

By [5, (2.2)], the system $q_{\rho|_{vD}} \perp q_{\rho|_{v'D}}$ is metabolic. Also, by Corollary 4.7, $q_{\rho|_{vD}} \perp q_{\rho|_{v'D}}$ is totally nonsingular. Hence, the system $q_{\rho|_W} \perp q_{\rho|_{W'}}$ is metabolic by Corollary 3.2. Note that $q_{\rho|_W}$ and $q_{\rho'|_{W'}}$ are both anisotropic. Therefore, by the induction hypothesis, $\rho|_W \simeq \rho'|_{W'}$, which implies that $\rho \simeq \rho'$. □

LEMMA 6.3. *Let $f : (V_1, q_1) \perp (V_1', q_1') \simeq (V_2, q_2) \perp (V_2', q_2')$ be an isometry of systems of quadratic forms over F. If q_i is trivial and q_i' is regular for $i = 1, 2$, then $q_1' \simeq q_2'$.*

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PROOF. Note that $\mathrm{rad}(q_1 \perp q_1') = V_1$ and $\mathrm{rad}(q_2 \perp q_2') = V_2$, hence $f(V_1) = V_2$. By dimension count, $\dim_F V_1' = \dim_F V_2'$. Consider the natural injection $i : V_1' \hookrightarrow V_1 \oplus V_1'$ and the natural projection $\pi : V_2 \oplus V_2' \to V_2'$. The equality $f(V_1) = V_2$ implies that the composition $\pi \circ f \circ i : V_1' \to V_2'$ is injective, and therefore it is an isomorphism by dimension count. It now readily follows that the map $\pi \circ f \circ i$ is an isometry and that $(V_1', q_1') \simeq (V_2', q_2')$. □

THEOREM 6.4. *Let (V, ρ) and (V', ρ') be two generalised quadratic spaces over (D, θ). Then $(V, \rho) \simeq (V', \rho')$ if and only if $(V, q_\rho) \simeq (V, q_{\rho'})$.*

PROOF. The 'only if' implication was proved in Lemma 4.2. Conversely, suppose that $q_\rho \simeq q_{\rho'}$. Write $\rho \simeq \rho_0 \perp \rho_{\mathrm{re}}$ and $\rho' \simeq \rho_0' \perp \rho_{\mathrm{re}}'$, where $\rho_0 = \rho|_{\mathrm{rad}\,\rho}$ and $\rho_0' = \rho'|_{\mathrm{rad}\,\rho'}$ are trivial and ρ_{re} and ρ_{re}' are regular. Then

$$q_\rho \simeq q_{\rho_0} \perp q_{\rho_{\mathrm{re}}} \quad \text{and} \quad q_{\rho'} \simeq q_{\rho_0'} \perp q_{\rho_{\mathrm{re}}'}.$$

Note that $q_{\rho_{\mathrm{re}}}$ and $q_{\rho_{\mathrm{re}}'}$ are regular by Corollary 4.7. Since the systems q_{ρ_0} and $q_{\rho_0'}$ are trivial, the isometry $q_\rho \simeq q_{\rho'}$ yields $q_{\rho_{\mathrm{re}}} \simeq q_{\rho_{\mathrm{re}}'}$, thanks to Lemma 6.3. Now, write $\rho_{\mathrm{re}} \simeq \rho_{\mathrm{an}} \perp \rho_{\mathrm{hyp}}$ and $\rho_{\mathrm{re}}' \simeq \rho_{\mathrm{an}}' \perp \rho_{\mathrm{hyp}}'$, where ρ_{an} and ρ_{an}' are anisotropic, and $\rho|_{\mathrm{hyp}}$ and ρ_{hyp}' are hyperbolic. Then $q_{\rho_{\mathrm{re}}} \simeq q_{\rho_{\mathrm{an}}} \perp q_{\rho_{\mathrm{hyp}}}$ and $q_{\rho_{\mathrm{re}}'} \simeq q_{\rho_{\mathrm{an}}'} \perp q_{\rho_{\mathrm{hyp}}'}$. The isometry $q_{\rho_{\mathrm{re}}} \simeq q_{\rho_{\mathrm{re}}'}$ implies that

$$q_{\rho_{\mathrm{an}}} \perp q_{\rho_{\mathrm{hyp}}} \simeq q_{\rho_{\mathrm{an}}'} \perp q_{\rho_{\mathrm{hyp}}'}. \tag{6.3}$$

Note that $q_{\rho_{\mathrm{hyp}}}$ and $q_{\rho_{\mathrm{hyp}}'}$ are metabolic and totally nonsingular by Proposition 4.8 and Corollary 4.7. Using Proposition 3.1, and by comparing Witt indices of both sides of (6.3), one concludes that $q_{\rho_{\mathrm{hyp}}}$ and $q_{\rho_{\mathrm{hyp}}'}$ have the same dimension over F. Hence, $\dim_D \rho_{\mathrm{hyp}} = \dim_D \rho_{\mathrm{hyp}}'$, which implies that $\rho_{\mathrm{hyp}} \simeq \rho_{\mathrm{hyp}}'$.

On the other hand, as $q_{\rho_{\mathrm{re}}} \simeq q_{\rho_{\mathrm{re}}'}$, the system $q_{\rho_{\mathrm{re}}} \perp q_{\rho_{\mathrm{re}}'}$ is metabolic by [5, (2.1)]. Hence,

$$q_{\rho_{\mathrm{an}}} \perp q_{\rho_{\mathrm{an}}'} \perp q_{\rho_{\mathrm{hyp}}} \perp q_{\rho_{\mathrm{hyp}}'},$$

is metabolic. Note that $q_{\rho_{\mathrm{hyp}}} \perp q_{\rho_{\mathrm{hyp}}'}$ is metabolic by Proposition 4.8 and is totally nonsingular by Corollary 4.7. Hence, Corollary 3.2 implies that $q_{\rho_{\mathrm{an}}} \perp q_{\rho_{\mathrm{an}}'}$ is also metabolic. By Proposition 6.2, we have $\rho_{\mathrm{an}} \simeq \rho_{\mathrm{an}}'$, which implies that $\rho \simeq \rho'$. □

Acknowledgement

The author thanks the referee for several useful comments.

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