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SOLVING LINEAR OPERATOR EQUATIONS

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1. Introduction. Let $\mathscr{X}$ be a complex Banach space and $\mathscr{L}(\mathscr{X})$ the algebra of bounded operators on $\mathscr{X}$. M. Rosenblum's theorem [**13**; **12**] (also discovered by M. G. Kreĭn, cf. [**9**]) states that (if A, B are fixed bounded operators) the spectrum of the operator $\mathscr{T}$ on $\mathscr{L}(\mathscr{X})$ defined by $\mathscr{T}(\mathscr{X}) = AX - XB$ is contained in $\sigma(A) - \sigma(B) = \{\alpha - \beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. In particular, the condition $\sigma(A) \cap \sigma(B) = \emptyset$ implies that for each $Y \in \mathscr{L}(\mathscr{X})$ there is a unique $X \in \mathscr{L}(\mathscr{X})$ such that $AX - XB = Y$. This does not completely settle the question of solvability of the equation $AX - XB = Y$: for example, if A is the backward unilateral shift and $B = 0$, then the equation has a solution (for any Y) even though $\sigma(B) \subseteq \sigma(A)$.

In this note we seek a sharper version. When the underlying space $\mathscr{X}$ is a Hilbert space, we may claim to have succeeded, for we give a simple necessary and sufficient condition (Theorem 5 below) for existence of a solution. We say something about uniqueness (Theorem 4) in any Banach space.

One proof of Rosenblum's theorem [**12**] goes as follows: defining commuting operators $\mathscr{A}$ and $\mathscr{B}$ by the equations $\mathscr{A}X = AX$ and $\mathscr{B}X = XB$ (so that the operator equation to be solved is $\mathscr{T}X = Y$ with $\mathscr{T} = \mathscr{A} - \mathscr{B}$) one uses the general fact that the spectrum of the sum of two commuting operators is contained in the sum of their spectra. Our procedure here is to use the corresponding general fact about approximate point spectrum (Theorem 2(i) below), which we prove by use of something like the well-known Berberian extension (see Section 2).

In the following, $\sigma_p(A)$ denotes the point spectrum of any operator A, and $\sigma_\pi(A)$ its approximate point spectrum. Also $\sigma_\delta(A)$ denotes its approximate defect spectrum, defined by $\sigma_\delta(A) = \{\lambda \in \mathbf{C} : A - \lambda \text{ is not onto}\}$. If A^* denotes the Banach-space adjoint of A, then [**14**, § 4.7]

$$\sigma_\pi(A) = \sigma_\delta(A^*), \sigma_\delta(A) = \sigma_\pi(A^*);$$

this duality plays an essential role in our application of our spectral inclusion theorem.

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There has been an independent and earlier treatment of analogues of

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Rosenblum's theorem in terms of one-sided spectra by Robin Harte [**6**; **7**], in the course of his very extensive study of left and right spectrum. To our Theorem 5, on conditions for the existence of a certain $X \in \mathscr{L}(\mathscr{X})$, corresponds his result [**7**, Theorem 3.4] on conditions for the existence of an element of $\mathscr{L}(\mathscr{L}(\mathscr{X}))$ which may or may not arise from one of our X. We do not see how to deduce our result from his. We suppose that, even were such a deduction to be found, our treatment here would retain interest.

Added in proof. We have recently seen the University of Virginia Master's Thesis of B. B. Borrel, 1966, in which the question of sharpening Rosenblum's theorem is treated, and some results obtained in special cases.

2. The Berberian-Quigley extension. In the case where $\mathscr{X}$ is a Hilbert space, S. Berberian [**2**] showed (cf. Calkin [**3**]) that there exists a Hilbert space $\mathscr{X}^0$ containing $\mathscr{X}$ as a subspace, and there exists a homomorphism from operators $A \in \mathscr{L}(\mathscr{X})$ to operators $A^0 \in \mathscr{L}(\mathscr{X}^0)$, such that A^0 is an extension of A and $\sigma_p(A^0) = \sigma_\pi(A^0) = \sigma_\pi(A)$. The analogous construction for Banach spaces (based on an idea introduced by F. D. Quigley) is a little simpler because in a Banach space there is less to prove, though the statement looks the same:

THEOREM 1. *Given any Banach space $\mathscr{X}$, there is an imbedding of $\mathscr{X}$ into a larger Banach space $\mathscr{X}^0$, and a mapping $A \rightarrow A^0$ of $\mathscr{L}(\mathscr{X})$ into $\mathscr{L}(\mathscr{X}^0)$ which is an isometric isomorphism such that every A^0 is an extension of A and*

$$\sigma_p(A^0) = \sigma_\pi(A^0) = \sigma_\pi(A).$$

Rather than give details here, we refer to R. A. Hirschfeld [**8**] or to a recent discussion suitable to our purposes [**4**].

3. Approximate point spectrum of a sum of commuting operators.

THEOREM 2. *If $A, B \in \mathscr{L}(\mathscr{X})$ and $AB = BA$, then*

(i) $\sigma_\pi(A + B) \subseteq \sigma_\pi(A) + \sigma_\pi(B) = \{\alpha + \beta : \alpha \in \sigma_\pi(A), \beta \in \sigma_\pi(B)\}$;

(ii) $\sigma_\pi(AB) \subseteq \sigma_\pi(A)\sigma_\pi(B) = \{\alpha\beta : \alpha \in \sigma_\pi(A), \beta \in \sigma_\pi(B)\}$.

Proof. Let $\lambda \in \sigma_\pi(A + B)$. Then, by Theorem 1, $\lambda \in \sigma_p((A + B)^0) = \sigma_p(A^0 + B^0)$. Let

$$\mathscr{M} = \{\mathbf{x} \in \mathscr{X}^0 : (A^0 + B^0)\mathbf{x} = \lambda\mathbf{x}\}.$$

Again by Theorem 1, A^0 and B^0 commute; therefore $\mathscr{M}$ is invariant under each of A^0 and B^0. Note that $B^0|\mathscr{M} = (\lambda - A^0)|\mathscr{M}$. Choosing, then, $\alpha \in \sigma_\pi(A^0|\mathscr{M})$, we will have $\lambda - \alpha \in \sigma_\pi(B^0|\mathscr{M})$, therefore

$$\lambda = \alpha + (\lambda - \alpha) \in \sigma_\pi(A^0|\mathscr{M}) + \sigma_\pi(B^0|\mathscr{M}) \subseteq \sigma_\pi(A^0) + \sigma_\pi(B^0) = \sigma_\pi(A) + \sigma_\pi(B).$$

Next consider $\sigma_\pi(AB)$. As above, take $\lambda \in \sigma_\pi(AB)$, and let

$$\mathscr{M} = \{\mathbf{x} \in \mathscr{X}^0 : A^0B^0\mathbf{x} = \lambda\mathbf{x}\}.$$

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Then $\mathscr{M}$ is invariant under A^0 and B^0. If $\lambda = 0$ then $\sigma_\pi(A^0)$ or $\sigma_\pi(B^0)$ contains 0, so $0 \in \sigma_\pi(A)\sigma_\pi(B)$. If $\lambda \neq 0$, then $A^0|\mathscr{M}$ has the two-sided inverse $\lambda^{-1}B^0|\mathscr{M}$. Choose α in the boundary of $\sigma(A^0|\mathscr{M})$; then λ/α is in the boundary of $\sigma(B^0|\mathscr{M})$. But this implies [**14**, 5.1-D] that $\alpha \in \sigma_\pi(A^0|\mathscr{M})$, $\lambda/\alpha \in \sigma_\pi(B^0|\mathscr{M})$, giving in turn $\lambda \in \sigma_\pi(A^0)\sigma_\pi(B^0) = \sigma_\pi(A)\sigma_\pi(B)$.

The duality between approximate point spectrum and approximate defect spectrum, mentioned in the Introduction, gives this immediate consequence.

COROLLARY 1. *If $A, B \in \mathscr{L}(\mathscr{X})$ and $AB = BA$, then*
(i) $\sigma_\delta(A + B) \subseteq \sigma_\delta(A) + \sigma_\delta(B)$;
(ii) $\sigma_\delta(AB) \subseteq \sigma_\delta(A)\sigma_\delta(B)$.

4. Application to operator equations. Now to return to the problem from which we departed, we want to relate the spectra of operators to the spectra of their left regular and right regular representations.

LEMMA. *Let $A, B \in \mathscr{L}(\mathscr{X})$, and define operators $\mathscr{A}, \mathscr{B}$ on $\mathscr{L}(\mathscr{X})$ by $\mathscr{A}X = AX$, $\mathscr{B}X = XB$. Then $\sigma_\pi(\mathscr{A}) = \sigma_\pi(A)$ and $\sigma_\pi(\mathscr{B}) = \sigma_\delta(B)$. Moreover $\sigma_\delta(\mathscr{A}) \supseteq \sigma_\delta(A)$ and $\sigma_\delta(\mathscr{B}) \supseteq \sigma_\pi(B)$, with equality in case $\mathscr{X}$ is a Hilbert space.*

There are eight assertions in the Lemma; all are known, but we illustrate by proving two of them.

To prove that $\lambda \in \sigma_\pi(A)$ implies $\lambda \in \sigma_\pi(\mathscr{A})$, we take $\lambda = 0$ without loss of generality. Suppose then that A is not boundedly invertible on $\mathscr{R}(A)$, but that $\mathscr{A}$ is boundedly invertible on $\mathscr{R}(\mathscr{A})$. The latter supposition means that there exists $m > 0$ such that $||AX|| \geqq m||X||$ for all $X \in \mathscr{L}(\mathscr{X})$. By the first supposition, we can choose a unit vector $x \in \mathscr{X}$ such that $||Ax|| < m$. Now we set $X = xx^*$, where x^* is a unit vector in $\mathscr{X}^*$ such that $x^*x = 1$. Then $||AX|| = ||Ax|| < m = m||X||$, a contradiction.

To prove that $0 \in \sigma_\delta(\mathscr{B})$ implies $0 \in \sigma_\pi(B)$ in case $\mathscr{X}$ is a Hilbert space, we again reason indirectly. If $0 \notin \sigma_\pi(B)$, then B maps $\mathscr{X}$ $1-1$ and invertibly onto $\mathscr{R}(B)$. Let B^+ be the operator whose restriction to $\mathscr{R}(B)$ is the inverse to B and whose restriction to $\mathscr{R}(B)^\perp$ is 0. Then the equation $XB = C$ has a solution in $\mathscr{L}(\mathscr{X})$ for any $C \in \mathscr{L}(\mathscr{X})$, to wit, $X = CB^+$. In other words, $\mathscr{R}(\mathscr{B})$ is all of $\mathscr{L}(\mathscr{X})$, that is, $0 \notin \sigma_\delta(\mathscr{B})$, as desired.

THEOREM 3. *Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ each be a commuting family of operators on $\mathscr{X}$, and define the operator $\mathscr{T}$ on $\mathscr{L}(\mathscr{X})$ by $\mathscr{T}X = \sum_{i=1}^n A_iXB_i$. Then*

$$\sigma_\pi(\mathscr{T}) \subseteq \left\{ \sum_{i=1}^n \alpha_i\beta_i : \alpha_i \in \sigma_\pi(A_i), \beta_i \in \sigma_\delta(B_i)\right\}.$$

If $\mathscr{X}$ is a Hilbert space, then

$$\sigma_\delta(\mathscr{T}) \subseteq \left\{ \sum_{i=1}^n \alpha_i\beta_i : \alpha_i \in \sigma_\delta(A_i), \beta_i \in \sigma_\pi(B_i)\right\}.$$

For the proof, apply Theorem 2(i) (or in the case of the second conclusion, Corollary 1(i)), not to the operators A_i, B_i themselves, but to the left- or right-multiplication operators obtained from them, whose sum is $\mathcal{T}$; and then invoke the Lemma.

In the particular case of the equation $AX - XB = Y$, we can show that the conclusions drawn from Theorem 3 are sharp; this is the content of the following two theorems.

THEOREM 4. *In order that there exist a constant $m > 0$ such that $AX - XB = Y$ implies $||Y|| \geqq m||X||$, it is necessary and sufficient that $\sigma_\pi(A) \cap \sigma_\delta(B) = \emptyset$.*

Proof. For the particular $\mathcal{T}$ which is now in question, given by $\mathcal{T}X = AX - XB$, the first assertion of Theorem 3 tells us that $\sigma_\pi(\mathcal{T}) \subseteq \sigma_\pi(A) - \sigma_\delta(B)$; therefore the assumption $\sigma_\pi(A) \cap \sigma_\delta(B) = \emptyset$ entails $0 \notin \sigma_\pi(\mathcal{T})$. This proves sufficiency.

The converse (here and in Theorem 5 below) cannot rely at all on the results of § 3, whose inclusions can not be reversed, but must rely on the special nature of the commuting operators under consideration. Assume, then, $m > 0$ such that $||AX - XB|| \geqq m||X||$ for all X. Suppose if possible that $\sigma_\pi(A) \cap \sigma_\delta(B) \neq \emptyset$, or, without loss of generality, that $0 \in \sigma_\pi(A) \cap \sigma_\delta(B)$. That means that we can choose a unit vector $x \in \mathcal{X}$ and a unit vector $y^* \in \mathcal{X}^*$ so as to get $||Ax||$ and $||B^*y^*||$ as small as desired, so let us make them each $<m/2$. Set $X = xy^*$. Then $||X|| = 1$ but $||AX - XB|| \leqq ||Ax||\,||y^*|| + ||x||\,||B^*y^*|| < m$, a contradiction.

THEOREM 5. *If $AX - XB = Y$ has a solution X for every Y, then $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$. In case $\mathcal{X}$ is a Hilbert space, the converse holds: $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$ implies that $AX = XB - Y$ has a solution for every Y.*

Proof. If $\mathcal{X}$ is a Hilbert space then the second conclusion of Theorem 3 specializes to $\sigma_\delta(\mathcal{T}) \subseteq \sigma_\delta(A) - \sigma_\pi(B)$; therefore the assumption

$$\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$$

entails $0 \notin \sigma_\delta(\mathcal{T})$. This proves the second assertion in Theorem 5.

The proof of the first assertion will use again the notion of lower bound of an operator. We are now assuming that the commutator map $\mathcal{T}$ is onto $\mathcal{L}(\mathcal{X})$. Therefore its range is closed, so there is a constant $m > 0$ such that, for all X, $||\mathcal{T}X|| \geqq m\,\mathrm{dist}(X, \mathcal{N}(\mathcal{T}))$; hence there is a constant $m > 0$ such that, for every $Y \in \mathcal{R}(\mathcal{T}) = \mathcal{L}(\mathcal{X})$, there is at least one solution X of $\mathcal{T}X = Y$ which satisfies also $m||X|| \leqq ||Y||$.

Suppose if possible that $0 \in \sigma_\delta(A) \cap \sigma_\pi(B)$ (as before, this is sufficiently general). Then there exist unit vectors $x^* \in \mathcal{X}^*$ and $y \in \mathcal{X}$ making $||A^*x^*||$ and $||By||$ as small as desired, so let us take them each $<m/2$. Then for any $Y \in \mathcal{L}(\mathcal{X})$, choosing the corresponding X in the way explained in the preceding paragraph,

$$\begin{aligned} ||x^*Yy|| = ||x^*AXy - x^*XBy|| &\leqq ||A^*x^*||\,||Xy|| + ||X^*x^*||\,||By|| \\ &< \tfrac{1}{2}m(||X|| + ||X^*||) \leqq ||Y||. \end{aligned}$$

But this is absurd, because there exists Y with $||x^* Yy|| = ||Y||$. This completes the proof.

Here is an illustration of the way in which known applications of Rosenblum's theorem can be slightly strengthened using the theorems of this section.

COROLLARY 2. *If A, B, C are operators on a Hilbert space $\mathscr{X}$, and if $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$, then the operator $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ on $\mathscr{X} \oplus \mathscr{X}$ is similar to the operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.*

Proof. By Theorem 5, there is some operator X such that $AX - XB = -C$. Then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

and $\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}$ is invertible.

Remark. All the considerations of this section can be generalized to the case where $A \in \mathscr{L}(\mathscr{X})$ and $B \in \mathscr{L}(\mathscr{Y})$ for different spaces $\mathscr{X}, \mathscr{Y}$. Then $\mathscr{T}$ is an operator on the space of bounded operators from $\mathscr{Y}$ to $\mathscr{X}$. Only notational changes are required to make the above proofs cover this more general situation.

5. Extension to other Banach spaces. The argument given in the Lemma of Section 4 to prove that $\sigma_\delta(\mathscr{B}) \subseteq \sigma_\pi(B)$ really used the hypothesis that $\mathscr{X}$ is a Hilbert space. Indeed, from the knowledge that B is a Banach-space isomorphism of $\mathscr{X}$ onto $\mathscr{R}(B)$, it does not follow that B has a left-inverse in $\mathscr{L}(\mathscr{X})$, except in the too-special case that $\mathscr{R}(B)$ possesses a topological complement.

A similar obstacle is encountered in trying to establish equality of $\sigma_\delta(\mathscr{A})$ with $\sigma_\delta(A)$: assuming $0 \notin \sigma_\delta(A)$, or $\mathscr{R}(A) = \mathscr{X}$, we want to conclude that such an equation as $AX = 1$ has a solution; but $\mathscr{R}(X)$, for such X, would be a closed complement to $\mathscr{N}(\mathscr{A})$, so we can not expect such a conclusion in general.

Is there then no hope for weakening the special hypothesis upon $\mathscr{X}$ where it occurs in Section 4?

A. S. Markus pointed out to us that even in Theorem 5 some hypothesis must be imposed upon $\mathscr{X}$. Indeed, let B be any operator which would not work in the Lemma, that is, assume without loss of generality that B maps $\mathscr{X}$ $1 - 1$ onto a non-complemented subspace [**5**, p. 191]. Then one need merely take $A = 0$ to produce an example with $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$ yet with $AX - XB = 1$ insoluble.

This remark too has its analogue with the role of $\sigma_\pi(B)$ taken by $\sigma_\delta(A)$.

Now since every Banach space not isomorphic to Hilbert space has some closed subspace without any topological complement [**10**], it might seem that only Hilbert space behaves in the way described in the Lemma. But P. Wojtaszczyk observes that on the contrary, a Banach space with non-complemented subspaces may still admit no B which would not work in the Lemma; that is, it may have non-complemented subspaces none of which is isomorphic to the whole space. Using [**11**] he can prove this for some of the most familiar non-Hilbert Banach spaces.

Thus, the possibility remains open for further investigation that the implication in Theorem 5 may go both ways for a considerable class of Banach spaces.

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