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Values at T-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables

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Abstract

We consider twisted multivariable zeta series associated to polynomials of several variables. We introduce a new class of polynomials, namely *HDF*, that contains strictly nondegenerate and hypoelliptic polynomials. For polynomials belonging to the *HDF* class, we show that we can extend holomorphically our series to $\mathbb{C}^T$. Then, thanks to a new principle called 'the Exchange Lemma', we give very simple formulae for the values of our series at T-tuples of negative integers. Finally, we make the p-adic interpolation of those values. Thus, we have generalized the results of Cassou-Noguès (that she used to construct the p-adic L-functions of totally real fields) in two ways: we consider multivariable series and our series are associated to more general polynomials. In addition, our proof is completely different.

Contents

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Introduction

Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $\mu_1, \dots, \mu_N$ be complex numbers of modulus 1. To these data we can associate the following multivariable zeta series:

$$Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T) = \sum_{m_1 \geqslant 1, \dots, m_N \geqslant 1} \frac{(\prod_{n=1}^N \mu_n^{m_n}) Q(m_1, \dots, m_N)}{\prod_{t=1}^T P_t(m_1, \dots, m_N)^{s_t}}$$

where $(s_1, \dots, s_T) \in \mathbb{C}^T$.

In this article we will always assume that

$$\forall t \in \{1, \dots, T\}, \quad \forall \mathbf{x} \in [1, +\infty[^N, \quad P_t(\mathbf{x}) > 0 \quad \text{and} \quad \prod_{t=1}^T P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Then $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T)$ is an absolutely convergent series when $\Re(s_1), \dots, \Re(s_T)$ are sufficiently large. We will say that the series is twisted when all of the μ_t are different from one and non-twisted when they are all equal to one.

The complexity of these series lies in the polynomials P_t, so the authors studied these series for several classes of polynomials.

The issue of meromorphic continuation

The meromorphic continuation to $\mathbb{C}$ of these series was proved in the non-twisted and monovariable (i.e. $T = 1$) case by Mellin (P with positive coefficients [Mel01]), Mahler (elliptic case [Mah28]), Cassou-Noguès (positive coefficients case for polynomials with two variables [Cas83]), Sargos (non-degenerate case [Sar84]), Lichtin (hypoelliptic monovariable case [Lic88]) and Essouabri (H_0S case [Ess97]) Definitions of some of these classes are recalled at the beginning of § 1.

In fact, these results extend easily (in their respective classes) to the multivariable case (see, for example, [Lic91] and [Ess95, p. 74]).

When the μ_n are roots of unity, the meromorphic continuation is clearly a consequence of the non-twisted case, but when they are not, we have to use the path used in [Ess97]. As a conclusion, it is a simple adaptation of the work of Essouabri to see that under H_0S the series can be meromorphically extended to $\mathbb{C}^T$ for any $\mu_1, \dots, \mu_N$ of modulus 1.

Katsurada and Matsumoto [KM96], Akiyama and Ishikawa [AI02], Matsumoto and Tanigawa [MT03], Zhao [Zha00], Ishikawa [Ish02], and Egami and Matsumoto [EM02] gave simple proofs of the existence of meromorphic continuation. However, they only considered special cases of linear forms.

The issue of values at negative integers

The monovariable and non-twisted case when $P = P_1$ is a product of linear forms. Shintani [Shi76] showed that the negative integers are not poles and gave formulae for the values at those points. Thanks to this, he gave a new proof of a result of Klingen and Siegel: for any totally real number field $\mathbb{K}$, we have $\zeta_{\mathbb{K}}(-k) \in \mathbb{Q}$ for all $k \in \mathbb{N}$.

Eie also studied this case in [Eie96].

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In [Cas79], Cassou-Noguès studied the twisted case for $T = 1$ when P_1 is a product of linear forms. She gave formulae at negative integers adapted to p-adic interpolation. This allowed her to construct the p-adic L-functions associated to number fields and to solve crucial arithmetic conjectures.

In [Cas82] she generalized her work to the $T = 1$ polynomial with positive coefficients, still in the twisted case. Using similar methods, Chen and Eie (in [CE01]) gave very simple formulae for the values at negative integers, but they did not achieve the link with the formulae of Cassou-Noguès that are useful for p-adic interpolation.

The methods of Cassou-Noguès do not appear to extend easily to more general settings, that is, the case $T \geqslant 2$, or the case of degenerate polynomials.

The works of Akiyama, Egami and Tanigawa [AET01], Akiyama and Tanigawa [AT01], Arakawa and Kaneko [AK99], Apostol and Vu [AV84] deal with the values in the multivariable setting and non-twisted case. They deal with special cases of linear forms.

Presentation of this work

Although the H_0S class contains both non-degenerate and hypoelliptic polynomials, it is too large for our purposes. Indeed, one might hope that for any polynomial P belonging to H_0S, the continuation of any twisted zeta series $Z(Q, P, \mu, s)$ would be entire. We give an example P_{ex} that shows that this is *not* the case. This leads us to introduce a subset HDF of H_0S that still contains strictly all non-degenerate and hypoelliptic polynomials. The first main result of this paper shows that if $P_1, \dots, P_T$ belong to HDF, then any twisted series $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; s_1, \dots, s_T)$ has a holomorphic continuation to $\mathbb{C}^T$ (Theorem A). The second main result of this article is Theorem B. This gives a very simple expression for the value at any T-tuple of negative integers of these holomorphically continued series. This generalizes the result of Cassou-Noguès [Cas79, Cas82] to the multivariable case where the polynomials $P_1, \dots, P_T$, $T \geqslant 1$ belong to HDF. Our proof is quite different than that of Cassou-Noguès and is based on a simple 'Exchange Lemma'. This is a new idea whose proof can only be given in a multivariable setting. The formulae we obtain also generalize those of Chen and Eie in [CE01]. Using these formulae we are then able to prove our third main result, Theorem C. This shows that the values at T-tuples of negative integers of a large class of twisted series (in T variables) can be p-adically interpolated.

Notation

Set $\mathbb{N} = \{0, 1, 2, \dots\}, \mathbb{N}^* = \mathbb{N} - \{0\}$, $J = [1, +\infty[$, and $\mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$. The real part of $s \in \mathbb{C}$ will be denoted by $\Re(s) = \sigma$ and its imaginary part by $\Im(s) = \tau$. If $x \in \mathbb{Q}_p$, set $v_p(x) = \mathrm{ord}_p(x)$. Set $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^N$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$. For $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ we set $|\mathbf{x}| = |x_1| + \dots + |x_N|$. For $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{C}^N$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_+^N$ we set $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \dots z_N^{\alpha_N}$. For $t \in \{1, \dots, T\}$ we denote $\mathbf{e}_t = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^T$. Define $e \colon \mathbb{C} \to \mathbb{C}$ by $e(z) = \exp(2i\pi z)$. Given $P = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^N} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}} \in \mathbb{R}[X_1, \dots, X_N]$, we define $P^+ \in \mathbb{R}[X_1, \dots, X_N]$ by $P^+ = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^N} |a_{\boldsymbol{\alpha}}| \mathbf{X}^{\boldsymbol{\alpha}}$.

The notation $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ (uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$) means that there exists $A = A(\mathbf{y}) > 0$, that does not depend on $\mathbf{x}$ or λ, but could *a priori* depend on other parameters and, in particular, on $\mathbf{y}$, such that for all $\mathbf{x} \in X$ and all $\lambda \in \Lambda$, $|f(\lambda, \mathbf{y}, \mathbf{x})| \leqslant Ag(\mathbf{x})$. When there is no ambiguity, we will omit the word uniformly and the index $\mathbf{y}$. The notation $f \asymp g$ means that we have both $f \ll g$ and $g \ll f$.

Convention

In this work we will say that a series defined by a sum over $N \geqslant 1$ variables is convergent when it is absolutely convergent.

1. Statements of main results

Let us first recall a few definitions.

Definition 1.1. We say that $P \in \mathbb{R}[X_1, \dots, X_N]\backslash\{0\}$ is non-degenerate if $P(\mathbf{x}) \asymp P^+(\mathbf{x})$ ($\mathbf{x} \in J^N$).

Clearly the polynomials with positive coefficients are non-degenerate.

The following proposition characterizes the non-degenerate polynomials according to their growth performance on J^N. The proof is given in [Dec03].

Proposition 1.2. *Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfying $P(\mathbf{x}) > 0$ for all $\mathbf{x} \in J^N$. Then P is non-degenerate if and only if for all $\boldsymbol{\alpha} \in \mathbb{N}^N (\partial^{\boldsymbol{\alpha}} P/P)(\mathbf{x}) \ll \mathbf{x}^{-\boldsymbol{\alpha}}$ ($\mathbf{x} \in J^N$).*

Definition 1.3. We say that $P \in \mathbb{R}[X_1, \dots, X_N]$ is hypoelliptic if

$$\forall \mathbf{x} \in J^N,\ P(\mathbf{x}) > 0 \quad \text{and} \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^N \setminus \{\mathbf{0}\}, \quad \frac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} 0.$$

In [Ess97], Essouabri introduced a new class of polynomials as follows.

Definition 1.4. We say that $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfies H_0S if

$$\forall \mathbf{x} \in J^N,\ P(\mathbf{x}) > 0 \quad \text{and} \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^N,\ \frac{\partial^{\boldsymbol{\alpha}} P}{P}(\mathbf{x}) \ll 1 \quad (\mathbf{x} \in J^N).$$

It is clear that this class contains both non-degenerate and hypoelliptic polynomials on J^N. What is less clear is that this inclusion is strict. Essouabri gave the following example.

Example 1.5. Let $P_{ex} = (X - Y)^2 X + X \in \mathbb{R}[X, Y]$. Then P_{ex} satisfy H_0S but P is degenerate and is not hypoelliptic.

In the H_0S class, the extension of a twisted series is not always holomorphic.

Proposition 1.6. *We have that $Z(1, P_{ex}, -1, -1, \cdot)$ has a meromorphic extension to $\mathbb{C}$ with a single pole at $s = 1$, which is simple. The residue at $s = 1$ is equal to $\pi/\sinh(\pi)$.*

This will be proved in § 3.4.

Remark 1.7. It follows from the algebraic independence of π and e^{π} that $\pi/\sinh(\pi)$ is transcendental.

Thus, to show that a twisted series has a holomorphic extension to $\mathbb{C}$, we have to restrict to a subclass of H_0S. So we introduce a new class of polynomials.

Definition 1.8 (*HDF* hypothesis). Let $P \in \mathbb{R}[X_1, \dots, X_N]$. Then P is said to satisfy the weak decreasing hypothesis (denoted *HDF* in the rest of the article) if:

- for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$;
- there exists $\epsilon_0 > 0$ such that for $\boldsymbol{\alpha} \in \mathbb{N}^N$ and $n \in \{1, \dots, N\}$: $\alpha_n \geqslant 1 \Rightarrow (\partial^{\boldsymbol{\alpha}} P/P)(\mathbf{x}) \ll x_n^{-\epsilon_0}$ ($\mathbf{x} \in J^N$).

The proof of the first point of the following remark is easy and is given in [Dec03, p. 48]. Points (2), (3) and (4) are clear.

Remark 1.9.

(1) Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfy *HDF*. Let us denote $I = \{n \mid P \text{ depends effectively on } X_n\}$. Then

$$P(\mathbf{x}) \xrightarrow[\substack{\sum_{n \in I} x_n \to +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

(2) The class *HDF* is stable under product.

As a consequence, we have the following.

(3) If $P_1, \ldots, P_T$ satisfy *HDF*, then

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\mathbf{x}\in J^N]{\mathbf{x}\to+\infty} +\infty \iff \prod_{t=1}^{T} P_t(\mathbf{x}) \text{ depends effectively on all variables.}$$

(4) For $P_1, \ldots, P_T \in \mathbb{R}[X_1, \ldots, X_N]$ we have

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \text{ depends effectively on all variables}$$
$$\iff \text{for all } n \text{ there exists } t \text{ such that } P_t \text{ depends effectively on } X_n.$$

The condition on the right-hand side is very easy to verify.

It is clear from the preceding definitions that the *HDF* class is contained in H_0S and contains both hypoelliptic and non-degenerate polynomials. We are now going to give a simple method to construct polynomials satisfying *HDF* but that are degenerate and not hypoelliptic. So the *HDF* class is strictly larger that the union of the class of non-degenerate polynomials with the class of the hypoelliptic polynomials. The result is as follows.

LEMMA 1.10. *We assume that $P \in \mathbb{R}[X_1, \ldots, X_N]$ is non-degenerate and is not hypoelliptic. We assume that $Q \in \mathbb{R}[X_1, \ldots, X_N]$ is hypoelliptic and degenerate. Then PQ is degenerate and is not hypoelliptic.*

Furthermore, since the class *HDF* is stable under product, PQ satisfies *HDF*, so we have obtained what was required.

The preceding lemma is an obvious consequence of the following lemmas.

LEMMA 1.11. *Let P and $Q \in \mathbb{R}[X_1, \ldots, X_N]$. We assume that for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$ and that P is degenerate, but that Q is not. Then PQ is degenerate.*

The proof is in [Dec03].

LEMMA 1.12. *Let P and $Q \in \mathbb{R}[X_1, \ldots, X_N]$. We assume that P is hypoelliptic. We assume that Q satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \in J^N$ and is not hypoelliptic. Then PQ is not hypoelliptic.*

The proof is easy with the Leibniz formula.

We now come back to our series. Under the *HDF* hypothesis the twisted series Z extends holomorphically. More precisely we have the following.

THEOREM A. *Let $Q, P_1, \ldots, P_T \in \mathbb{R}[X_1, \ldots, X_N]$ and $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$.*
We assume that $P_1, \ldots, P_T$ satisfy HDF and that

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\mathbf{x}\in J^N]{|\mathbf{x}|\to+\infty} +\infty.$$

Then $Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; \cdot)$ extends to $\mathbb{C}^T$ as an entire function.

To study the values of $Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; \cdot)$ on $(-\mathbb{N})^T$, the key lemma is as follows.

EXCHANGE LEMMA. *Let $Q, P_1, \ldots, P_T, Q_1, \ldots, Q_{T'} \in \mathbb{R}[X_1, \ldots, X_N]$. We assume that:*

- *$P_1, \ldots, P_T, Q_1, \ldots, Q_{T'}$ satisfy HDF;*
- $\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\mathbf{x}\in J^N]{|\mathbf{x}|\to+\infty} +\infty$ *and* $\prod_{t=1}^{T'} Q_t(\mathbf{x}) \xrightarrow[\mathbf{x}\in J^N]{|\mathbf{x}|\to+\infty} +\infty.$

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Let $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$ *and* $\mathbf{k} = (k_1, \ldots, k_T) \in \mathbb{N}^T, \boldsymbol{\ell} = (\ell_1, \ldots, \ell_{T'}) \in \mathbb{N}^{T'}$*. Then*

$$Z\biggl(Q \prod_{t=1}^{T'} Q_t{}^{\ell_t}; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}\biggr) = Z\biggl(Q \prod_{t=1}^{T} P_t{}^{k_t}; Q_1, \ldots, Q_{T'}; \boldsymbol{\mu}; -\boldsymbol{\ell}\biggr).$$

Remark 1.13. (1) *Justification of the interest of this lemma.* Let us consider the case $T = T' = 1$ with $Q = 1$. Let us assume that P_1 is 'complicated' and that Q_1 is 'simple'. The Exchange Lemma gives $Z(Q_1^{\ell_1}; P_1; \boldsymbol{\mu}; -k_1) = Z(P_1^{k_1}; Q_1; \boldsymbol{\mu}; -\ell_1)$. In principle, the left-hand side should be difficult to evaluate, whereas the right-hand side should be easier to evaluate. The equation indicates that an *a priori* hard problem (evaluation of the left-hand side) is actually easier than one might think.

(2) *Justification of the study of multivariables series.* It is true that the Exchange Lemma is meaningful for series in $T = T' = 1$ variable. However, to prove the Exchange Lemma in the monovariable setting, we need to use series in $T + T' = 2$ variables. This justifies, if required, the use of multivariable series.

Remark 1.14. In the previous works, the existence of a holomorphic continuation and the calculus of the values were simultaneously worked out. Here it is absolutely not the case: we have two independent steps.

DEFINITION 1.15. For $\mu \in \mathbb{T}$, we set $\zeta_\mu(s) = Z(1; X; \mu; s) = \sum_{m \geqslant 1} (\mu^m / m^s)$.

To illustrate how to use the Exchange Lemma, we easily deduce a theorem giving the values of the general series Z at points $-\mathbf{k} \in (-\mathbb{N})^T$ in terms of the values at negative integers of a much simpler series ζ_μ. This result also extends those obtained by Cassou-Noguès [Cas82] and Chen and Eie [CE01].

THEOREM B. *Let* $Q, P_1, \ldots, P_T \in \mathbb{R}[X_1, \ldots, X_N]$*. We assume that:* $P_1, \ldots, P_T$ *satisfy HDF and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

Let $\mathbf{k} = (k_1, \ldots, k_T) \in \mathbb{N}^T$ *and write* $Q \prod_{t=1}^{T} P_t^{k_t} = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$*. Let* $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$*. Then* $Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \zeta_{\mu_n}(-\alpha_n)$.

An interesting corollary, arithmetic in nature, now follows as an immediate consequence of formulae for ζ_μ at negative integers (cf. Lemma 5.7) and of Theorem B.

COROLLARY 1.16. *Let* $\mathbb{K}$ *be a subfield of* $\mathbb{R}$*. Let* $Q, P_1, \ldots, P_T \in \mathbb{K}[X_1, \ldots, X_N]$*. We assume that* $P_1, \ldots, P_T$ *satisfy HDF and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in J^N}]{} +\infty.$$

For any $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$ *and* $\mathbf{k} \in \mathbb{N}^T$ *we have* $Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}) \in \mathbb{K}(\mu_1, \ldots, \mu_N)$.

The Exchange Lemma is a general principle of calculus of values at T-tuples of negative integers; we can also apply it to a class of integrals Y (see § 2 and § 4.3).

A suitable p-adic interpolation for the function $-\mathbf{k} \to Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k})$ is only possible provided that one restricts the series to lattice points $\mathbf{m}$ such that $p \nmid P_t(\mathbf{m})$ for each t. The second main result of the paper is the following. Its proof is based on Theorem B.

THEOREM C. *Let* p *be a prime number. We fix a field morphism from* $\mathbb{C}$ *into* $\mathbb{C}_p$ *(left implicit in the discussion and by means of which one calculates* $|x|_p$ *for any* $x \in \mathbb{C}$*). Let* $Q, P_1, \ldots, P_T \in \mathbb{Z}[X_1, \ldots, X_N]$ *and* $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$*. We assume that:*

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(i) $P_1, \dots, P_T$ *satisfy HDF, and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty \\ \mathbf{x}\in J^N}]{} +\infty;$$

(ii) *for all* $n \in \{1, \dots, N\}$, $|1-\mu_n|_p > p^{-1/p(p-1)}$.

We set

$$\tilde{Z}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \mathbf{s}) = \sum_{\substack{\mathbf{m}\in\mathbb{N}^{*N} \\ \forall t\in\{1,\dots,T\},\ p\nmid P_t(\mathbf{m})}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t}.$$

Let $\mathbf{r} \in \{0, \dots, p-2\}^T$. *Then there exists* $\tilde{Z}_p^{\mathbf{r}}(Q, P_1; \dots, P_T; \boldsymbol{\mu}; \cdot)\colon \mathbb{Z}_p{}^T \to \mathbb{C}_p$ *continuous such that for all* $\mathbf{k} \in \mathbb{N}^T$ *satisfying* $k_t \equiv r_t \mod (p-1)$ *for all* $t \in \{1, \dots, T\}$, *we have*

$$\tilde{Z}_p^{\mathbf{r}}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = \tilde{Z}(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -\mathbf{k}).$$

2. Analytic properties of certain functions Y defined by means of integrals

The proof of Theorem A, given in § 3, uses an integral representation of each twisted series $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \mathbf{s})$ as a finite sum of integrals $Y(\mathbf{s}) = Y(Q; P_1, \dots, P_T; f_1, \dots, f_N; \mathbf{s})$, defined in § 2.1. An important ingredient in the proof is therefore a precise description of the analytic continuation of each such integral Y in $\mathbf{s}$. The main result of this section is proved in § 2.3. This shows that if each P_t is in the class *HDF*, then each function Y can be extended from some open set in which each σ_t is sufficiently large to $\mathbb{C}^T$ as an entire function.

2.1 Precise definition of the functions Y

Definition 2.1. Let $Q, P_1, \dots, P_T \in \mathbb{C}[X_1, \dots, X_N]$ and $N_1 \in \{0, \dots, N\}$. We assume that for all $t \in \{1, \dots, T\}$ and all $\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}$, $P_t(\mathbf{x}) \notin \mathbb{R}_-$. Furthermore, we take $f\colon [-1,1]^{N_1} \to \mathbb{C}$ continuous and $f_{N_1+1}, \dots, f_N\colon [1,+\infty[\to \mathbb{C}$ continuous and bounded. For $\mathbf{s} \in \mathbb{C}^T$ we define

$$\begin{aligned} &Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \mathbf{s}) \\ &\quad = \int_{[-1,1]^{N_1}\times J^{N-N_1}} Q(\mathbf{x}) \biggl(\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t}\biggr) f(x_1, \dots, x_{N_1}) \biggl(\prod_{n=N_1+1}^{N} f_n(x_n)\biggr) d\mathbf{x}. \end{aligned}$$

Lemma 2.2. *Let* $Q, P_1, \dots, P_T \in \mathbb{C}[X_1, \dots, X_N]$ *and* $N_1 \in \{0, \dots, N\}$. *We assume the following.*

(a) *For all* $t \in \{1, \dots, T\}$ *we have:*
- *for all* $\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}$, $P_t(\mathbf{x}) \notin \mathbb{R}_-$;
- $|P_t(\mathbf{x})| \gg 1$ $(\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1})$.

(b) $\displaystyle\prod_{t=1}^{T} |P_t(\mathbf{x})| \xrightarrow[\substack{|\mathbf{x}|\to+\infty \\ \mathbf{x}\in[-1,1]^{N_1}\times J^{N-N_1}}]{} +\infty.$

Furthermore, we take $f\colon [-1,1]^{N_1} \to \mathbb{C}$ *continuous and* $f_{N_1+1}, \dots, f_N\colon [1,+\infty[\to \mathbb{C}$ *continuous and bounded. Then there exists* $\sigma_0 > 0$ *such that* $\mathbf{s} \mapsto Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \mathbf{s})$ *exists and is holomorphic on* $\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1, \dots, T\},\ \sigma_t > \sigma_0\}$.

Proof. (1) *Choice of an* ϵ. Thanks to the Tarski Saidenberg theorem there exists $\epsilon > 0$ such that

$$\prod_{t=1}^{T} |P_t(\mathbf{x})| \gg \biggl(\prod_{n=N_1+1}^{N} x_n\biggr)^{\epsilon} \quad (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}).$$

(2) *Proof of the existence of* σ_0. Let $\sigma_0 \in \mathbb{R}$, that will be fixed in the following. Let K be a compact of $\mathbb{C}^T$ included in $\{\mathbf{s} \in \mathbb{C}^T \mid \forall t \in \{1,\dots,T\},\ \sigma_t > \sigma_0\}$.

- Let $t \in \{1,\dots,T\}$. Then $|P_t(\mathbf{x})| \gg 1$ ($\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}$) so there exists $c>0$ such that for all $\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1} |P_t(\mathbf{x})| \geqslant c$. For all, $\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}, c^{-1}|P_t(\mathbf{x})| \geqslant 1$ so $\sigma_t > \sigma_0 \Rightarrow (c^{-1}|P_t(\mathbf{x})|)^{\sigma_t} \geqslant (c^{-1}|P_t(\mathbf{x})|)^{\sigma_0}$.
  Thus, we have $|P_t(\mathbf{x})|^{\sigma_t} \gg |P_t(\mathbf{x})|^{\sigma_0}$ ($\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}$, $\mathbf{s} \in K$). Then, $|P_t(\mathbf{x})^{s_t}| = |P_t(\mathbf{x})|^{\sigma_t} \exp[-\tau_t \arg P_t(\mathbf{x})]$ so
  $$|P_t(\mathbf{x})^{s_t}| \gg |P_t(\mathbf{x})|^{\sigma_t} \quad (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1},\ \mathbf{s} \in K).$$
  Thanks to what precedes, we deduce that $|P_t(\mathbf{x})^{-s_t}| \ll |P_t(\mathbf{x})|^{-\sigma_0}$.
- So we have
  $$\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} \ll \biggl(\prod_{t=1}^{T} |P_t(\mathbf{x})|\biggr)^{-\sigma_0} (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1},\ \mathbf{s} \in K).$$

From now on we assume that $\sigma_0 > 0$. Then
$$\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} \ll \biggl(\prod_{n=N_1+1}^{N} x_n\biggr)^{-\sigma_0\epsilon} \quad (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1},\ \mathbf{s} \in K).$$
We denote $q = \max\{\deg_{X_n} Q \mid N_1+1 \leqslant n \leqslant N\}$ (we can obviously assume that $Q \neq 0$). We obtain
$$Q(\mathbf{x})\biggl(\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t}\biggr) f(x_1,\dots,x_{N_1}) \prod_{n=N_1+1}^{N} f_n(x_n)$$
$$\ll \biggl(\prod_{n=N_1+1}^{N} x_n\biggr)^{q-\sigma_0\epsilon} \quad (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1},\ \mathbf{s} \in K).$$
We are led to make the following choice:
$$\sigma_0 = \frac{q+2}{\epsilon} > 0.$$
The theorem that guarantees the holomorphy of a function defined as an integral allows us to conclude. □

2.2 The $\mathcal{B}$ class

DEFINITION 2.3. For $r \in \mathbb{R}$ we define
$$\mathcal{B}(r) = \{f\colon [r,+\infty[\ \to \mathbb{C} \mid \exists (f_n)_{n\in\mathbb{N}},\ f_n\colon [r,+\infty[\ \to \mathbb{C},$$
$$C^\infty \text{ bounded satisfying } f_0 = f \text{ and } \forall n \in \mathbb{N},\ f'_{n+1} = f_n\}.$$

LEMMA 2.4. *Let $r \in \mathbb{R}$ and $f \in \mathcal{B}(r)$. Then we have the following.*

(1) *There is one and only one sequence $(f_n)_{n\in\mathbb{N}}$ where $f_n\colon [r,+\infty[\ \to \mathbb{C}$ such that:*
- *f_n is C^∞ bounded;*
- *$f_0 = f$;*
- *for all $n \in \mathbb{N}$, $f'_{n+1} = f_n$.*

(2) *For all $n \in \mathbb{N}$, $f_n \in \mathcal{B}(r)$.*

Proof. (1) Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ both satisfy the hypotheses of the lemma. Let us prove by induction on n that for all n, $f_n = g_n$. It is clear for $n=0$. If we have $f_n = g_n$, then $f''_{n+2} = f'_{n+1} = f_n = g_n = g'_{n+1} = g''_{n+2}$. As $f''_{n+2} = g''_{n+2}$, so $f_{n+2} - g_{n+2}$ is of the form $x \mapsto ax+b$. However, we are

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on $[r, +\infty[$ and $f_{n+2} - g_{n+2}$ is bounded, so it is constant, so its derivative is null, so $f_{n+1} - g_{n+1} = 0$, whence $f_{n+1} = g_{n+1}$.

(2) This is clear. □

The following lemma will not be used in the sequel, but it answers a natural question on the $\mathcal{B}(r)$ class. The proof (given in [Dec03]) is left as an exercise.

LEMMA 2.5. *Let $r \in \mathbb{R}$ and $f : [r, +\infty[\to \mathbb{C}$. Then*

$$f \in \mathcal{B}(r) \iff \forall n \in \mathbb{N},\ \exists g : [r, +\infty[\to \mathbb{C},$$

C^∞ bounded, such that $g^{(n)} = f$.

Let us give two examples of families of functions belonging to $\mathcal{B}(r)$: the first is the 'typical' example, the second will be used in the proof of Theorem A.

Example 2.6. Let $r \in \mathbb{R}$.

(1) Let $f : [r, +\infty[\to \mathbb{C}$, that is C^∞ and periodic with a null mean value. Then $f \in \mathcal{B}(r)$.

(2) Let $\alpha, \beta \in \mathbb{R}$ and $a \in \mathbb{C}$. We assume that $\beta \neq 0,\ \alpha/\beta \notin \mathbb{Z}$ and $|a| \neq 1$.

Then $f : [r, +\infty[\to \mathbb{C}$ defined by

$$f(x) = \frac{\exp(i\alpha x)}{1 - a\exp(i\beta x)}$$

belongs to $\mathcal{B}(r)$.

Proof. (1) The Fourier expansion of f gives the result.

(2a) If $|a| < 1$,

$$f(x) = \exp(i\alpha x)\sum_{k=0}^{+\infty} a^k \exp(ik\beta x) = \sum_{k=0}^{+\infty} a^k \exp(i(\alpha + k\beta)x).$$

So, for $n \in \mathbb{N}$, we define f_n by

$$f_n(x) = \sum_{k=0}^{+\infty} \frac{a^k}{(i(\alpha + k\beta))^n} \exp(i(\alpha + k\beta)x)$$

f_n is C^∞ and bounded, $f'_{n+1} = f_n,\ f_0 = f$; so $f \in \mathcal{B}(r)$.

(2b) If $|a| > 1$,

$$f(x) = \frac{a^{-1}\exp(-i\beta x)\exp(i\alpha x)}{a^{-1}\exp(-i\beta x) - 1} = -a^{-1}\frac{\exp(i(\alpha - \beta)x)}{1 - a^{-1}\exp(i(-\beta)x)}.$$

This reduces this case to the preceding case, so $f \in \mathcal{B}(r)$. □

2.3 Under suitable hypothesis the twisted integrals Y holomorphically extend to $\mathbb{C}^T$

The aim of this subsection is to prove the following theorem.

THEOREM 2.7. *Let $Q, P_1, \ldots, P_T \in \mathbb{C}[X_1, \ldots, X_N]$ and $N_1 \in \{0, \ldots, N\}$. We assume the following.*

(a) *For all $t \in \{1, \ldots, T\}$ we have:*
- *for all $\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1},\ P_t(\mathbf{x}) \notin \mathbb{R}_-$;*
- $|P_t(\mathbf{x})| \gg 1$ $(\mathbf{x} \in [-1, 1]^{N_1} \times J^{N-N_1})$.

(b) $$\prod_{t=1}^{T} |P_t(\mathbf{x})| \xrightarrow[\substack{|\mathbf{x}| \to +\infty \\ \mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}}]{} +\infty.$$

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(c) *There exists* $\epsilon_0 > 0$ *such that for* $\boldsymbol{\alpha} \in \{0\}^{N_1} \times \mathbb{N}^{N-N_1}$ *and* $n \in \{N_1+1, \dots, N\}$ *we have*

$$\alpha_n \geqslant 1 \Rightarrow \forall t \in \{1, \dots, T\}, \ \frac{\partial^{\boldsymbol{\alpha}} P_t}{P_t}(\mathbf{x}) \ll x_n^{-\epsilon_0} \quad (\mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}).$$

In addition, we consider $f\colon [-1,1]^{N_1} \to \mathbb{C}$ *continuous and* $f_{N_1+1}, \dots, f_N \in \mathcal{B}(1)$. *Then the following property is true for all* $0 \leqslant N_1 \leqslant N$:

$\mathcal{P}(N_1, N) \overset{\text{def}}{=} Y(Q; P_1, \dots, P_T; f_{N_1+1}, \dots, f_N; f; \cdot)$ *has an analytic extension to* $\mathbb{C}^T$ *as an entire function.*

Remark 2.8. When $N_1 = 0$, the hypothesis (c) is nothing but *HDF*.

Proof. (1) *Proof of the assertion* $\mathcal{P}(0, N)$. Let us agree on the following.

- We will say that a function Y is an entire combination of the functions $Y_1, \dots, Y_k$ if there exists entire functions $\lambda, \lambda_1, \dots, \lambda_k\colon \mathbb{C}^T \to \mathbb{C}$ such that $Y = \lambda + \sum_{i=1}^{k} \lambda_i Y_i$.
- The polynomials $P_1, \dots, P_T$ are fixed for the whole proof, so we will write $Y(Q; f_1, \dots, f_N; \cdot)$ for $Y(Q; P_1, \dots, P_T; f_1, \dots, f_N; \cdot)$.
- Here $\mathcal{B}$ means $\mathcal{B}(1)$.

The proof is by induction on N.

Since $\mathcal{P}(0,0)$ is obvious, it suffices to show that the implication $\mathcal{P}(0, N-1) \Rightarrow \mathcal{P}(0, N)$ is true. The proof of this assertion will then easily be seen to apply for any other value for N_1 (the details are left to the reader). The argument is made up of ten steps.

Step 1. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then $Y(Q, f_1, \dots, f_N, \mathbf{s})$ is an entire combination of $Y(\partial Q/\partial x_1, f_1, \dots, f_N, \mathbf{s})$ and of functions of the type $Y(Q(\partial P_t/\partial x_1); g_1, \dots, g_N; \mathbf{s} + \mathbf{e}_t)$ where $t \in \{1, \dots, T\}$ and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 1. We have $f_1 \in \mathcal{B}$ so, thanks to Lemma 2.4, a sequence of functions belonging to $\mathcal{B}$ is associated to f_1. We denote the first term of this sequence by f_1^1. Then

$$\begin{aligned} Y(Q; f_1, \dots; f_N; \mathbf{s}) &= \int_{J^N} Q(\mathbf{x}) \prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} \prod_{n=1}^{N} f_n(x_n)\, d\mathbf{x} \\ &= \int_{J^{N-1}} \left\{ \int_1^{+\infty} Q(\mathbf{x}) \left(\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} \right) f_1(x_1)\, dx_1 \right\} \prod_{n=2}^{N} f_n(x_n) \prod_{n=2}^{N} dx_n. \end{aligned}$$

By means of an integration by parts with respect to x_1, the expression between braces is the difference between

$$\left[Q(\mathbf{x}) \left(\prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} \right) f_1^1(x_1) \right]_{x_1=1}^{x_1=+\infty}$$

and

$$\int_1^{+\infty} \left(\frac{\partial Q}{\partial x_1}(\mathbf{x}) \prod_{t=1}^{T} P_t(\mathbf{x})^{-s_t} + Q(\mathbf{x}) \sum_{t=1}^{T} (-s_t) \frac{\partial P_t}{\partial x_1}(\mathbf{x}) P_t(\mathbf{x})^{-(s_t+1)} \prod_{r \neq t} P_r(\mathbf{x})^{-s_r} \right) f_1(x_1)\, dx_1.$$

We deduce from this that

$$\begin{aligned} Y(Q; f_1, \dots, f_N; \mathbf{s}) = &- \int_{J^{N-1}} Q(1, x_2, \dots, x_N) \left(\prod_{t=1}^{T} P_t(1, x_2, \dots, x_N)^{-s_t} \right) f_1^1(1) \prod_{n=2}^{N} f_n(x_n) \prod_{n=2}^{N} dx_n \\ &- Y\left(\frac{\partial Q}{\partial x_1}; f_1, \dots, f_N; \mathbf{s} \right) + \sum_{t=1}^{T} s_t Y\left(Q \frac{\partial P_t}{\partial x_1}; f_1^1, f_2, \dots, f_N; \mathbf{s} + \mathbf{e}_t \right). \end{aligned}$$

The polynomials of $N-1$ variables $P_1(1,X_2,\ldots,X_N),\ldots,P_T(1,X_2,\ldots,X_N)$ satisfy the hypothesis in $\mathcal{P}(0,N-1)$. Thus, the induction hypothesis implies that the term defined by an integral over J^{N-1} admits a holomorphic continuation to $\mathbb{C}^T$. This fact consequently implies the assertion of Step 1.

Step 2. Let $Q \in \mathbb{C}[X_1,\ldots,X_N]$ and $f_1,\ldots,f_N \in \mathcal{B}$. Then the following property is true for all $d \geqslant 1$:

$P_2(d,N) \overset{\text{def}}{=} Y(Q,f_1,\ldots,f_N,\mathbf{s})$ is an entire combination of $Y(\partial^d Q/\partial x_1^d; f_1,\ldots,f_N;\mathbf{s})$ and of functions of the type

$$Y((\partial^i Q/\partial x_1^i)(\partial P_t/\partial x_1); g_1,\ldots,g_N;\mathbf{s}+\mathbf{e}_t)$$

where $i \in \mathbb{N}, t \in \{1,\ldots,T\}$, and $g_1,\ldots,g_N \in \mathcal{B}$.

Proof of Step 2. The proof is by induction on d. The assertion $P_2(1,N)$ is implied by Step 1. The implication $P_2(d,N) \Rightarrow P_2(d+1,N)$ is proved by combining Step 1, applied to the polynomial $\partial^d Q/\partial x_1^d$, with the result that is assumed to be true in the property $P_2(d,N)$.

Step 3. Let $Q \in \mathbb{C}[X_1,\ldots,X_N]$ and $f_1,\ldots,f_N \in \mathcal{B}$. Then for all $n \in \{1,\ldots,N\}$, $Y(Q;f_1,\ldots,f_N;\mathbf{s})$ is an entire combination of functions of the type $Y((\partial^i Q/\partial x_n^i)(\partial P_t/\partial x_n); g_1,\ldots,g_N;\mathbf{s}+\mathbf{e}_t)$ where $i \in \mathbb{N}, t \in \{1,\ldots,T\}$ and $g_1,\ldots,g_N \in \mathcal{B}$.

Proof of Step 3. Of course, it is enough to deal with the case $n=1$. In order to deduce the result for $n=1$, it is sufficient to apply Step 2 with $d = \deg_{X_1} Q + 1$.

Step 4. For $n \in \{1,\ldots,N\}, \mathbf{u} = (u_1,\ldots,u_T) \in \mathbb{N}^T$ and $Q \in \mathbb{C}[X_1,\ldots,X_N]$, we define $\mathcal{E}_{\mathbf{u}}^n(Q)$ to be the subspace of $\mathbb{C}[X_1,\ldots,X_N]$ generated by the polynomials of the form

$$\partial^{\boldsymbol{\beta}} Q \prod_{k=1}^{n} \frac{\partial^{|\boldsymbol{\alpha}_k|+1} P_{t_k}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_k} \partial x_k}$$

where $\boldsymbol{\beta} \in \mathbb{N}^N, \boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n \in \mathbb{N}^N$ and $t_1,\ldots,t_n \in \{1,\ldots,T\}$ verify that for all $t \in \{1,\ldots,T\}$

$$u_t = \operatorname{card}\{k \in \{1,\ldots,n\} \mid t_k = t\}.$$

It is clear that $n \neq |\mathbf{u}| \Rightarrow \mathcal{E}_{\mathbf{u}}^n(Q) = \{0\}$.

The following two observations are satisfied:

- $\mathcal{E}_{\mathbf{u}}^n(Q)$ is stable under derivation;
- for any $n \in \{1,\ldots,N-1\},\ t \in \{1,\ldots,T\}$ and $Q \in \mathbb{C}[X_1,\ldots,X_N]$, we have

$$\frac{\partial P_t}{\partial x_{n+1}} \mathcal{E}_{\mathbf{u}}^n(Q) \subset \mathcal{E}_{\mathbf{u}+\mathbf{e}_t}^{n+1}(Q).$$

Step 5. Let $n \in \{1,\ldots,N\}$, $Q \in \mathbb{C}[X_1,\ldots,X_N]$ and $f_1,\ldots,f_N \in \mathcal{B}$. Define the property $P_5(n,N)$: $Y(Q;f_1,\ldots,f_N;\mathbf{s})$ is an entire combination of functions of the type $Y(R;g_1,\ldots,g_N;\mathbf{s}+\mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T, R \in \mathcal{E}_{\mathbf{u}}^n(Q)$ and $g_1,\ldots,g_N \in \mathcal{B}$.

Then $P_5(n,N)$ is true for all $n \in \{1,\ldots,N\}$.

Proof of Step 5. The proof is by induction on n. Step 3 shows that the property $P_5(1,N)$ is true. Let us assume that $P_5(n,N)$ is true for any $n \in \{1,\ldots,N-1\}$. Thus, $Y(Q;f_1,\ldots,f_N;\mathbf{s})$ is an entire combination of functions of the type: $Y(R;g_1,\ldots,g_N;\mathbf{s}+\mathbf{u})$ where $\mathbf{u} \in \mathbb{N}^T, R \in \mathcal{E}_{\mathbf{u}}^n(Q)$ and $g_1,\ldots,g_N \in \mathcal{B}$. By Step 3, $Y(R,g_1,\ldots,g_N,\mathbf{s}+\mathbf{u})$ is an entire combination of functions of the type

$$Y\left(\frac{\partial^i R}{\partial x_{n+1}^i} \frac{\partial P_t}{\partial x_{n+1}}; h_1,\ldots,h_N;\mathbf{s}+\mathbf{u}+\mathbf{e}_t\right)$$

where $i \in \mathbb{N}, t \in \{1,\ldots,T\}$ and $h_1,\ldots,h_N \in \mathcal{B}$.

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Thanks to the two observations made in Step 4,

$$\frac{\partial^i R}{\partial x_{n+1}^i}\frac{\partial P_t}{\partial x_{n+1}} \in \mathcal{E}^{n+1}_{\mathbf{u}+\mathbf{e}_t}(Q).$$

This now shows that $P_5(n+1,N)$ is also true.

Step 6. For $\mathbf{u}\in\mathbb{N}^T$ and $Q\in\mathbb{C}[X_1,\dots,X_N]$, we define $\mathcal{E}_{\mathbf{u}}(Q)$ to denote the subspace $\mathbb{C}[X_1,\dots,X_N]$ generated by all polynomials of the form $\partial^{\boldsymbol{\beta}}Q\prod_{t=1}^{T}\prod_{k\in F_t}\partial^{f_t(k)}P_t$, where:

- $\boldsymbol{\beta}\in\mathbb{N}^N$;
- the F_t are finite and pairwise disjoint subsets of $\mathbb{N}$, satisfying $|F_t|=u_t$;
- for all $t\in\{1,\dots,T\}$, f_t is a function from F_t to $\mathbb{N}^N$;
- we can associate to the f_t finite and pairwise disjoint subsets of $\mathbb{N}$, $D_1,\dots,D_N$ such that
  - $|D_1|=\cdots=|D_N|$,
  - $\bigsqcup_{n=1}^{N}D_n=\bigsqcup_{t=1}^{T}F_t$,
  - $t\in\{1,\dots,T\},\ n\in\{1,\dots,N\}$ and $k\in D_n\cap F_t\Rightarrow f_t(k)\in\mathbb{N}^{n-1}\times\mathbb{N}^*\times\mathbb{N}^{N-n}$.

We note that $\mathcal{E}_{\mathbf{u}}(Q)$ is stable under derivation.

Example 2.9 (The case $T=4$ and $N=3$). Let us take $\mathbf{u}=(1,3,2,3)$ and $F_1=\{1\}$, $F_2=\{2,3,4\}$ $F_3=\{5,6\}$ and $F_4=\{7,8,9\}$. Let us take f_1,f_2,f_3 and f_4 defined as follows:

- $f_1\colon F_1\to\mathbb{N}^3$ is defined by $f_1(1)=(1,0,0)$;
- $f_2\colon F_2\to\mathbb{N}^3$ is defined by $f_2(2)=(1,2,3),\ f_2(3)=(0,2,1),\ f_2(4)=(2,0,1)$;
- $f_3\colon F_3\to\mathbb{N}^3$ is defined by $f_3(5)=(2,1,0),\ f_3(6)=(0,3,0)$;
- $f_4\colon F_4\to\mathbb{N}^3$ is defined by: $f_4(7)=(0,1,2),\ f_4(8)=(4,1,2),\ f_4(9)=(1,0,2)$.

Then it is easy to check that $D_1=\{1,2,5\}$, $D_2=\{3,6,7\}$, $D_3=\{4,8,9\}$ satisfy the conditions we require.

Step 7. Let $\mathbf{u},\mathbf{v}\in\mathbb{N}^T$ and $R,S\in\mathbb{C}[X_1,\dots,X_N]$. Then $R\in\mathcal{E}_{\mathbf{u}}(Q)$ and $S\in\mathcal{E}_{\mathbf{v}}(R)$ implies $S\in\mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

Proof of Step 7. Let S be an entire combination of terms of the form $\partial^{\boldsymbol{\beta}}R\prod_{t=1}^{T}\prod_{k\in F'_t}\partial^{f'_t(k)}P_t$, where:

- $\boldsymbol{\beta}\in\mathbb{N}^N, |F'_t|=v_t, f'_t\colon F'_t\to\mathbb{N}^N$ and $D'_1,\dots,D'_N$ are as in Step 6;
- $R\in\mathcal{E}_{\mathbf{u}}(Q)$, so $\partial^{\boldsymbol{\beta}}R\in\mathcal{E}_{\mathbf{u}}(Q)$, so $\partial^{\boldsymbol{\beta}}R$ is a linear combination of terms of the form

  $$\partial^{\boldsymbol{\gamma}}Q\prod_{t=1}^{T}\prod_{k\in F_t}\partial^{f_t(k)}P_t,$$

  where $\boldsymbol{\gamma}\in\mathbb{N}^N, |F_t|=u_t, f_t\colon F_t\to\mathbb{N}^N$ and $D_1,\dots,D_N$ are as in Step 6.

We can assume that for all $t_1,t_2,\ F_{t_1}\cap F'_{t_2}=\emptyset$. This implies that for all n,t $D_n\cap F'_t=F_t\cap D'_n=\emptyset$ and for all $n,n',\ D_n\cap D'_{n'}=\emptyset$. So, as to conclude, it is enough for us to prove that

$$U\stackrel{\text{def}}{=}\partial^{\boldsymbol{\gamma}}Q\biggl(\prod_{t=1}^{T}\prod_{k\in F_t}\partial^{f_t(k)}P_t\biggr)\biggl(\prod_{t=1}^{T}\prod_{k\in F'_t}\partial^{f'_t(k)}P_t\biggr)$$

is in $\mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

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For $t \in \{1, \dots, T\}$, we define $g_t \colon F_t \sqcup F'_t \to \mathbb{N}^N$ by

$$g_t(k) = \begin{cases} f_t(k) & \text{if } k \in F_t, \\ f'_t(k) & \text{if } k \in F'_t. \end{cases}$$

Then

$$U = \partial^{\gamma} Q \prod_{t=1}^{T} \prod_{k \in F_t \sqcup F'_t} \partial^{g_t(k)} P_t.$$

Thanks to this expression we now show that $U \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$. The following points justify this assertion:

- the $F_t \sqcup F'_t$ are pairwise disjoint and for all $t \in \{1, \dots, T\}$, $|F_t \sqcup F'_t| = u_t + v_t$;
- the $D_n \sqcup D'_n$ are pairwise disjoint, $\bigsqcup_{t=1}^{T}(F_t \sqcup F'_t) = \bigsqcup_{n=1}^{N}(D_n \sqcup D'_n)$, and$|D_1 \sqcup D'_1| = \cdots = |D_N \sqcup D'_N|$;
- if $k \in (D_n \sqcup D'_n) \cap (F_t \sqcup F'_t) = (D_n \cap F_t) \sqcup (D'_n \cap F'_t)$, then either,
  - $k \in D_n \cap F_t$ and then $g_t(k) = f_t(k) \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$, or
  - $k \in D'_n \cap F'_t$ and then $g_t(k) = f'_t(k) \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$.

So, we actually obtain the conclusion that $U \in \mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$.

Step 8. We have $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $\mathbf{u} \in \mathbb{N}^T \Rightarrow \mathcal{E}^N_{\mathbf{u}}(Q) \subset \mathcal{E}_{\mathbf{u}}(Q)$.

Proof of Step 8. We set

$$S = \partial^{\boldsymbol{\beta}} Q \prod_{k=1}^{N} \frac{\partial^{|\boldsymbol{\alpha}_k|+1} P_{t_k}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_k} \partial x_k},$$

where $\boldsymbol{\beta} \in \mathbb{N}^N, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_N \in \mathbb{N}^N$ and $t_1, \dots, t_N \in \{1, \dots, T\}$ satisfy: $u_t = \operatorname{card}\{k \in \{1, \dots, N\} \mid t_k = t\}$ for all $t \in \{1, \dots, T\}$. So, as to conclude, it is enough to show that $S \in \mathcal{E}_{\mathbf{u}}(Q)$.

For $t \in \{1, \dots, T\}$, we set $F_t = \{k \in \{1, \dots, N\} \mid t_k = t\}$, so that $|F_t| = u_t$. We see that the F_t are pairwise disjoint, and that $\bigsqcup_{t=1}^{T} F_t = \{1, \dots, N\}$. We define $f_t \colon F_t \to \mathbb{N}^N$ by $f_t(k) = \boldsymbol{\alpha}_k + \mathbf{e}_k$. We set $D_n = \{n\}$. Then we see that:

- $|D_1| = \cdots = |D_N|$;
- $\displaystyle\bigsqcup_{n=1}^{N} D_n = \{1, \dots, N\}$;
- if $k \in D_n \cap F_t$, then $k = n$, which implies $f_t(k) = \boldsymbol{\alpha}_n + \mathbf{e}_n \in \mathbb{N}^{n-1} \times \mathbb{N}^* \times \mathbb{N}^{N-n}$.

Thus,

$$S = \partial^{\boldsymbol{\beta}} Q \prod_{t=1}^{T} \prod_{k \in F_t} \frac{\partial^{|\boldsymbol{\alpha}_k|+1} P_{t_k}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_k} \partial x_k} = \partial^{\boldsymbol{\beta}} Q \prod_{t=1}^{T} \prod_{k \in F_t} \partial^{f_t(k)} P_t.$$

This implies $S \in \mathcal{E}_{\mathbf{u}}(Q)$.

Step 9. Let $Q \in \mathbb{C}[X_1, \dots, X_N]$ and $f_1, \dots, f_N \in \mathcal{B}$. Then the following property is true for all $m \geqslant 1$:

$P_9(m, N) \overset{\text{def}}{=} Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R; g_1, \dots, g_N; \mathbf{s} + \mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T, |\mathbf{u}| = mN, R \in \mathcal{E}_{\mathbf{u}}(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$.

Proof of Step 9. The proof is by induction on $m \geqslant 1$.

- $P_9(1, N)$ *is true.* Thanks to Step 5, $Y(Q; f_1, \dots, f_N; \mathbf{s})$ is an entire combination of functions of the type $Y(R, g_1, \dots, g_N, \mathbf{s} + \mathbf{u})$, where $\mathbf{u} \in \mathbb{N}^T, R \in \mathcal{E}^N_{\mathbf{u}}(Q)$ and $g_1, \dots, g_N \in \mathcal{B}$. We can assume $|\mathbf{u}| = N$ since $|\mathbf{u}| \neq N$ would imply $\mathcal{E}^N_{\mathbf{u}}(Q) = \{0\}$. Thus, Step 8 gives the result.

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- $P_9(m,N) \Rightarrow P_9(m+1,N)$ *is true*. Let us assume that $Y(Q;f_1,\dots,f_N;\mathbf{s})$ is an entire combination of functions of the type $Y(R;g_1,\dots,g_N;\mathbf{s}+\mathbf{u})$ where $\mathbf{u}\in\mathbb{N}^T, |\mathbf{u}|=mN, R\in\mathcal{E}_{\mathbf{u}}(Q)$ and $g_1,\dots,g_N\in\mathcal{B}$. The application of the argument with $m=1$ shows that $Y(R;g_1,\dots,g_N;\mathbf{s}+\mathbf{u})$ is an entire combination of functions of the type $Y(S;h_1,\dots,h_N;\mathbf{s}+\mathbf{u}+\mathbf{v})$ where $\mathbf{v}\in\mathbb{N}^T, |\mathbf{v}|=N, S\in\mathcal{E}_{\mathbf{v}}(R)$ and $h_1,\dots,h_N\in\mathcal{B}$. Step 7 then implies that $S\in\mathcal{E}_{\mathbf{u}+\mathbf{v}}(Q)$. Since $|\mathbf{u}+\mathbf{v}|=(m+1)N$, this shows that $P_9(m+1,N)$ is true.

Step 10 (Conclusion of the proof). We fix $Q\in\mathbb{C}[X_1,\dots,X_N]\setminus\{0\}$ and $f_1,\dots,f_N\in\mathcal{B}$ until the end. Let $m\geqslant 1$. Thanks to Step 9, $Y(Q;f_1,\dots,f_N;\mathbf{s})$ is an entire combination of functions of the type $Y(R;g_1,\dots,g_N;\mathbf{s}+\mathbf{u})$ where $\mathbf{u}\in\mathbb{N}^T, |\mathbf{u}|=mN, R\in\mathcal{E}_{\mathbf{u}}(Q)$ and $g_1,\dots,g_N\in\mathcal{B}$.

Since $R\in\mathcal{E}_{\mathbf{u}}(Q)$, it follows that R is equal to a linear combination of polynomials of the type $\partial^{\boldsymbol{\beta}}Q\prod_{t=1}^T\prod_{k\in F_t}\partial^{f_t(k)}P_t$ with $\boldsymbol{\beta}\in\mathbb{N}^N,\ |F_t|=u_t,\ f_t\colon F_t\to\mathbb{N}^N$ and $D_1,\dots,D_N$ as in Step 6.

We have $|D_n|=m$ for all $n\in\{1,\dots,N\}$. It then follows that

$$\begin{aligned}\prod_{t=1}^T\prod_{k\in F_t}\partial^{f_t(k)}P_t(\mathbf{x}) &= \prod_{t=1}^T\prod_{n=1}^N\prod_{k\in F_t\cap D_n}\partial^{f_t(k)}P_t(\mathbf{x})\\ &\ll \prod_{t=1}^T\prod_{n=1}^N\prod_{k\in F_t\cap D_n}x_n^{-\epsilon_0}P_t(\mathbf{x}) \quad (\mathbf{x}\in J^N)\\ &\ll \prod_{t=1}^T\prod_{n=1}^N(x_n^{-\epsilon_0}P_t(\mathbf{x}))^{|F_t\cap D_n|} \quad (\mathbf{x}\in J^N)\\ &\ll \prod_{n=1}^N\prod_{t=1}^T x_n^{-\epsilon_0|F_t\cap D_n|}\prod_{t=1}^T\prod_{n=1}^N P_t(\mathbf{x})^{|F_t\cap D_n|} \quad (\mathbf{x}\in J^N)\\ &\ll \prod_{n=1}^N x_n^{-\epsilon_0|D_n|}\prod_{t=1}^T P_t(\mathbf{x})^{|F_t|} \quad (\mathbf{x}\in J^N)\\ &\ll \prod_{n=1}^N x_n^{-\epsilon_0 m}\prod_{t=1}^T P_t(\mathbf{x})^{u_t} \quad (\mathbf{x}\in J^N).\end{aligned}$$

We set $q=\max\{\deg_{X_n}Q\mid 1\leqslant n\leqslant N\}$. We also set $p=\max\{\deg_{X_n}P_t\mid 1\leqslant n\leqslant N,\ 1\leqslant t\leqslant T\}$. We introduce a parameter $a>0$ whose value will be determined in the following.

Let K be a compact subset of $\mathbb{C}^T$ included in $\{\mathbf{s}\in\mathbb{C}^T\mid\forall t\in\{1,\dots,T\},\ \sigma_t>-a\}$.

- Let $t\in\{1,\dots,T\}$. As in the proof of the existence of σ_0 in the proof of the Lemma 2.2,

  $$|P_t(\mathbf{x})^{-s_t}|\ll|P_t(\mathbf{x})|^a \quad (\mathbf{x}\in J^N,\ \mathbf{s}\in K).$$

  Since $a>0$, it follows that

  $$P_t(\mathbf{x})^a\ll\biggl(\prod_{n=1}^N x_n\biggr)^{pa} \quad (\mathbf{x}\in J^N).$$

- From the previous inequalities we deduce that

  $$P_t(\mathbf{x})^{-s_t}\ll\biggl(\prod_{n=1}^N x_n\biggr)^{pa} \quad (\mathbf{x}\in J^N,\ \mathbf{s}\in K).$$

We set

$$S=\partial^{\boldsymbol{\beta}}Q\prod_{t=1}^T\prod_{k\in F_t}\partial^{f_t(k)}P_t.$$

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By combining the preceding estimates, we obtain

$$\begin{aligned}S(\mathbf{x})\prod_{t=1}^{T}P_t(\mathbf{x})^{-(s_t+u_t)} &\ll \partial^{\boldsymbol{\beta}}Q(\mathbf{x})\prod_{n=1}^{N}x_n^{-\epsilon_0 m}\prod_{t=1}^{T}P_t(\mathbf{x})^{u_t}\prod_{t=1}^{T}P_t(\mathbf{x})^{-(s_t+u_t)} \quad (\mathbf{x}\in J^N,\ \mathbf{s}\in K)\\ &\ll \biggl(\prod_{n=1}^{N}x_n\biggr)^{q}\biggl(\prod_{n=1}^{N}x_n\biggr)^{-\epsilon_0 m}\biggl(\prod_{n=1}^{N}x_n\biggr)^{Tpa} \quad (\mathbf{x}\in J^N,\ \mathbf{s}\in K)\\ &\ll \biggl(\prod_{n=1}^{N}x_n\biggr)^{q+Tpa-\epsilon_0 m} \quad (\mathbf{x}\in J^N,\ \mathbf{s}\in K).\end{aligned}$$

From now on we choose m so that $m>(q+2)/\epsilon_0$. We then choose a so that $a=(\epsilon_0 m-(q+2))/Tp$ (clearly $a>0$). The above estimates then show that $Y(S;g_1,\dots,g_N;\mathbf{s}+\mathbf{u})$ is holomorphic on

$$\{\mathbf{s}\in\mathbb{C}^T \mid \forall t\in\{1,\dots,T\},\ \sigma_t>-a\}.$$

Since R is a linear combination of S as above, it then follows that $Y(Q,f_1,\dots,f_N,\cdot)$ can be extended analytically to

$$\biggl\{\mathbf{s}\in\mathbb{C}^T \mid \forall t\in\{1,\dots,T\},\ \sigma_t>\frac{q+2-\epsilon_0 m}{Tp}\biggr\}.$$

Since this is true for all $m>(q+2)/\epsilon_0$, one concludes that $Y(Q;f_1,\dots,f_N;\cdot)$ can be extended to $\mathbb{C}^T$ as an analytic function.

This completes the proof that $\mathcal{P}(0,N-1)\Rightarrow\mathcal{P}(0,N)$ is true. Thus, $\mathcal{P}(0,N)$ is true for all N. The details needed to verify the similar argument when $N_1\geqslant 1$ are left to the reader. □

2.4 Proof that $Y(1,P_{ex},x\mapsto e^{ix},y\mapsto e^{-iy},\cdot)$ has a pole

As the following example shows, the H_0S hypothesis is not enough to guarantee the holomorphy of the continuation of the twisted Y.

Example 2.10. We define $f_1\colon J\to\mathbb{C}$ by $f_1(x)=e^{ix}$ and $f_2\colon J\to\mathbb{C}$ by $f_2(y)=e^{-iy}$; f_1 and f_2 belong to $\mathcal{B}(1)$. Then $Y(1;P;f_1,f_2;\cdot)$ has a meromorphic extension to $\mathbb{C}$ with a single pole at $s=1$ which is simple. The residue at $s=1$ is equal to π/e.

Proof. By definition,

$$Y(1;P;f_1,f_2;s)=\int_{J^2}P(x,y)^{-s}e^{i(x-y)}\,dx\,dy.$$

We set

$$Y_1(s)=\int_{\{(x,y)\,|\,1<x<y\}}P(x,y)^{-s}e^{i(x-y)}\,dx\,dy.$$

Let $g_1\colon]1,+\infty[\,\times\,\mathbb{R}_+^*\to\{(x,y)\mid 1<x<y\}$ be defined by $g_1(u,v)=(u,u+v)$; g_1 is a diffeomorphism with Jacobian equal to 1. Thanks to g_1, we see that

$$Y_1(s)=\int_{]1,+\infty[\,\times\,\mathbb{R}_+^*}[(u-(u+v))^2u+u]^{-s}e^{i(u-(u+v))}\,du\,dv.$$

So

$$Y_1(s)=\int_1^{+\infty}u^{-s}\,du\int_0^{+\infty}(v^2+1)^{-s}e^{-iv}\,dv=\frac{1}{s-1}\int_0^{+\infty}(v^2+1)^{-s}e^{-iv}\,dv.$$

We now set

$$Y_2(s)=\int_{\{(x,y)|1<y<x\}}P(x,y)^{-s}e^{i(x-y)}\,dx\,dy.$$

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Let $g_2\colon]1,+\infty[\ \times\ \mathbb{R}_+^* \to \{(x,y)\mid 1<y<x\}$ be defined by $g_2(u,v)=(u+v,u)$; g_2 is also a diffeomorphism with Jacobian equal to -1. Thanks to g_2, we see that

$$Y_2(s)=\int_{]1,+\infty[\ \times\ \mathbb{R}_+^*}[(u+v-u)^2(u+v)+(u+v)]^{-s}e^{i(u+v-u)}\,du\,dv.$$

Thus,

$$\begin{aligned}Y_2(s)&=\int_{]1,+\infty[\ \times\ \mathbb{R}_+^*}(v^2+1)^{-s}(u+v)^{-s}e^{iv}\,du\,dv=\int_0^{+\infty}(v^2+1)^{-s}e^{iv}\biggl\{\int_1^{+\infty}(u+v)^{-s}\,du\biggr\}\,dv\\&=\int_0^{+\infty}(v^2+1)^{-s}e^{iv}\frac{(1+v)^{-s+1}}{s-1}\,dv=\frac{1}{s-1}\int_0^{+\infty}(v^2+1)^{-s}(v+1)^{-s+1}e^{iv}\,dv.\end{aligned}$$

Let us now set

$$Y(s)=\int_0^{+\infty}(v^2+1)^{-s}e^{-iv}dv+\int_0^{+\infty}(v^2+1)^{-s}(v+1)^{-s+1}e^{iv}\,dv.$$

Thanks to Theorem 2.7, Y has a holomorphic continuation to $\mathbb{C}$. Since $Y(1;P,f_1,f_2;s)=(s-1)^{-1}Y(s)$, we now evaluate $Y(1)$ as follows:

$$Y(1)=\int_0^{+\infty}(v^2+1)^{-1}e^{-iv}\,dv+\int_0^{+\infty}(v^2+1)^{-1}e^{iv}\,dv=\int_{-\infty}^{+\infty}(v^2+1)^{-1}e^{iv}\,dv.$$

Showing that this integral is equal to π/e is a classical application of the residue theorem, and so we are through. □

3. Analytic properties of the series Z

The main result of this section is Theorem A, proved in § 3.3. The proof is based on a simple integral representation for the sum of values of any holomorphic function at integral points, proved in § 3.2, and on the main result of § 2.

3.1 Holomorphy of Z on this set of convergence

In this subsection we establish some easy properties of the set of convergence of the series defining Z. The proofs are easy and can be found in [Dec03].

Definition 3.1. Let $Q,P_1,\dots,P_T\in\mathbb{R}[X_1,\dots,X_N]$ satisfy $P_t(\mathbf{x})>0$ for all $t\in\{1,\dots,T\}$ and all $\mathbf{x}\in J^N$. We set

$$\mathcal{C}(Q,P_1,\dots,P_T)=\{(\sigma_1,\dots,\sigma_T)\in\mathbb{R}^T\mid Z(Q,P_1,\dots,P_T,\mathbf{1},\sigma_1,\dots,\sigma_T)\text{ converges}\}.$$

The set of convergence of Z does not depend on $\boldsymbol{\mu}$.

Remark 3.2. If, moreover, $\boldsymbol{\mu}$ belongs to $\mathbb{T}^N$, then we have

$$Z(Q,P_1,\dots,P_T,\boldsymbol{\mu},s_1,\dots,s_T)\text{ converges}\iff(\sigma_1,\dots,\sigma_T)\in\mathcal{C}(Q,P_1,\dots,P_T).$$

Proposition 3.3. *Let $Q,P_1,\dots,P_T\in\mathbb{R}[X_1,\dots,X_N]$ such that $P_t(\mathbf{x})\gg 1$ ($\mathbf{x}\in J^N$) for all $t\in\{1,\dots,T\}$. Let $1\leqslant T_0\leqslant T$. We assume that*

$$\prod_{t=1}^{T_0}P_t(\mathbf{x})\xrightarrow[\substack{|\mathbf{x}|\to+\infty\\ \mathbf{x}\in J^N}]{}+\infty.$$

Let $\sigma_{T_0+1},\dots,\sigma_T\in\mathbb{R}$. Then there exists $\sigma_0\in\mathbb{R}$ such that: $\sigma_1,\dots,\sigma_{T_0}\geqslant\sigma_0\Rightarrow(\sigma_0,\dots,\sigma_{T_0},\sigma_{T_0+1},\dots,\sigma_T)\in\operatorname{int}(\mathcal{C}(Q,P_1,\dots,P_T))$.

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Proposition 3.4. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ satisfy $P_t(\mathbf{x}) \gg 1$ ($\mathbf{x} \in J^N$) for all $t \in \{1, \dots, T\}$. Let $\boldsymbol{\mu} \in \mathbb{T}^N$. Then $Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ is holomorphic on $\operatorname{int}(\mathcal{C}(Q, P_1, \dots, P_T)) + i\mathbb{R}^T$.*

Remark 3.5. Since $\operatorname{int}(\mathcal{C}(Q, P_1, \dots, P_T)) + i\mathbb{R}^T$ is convex and, therefore, connex, we can speak without ambiguity of the meromorphic continuation of $Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ (if it exists).

3.2 An integral representation for a sum

Notation/Definition 3.6. For $\epsilon > 0$, we define

$$\lambda_\epsilon \colon [\tfrac{3}{2}, +\infty[\to \mathbb{C} \quad \text{by } \lambda_\epsilon(x) = x + i\epsilon \quad \text{and} \quad \overline{\lambda_\epsilon} \colon [\tfrac{3}{2}, +\infty[\to \mathbb{C} \quad \text{by } \overline{\lambda_\epsilon} = x - i\epsilon.$$

Let k denote an integer belonging to $[2, +\infty[$ and set

$$\lambda_{\epsilon,k} = \lambda_{\epsilon|[3/2,k+1/2]} \quad \text{and} \quad \overline{\lambda_{\epsilon,k}} = \overline{\lambda}_{\epsilon|[3/2,k+1/2]}.$$

We define $\gamma_{\epsilon,k} \colon [-1, 1] \to \mathbb{C}$ by $\gamma_{\epsilon,k}(x) = k + \frac{1}{2} + i\epsilon x$ (even for $k = 1$).

The following is a straightforward application of residue calculus and induction.

Lemma 3.7. *Let U be an open set of $\mathbb{C}$ containing $[\frac{3}{2}, k + \frac{1}{2}] + i[-\epsilon, \epsilon]$.*

- *Let $f \colon U \to \mathbb{C}$ be holomorphic. Then*

$$\sum_{m=2}^{k} f(m) = -\int_{\gamma_{\epsilon,1}} \frac{f(z)}{e(z) - 1}\, dz + \int_{\overline{\lambda_{\epsilon,k}}} \frac{f(z)}{e(z) - 1}\, dz + \int_{\gamma_{\epsilon,k}} \frac{f(z)}{e(z) - 1}\, dz - \int_{\lambda_{\epsilon,k}} \frac{f(z)}{e(z) - 1}\, dz.$$

- *Let $f \colon U^N \to \mathbb{C}$ be holomorphic. For $\tau \in \mathcal{S}_N$, we define $f_\tau \colon U^N \to \mathbb{C}$ by $f_\tau(z_1, \dots, z_N) = f(z_{\tau(1)}, \dots, z_{\tau(N)})$. Then $\sum_{\mathbf{m} \in \{2,\dots,k\}^N} f(\mathbf{m})$ is a sum of a finite numbers of terms, each of the form*

$$\pm \int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon,k})^{N_2} \times (\overline{\lambda_{\epsilon,k}})^{N_3} \times (\gamma_{\epsilon,k})^{N_4}} f_\tau(z_1, \dots, z_N) \prod_{n=1}^{N} \frac{1}{e(z_n) - 1}\, d\mathbf{z}$$

where $N_1, N_2, N_3, N_4 \in \mathbb{N}$ satisfy $N_1 + N_2 + N_3 + N_4 = N$, and $\tau \in \mathcal{S}_N$.

3.3 Proof of Theorem A

Before applying Lemma 3.7 to the proof of the theorem, two preliminaries are needed.

The next result follows from [Ess95]. The complete proof is given in [Dec03].

Lemma 3.8. *Let $P \in \mathbb{R}[X_1, \dots, X_N]$ satisfying:*

(i) *for all $\mathbf{x} \in J^N$, $P(\mathbf{x}) > 0$;*

(ii) *there exists $\epsilon_0 > 0$ such that for all $\boldsymbol{\alpha} \in \mathbb{N}^N, \alpha_n \geqslant 1 \Rightarrow \partial^{\boldsymbol{\alpha}} P(\mathbf{x}) \ll x_n^{-\epsilon_0} P(\mathbf{x})$ ($\mathbf{x} \in J^N$).*

Then there exists $\epsilon > 0$ such that:

(i′) *$\mathbf{x} \in J^N$ and $\mathbf{y} \in [-2\epsilon, 2\epsilon]^N \Rightarrow \Re(P(\mathbf{x} + i\mathbf{y})) \geqslant \frac{1}{2} P(\mathbf{x})$;*

(ii′) *for all $\boldsymbol{\alpha} \in \mathbb{N}^N, \alpha_n \geqslant 1 \Rightarrow \partial^{\boldsymbol{\alpha}} P(\mathbf{x} + i\mathbf{y}) \ll x_n^{-\epsilon_0} P(\mathbf{x})$ ($\mathbf{x} \in J^N$, $\mathbf{y} \in [-2\epsilon, 2\epsilon]^N$).*

The second preliminary result is evident.

Lemma 3.9. *We can partition $\mathbb{N}^{*N}$ in the following way:*

$$\mathbb{N}^{*N} = \bigsqcup_{c=1}^{C} A_c, \text{ where for all } c,\ A_c \text{ is of the form } \prod_{n=1}^{N} B_n \text{ with } B_n = \{1\} \text{ or } B_n = [2, +\infty[\cap \mathbb{N}.$$

Proof of Theorem A. The proof is divided into two steps.

Step 1. We have that

$$\mathbf{s} \mapsto Z^*(\mathbf{s}) \overset{\text{def}}{=} \sum_{\mathbf{m} \geqslant \mathbf{2}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t}$$

can be holomorphically extended to $\mathbb{C}^T$.

Proof of Step 1. Since each P_t belongs to *HDF* it follows that we can choose $\epsilon_0 > 0$ such that:

- $\prod_{t=1}^{T} P_t(\mathbf{x}) \gg \prod_{n=1}^{N} x_n^{\epsilon_0}$ $(\mathbf{x} \in J^N)$;
- $\boldsymbol{\alpha} \in \mathbb{N}^N,\ \alpha_n \geqslant 1 \Rightarrow (\partial^{\boldsymbol{\alpha}} P_t / P_t)(\mathbf{x}) \ll x_n^{-\epsilon_0}$ $(\mathbf{x} \in J^N)$.

There exists $\sigma_0 > 0$ such that if $\sigma_1, \dots, \sigma_T > \sigma_0$, then $Z^*(\mathbf{s})$ converges. Starting with any $\mathbf{s}$ whose real part belongs to this set, one then proceeds as follows.

Applying Lemma 3.8, we obtain for each $t \in \{1, \dots, T\}$ an $\epsilon_t > 0$, and then set $\epsilon = \min_t \{\epsilon_t\}$. For $\mathbf{s} \in \mathbb{C}^N$, we define $f_{\mathbf{s}} \colon (]1, +\infty[+i] - 2\epsilon, 2\epsilon[)^N \to \mathbb{C}$ by $f_{\mathbf{s}}(\mathbf{z}) = Q(\mathbf{z}) \prod_{t=1}^{T} P_t(\mathbf{z})^{-s_t} \prod_{n=1}^{N} e^{i\theta_n z_n}$, where we have chosen $\theta_n \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ so that $\mu_n = e^{i\theta_n}$ for each n. Thus, for each integer $k \geqslant 2$,

$$\sum_{\mathbf{m} \in \{2, \dots, k\}^N} Q(\mathbf{m}) \prod_{n=1}^{N} e^{i\theta_n m_n} \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} = \sum_{\mathbf{m} \in \{2, \dots, k\}^N} f_{\mathbf{s}}(\mathbf{m}).$$

By applying Lemma 3.7 to $f_{\mathbf{s}}$ we conclude that $\sum_{\mathbf{m} \in \{2, \dots, k\}^N} f_{\mathbf{s}}(\mathbf{m})$ can be written as a sum/difference of finitely many integrals, each of which is indexed by a permutation τ on $\{1, \dots, N\}$ and a choice of $N_1, N_2, N_3, N_4 \in \mathbb{N}$ whose sum equals N. We will assume that τ is the identity since the argument is the same for any other permutation. Each integral is therefore an expression of the form

$$\int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon,k})^{N_2} \times (\overline{\lambda_{\epsilon,k}})^{N_3} \times (\gamma_{\epsilon,k})^{N_4}} Q(\mathbf{z}) \prod_{t=1}^{T} P_t(\mathbf{z})^{-s_t} \prod_{n=1}^{N} \frac{\exp(i\theta_n z_n)}{e(z_n) - 1}\, d\mathbf{z}.$$

We now conclude by dominated convergence that there exists $r > 0$ such that any integral with $N_4 \geqslant 1$ tends to zero (as $k \to \infty$) on $\{\mathbf{s} \in \mathbb{C}^T : \sigma_1, \dots, \sigma_T > r\}$. Thus, in this open set, Z^* is a linear combination of integrals of the form $Y^{N_1, N_2, N_3}(\mathbf{s})$ where

$$Y^{N_1, N_2, N_3}(\mathbf{s}) \overset{\text{def}}{=} \int_{(\gamma_{\epsilon,1})^{N_1} \times (\lambda_{\epsilon})^{N_2} \times (\overline{\lambda_{\epsilon,k}})^{N_3}} Q(\mathbf{z}) \prod_{t=1}^{T} P_t(\mathbf{z})^{-s_t} \prod_{n=1}^{N} \frac{\exp(i\theta_n z_n)}{e(z_n) - 1}\, d\mathbf{z}.$$

To finish the proof of Theorem A, it suffices to show that any $Y^{N_1, N_2, N_3}(\mathbf{s})$ satisfies the hypotheses of Theorem 2.7.

- For $1 \leqslant n \leqslant N_1$, define $f_n \colon [-1, 1] \to \mathbb{C}$ by
$$f_n(x) = \frac{\exp(i\theta_n \gamma_{\epsilon,1}(x))}{e(\gamma_{\epsilon,1}(x)) - 1} = \frac{\exp(i\theta_n(3/2 + i\epsilon x))}{\exp(2i\pi(3/2 + i\epsilon x)) - 1} = -\exp\left(\frac{3}{2} i\theta_n\right) \frac{\exp(-\epsilon\theta_n x)}{\exp(-2\pi\epsilon x) + 1}.$$
The function $f \colon [-1, 1]^{N_1} \to \mathbb{C}$ defined by $f(x_1, \dots, x_{N_1}) = \prod_{n=1}^{N_1} f_n(x_n)$ is evidently continuous.
- For $N_1 + 1 \leqslant n \leqslant N_1 + N_2$, define $f_n \colon [\frac{3}{2}, +\infty[\to \mathbb{C}$ by
$$f_n(x) = \frac{\exp(i\theta_n \lambda_\epsilon(x))}{e(\lambda_\epsilon(x)) - 1} = \frac{\exp(i\theta_n(x + i\epsilon))}{\exp(2i\pi(x + i\epsilon)) - 1} = -\exp(-\epsilon\theta_n) \frac{\exp(i\theta_n x)}{1 - \exp(-2\pi\epsilon)\exp(i2\pi x)}.$$
- For $N_1 + N_2 + 1 \leqslant n \leqslant N$, define $f_n \colon [\frac{3}{2}, +\infty[\to \mathbb{C}$ by
$$f_n(x) = \frac{\exp(i\theta_n \overline{\lambda}_\epsilon(x))}{e(\overline{\lambda}_\epsilon(x)) - 1} = \frac{\exp(i\theta_n(x - i\epsilon))}{\exp(2i\pi(x - i\epsilon)) - 1} = -\exp(\epsilon\theta_n) \frac{\exp(i\theta_n x)}{1 - \exp(2\pi\epsilon)\exp(i2\pi x)}.$$
Since $\theta_n / 2\pi \notin \mathbb{Z}$, it follows that $f_n \in \mathcal{B}(\frac{3}{2})$ for any $n \geqslant N_1 + 1$.

For any $P \in \mathbb{C}[X_1, \ldots, X_N]$ and N_1, N_2, N_3 of sum N, we define $P^{N_1,N_2,N_3} \in \mathbb{C}[X_1, \ldots, X_N]$ by

$$\begin{aligned} P^{N_1,N_2,N_3}(\mathbf{x}) &= P(\gamma_{\epsilon,1}(x_1), \ldots, \gamma_{\epsilon,1}(x_{N_1}), \lambda_\epsilon(x_{N_1+1}), \ldots, \lambda_\epsilon(x_{N_1+N_2}), \overline{\lambda_\epsilon}(x_{N_1+N_2+1}), \ldots, \overline{\lambda_\epsilon}(x_N)) \\ &= P(\tfrac{3}{2} + i\epsilon x_1, \ldots, \tfrac{3}{2} + i\epsilon x_{N_1}, x_{N_1+1} + i\epsilon, \ldots, x_{N_1+N_2} + i\epsilon, x_{N_1+N_2+1} - i\epsilon, \ldots, x_N - i\epsilon). \end{aligned}$$

Applying this to each P_t and using the defining property of ϵ from Lemma 3.8, it follows that

$$P_t^{N_1,N_2,N_3}(\mathbf{x}) = P_t((\tfrac{3}{2}, \ldots, \tfrac{3}{2}, x_{N_1+1}, \ldots, x_N) + i(\epsilon x_1, \ldots, \epsilon x_{N_1}, \epsilon, \ldots, \epsilon, -\epsilon, \ldots, -\epsilon))$$

satisfies

$$\Re(P_t^{N_1,N_2,N_3}(x_1, \ldots, x_N)) \geqslant \tfrac{1}{2} P_t(\tfrac{3}{2}, \ldots, \tfrac{3}{2}, x_{N_1+1}, \ldots, x_N) \quad \forall \mathbf{x} \in [-1,1]^{N_1} \times J^{N-N_1}.$$

Thus, for all $\mathbf{x} \in [-1,1]^{N_1} \times [\tfrac{3}{2}, +\infty[^{N-N_1}$, we have:

- $\Re(P_t^{N_1,N_2,N_3}(\mathbf{x})) > 0$;
- $|P_t^{N_1,N_2,N_3}(\mathbf{x})| \geqslant \tfrac{1}{2} P_t(\tfrac{3}{2}, \ldots, \tfrac{3}{2}, x_{N_1+1}, \ldots, x_N)$; and
- $\prod_{t=1}^{T} |P_t^{N_1,N_2,N_3}(\mathbf{x})| \gg \prod_{n=N_1+1}^{N} x_n^{\epsilon_0} (\mathbf{x} \in [-1,1]^{N_1} \times [\tfrac{3}{2}, +\infty)^{N-N_1})$.

Finally, if $\boldsymbol{\alpha} \in \{0\}^{N_1} \times \mathbb{N}^{N-N_1}$ and $N_1 + 1 \leqslant n \leqslant N$ is such that $\alpha_n \geqslant 1$, then it also follows from Lemma 3.8 that

$$\begin{aligned} \partial^{\boldsymbol{\alpha}} P_t^{N_1,N_2,N_3}(\mathbf{x}) &\ll x_n^{-\epsilon_0} P_t(\tfrac{3}{2}, \ldots, \tfrac{3}{2}, x_{N_1+1}, \ldots, x_N) \quad (\mathbf{x} \in [-1,1]^{N_1} \times [\tfrac{3}{2}, +\infty[^{N-N_1}) \\ &\ll x_n^{-\epsilon_0} |P_t^{N_1,N_2,N_3}(\mathbf{x})| \quad (\mathbf{x} \in [-1,1]^{N_1} \times [\tfrac{3}{2}, +\infty[^{N-N_1}). \end{aligned}$$

Since

$$Y^{N_1,N_2,N_3}(\mathbf{s}) = (i\epsilon)^{N_1} \int_{[-1,1]^{N_1} \times [3/2,+\infty[^{N-N_1}} Q^{N_1,N_2,N_3}(\mathbf{x}) \prod_{t=1}^{T} P_t^{N_1,N_2,N_3}(\mathbf{x})^{-s_t} \prod_{n=1}^{N} f_n(x_n)\, d\mathbf{x}$$

the hypotheses of Theorem 2.7 guarantee the existence of an holomorphic continuation for each $Y^{N_1,N_2,N_3}(\mathbf{s})$ to $\mathbb{C}^T$. This completes the proof of Step 1.

Step 2 (Conclusion). A simple induction argument (on N) completes the proof of Theorem A.

- For $N = 1$, we only need to write

  $$Z(Q; P_1, \ldots, P_T; \mu; \mathbf{s}) = \mu Q(1) \prod_{t=1}^{T} P_t(1)^{-s_t} + \sum_{m \geqslant 2} \mu^m Q(m) \prod_{t=1}^{T} P_t(m)^{-s_t}$$

  and then we apply Step 1.
- If the result is true for each any number of variables between 1 and $N - 1$, then, thanks to Lemma 3.9 and Step 1, we see that it is true for N variables. □

3.4 Proof that $Z(1, P_{ex}, -1, -1, \cdot)$ has a pole

Proof. For this proof we set $Z(s) = Z(1; P; -1, -1; s)$. Thus,

$$\begin{aligned} Z(s) &= \sum_{m,n \geqslant 1} (-1)^m (-1)^n [(m-n)^2 m + m]^{-s} \\ &= \sum_{m,n \geqslant 1} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s} \\ &= \sum_{1 \leqslant m \leqslant n} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s} + \sum_{1 \leqslant n < m} (-1)^{m-n} m^{-s} [(m-n)^2 + 1]^{-s}. \end{aligned}$$

By setting $n = m + u$ in the first sum and $m = n + u$ in the second sum, we obtain

$$\begin{aligned} Z(s) &= \sum_{\substack{m\geqslant 1\\ u\geqslant 0}} (-1)^u m^{-s}(u^2+1)^{-s} + \sum_{n,u\geqslant 1} (-1)^u (n+u)^{-s}(u^2+1)^{-s} \\ &= \zeta(s)\sum_{u\geqslant 0}(-1)^u(u^2+1)^{-s} + \sum_{u\geqslant 1}(-1)^u(u^2+1)^{-s}\sum_{n\geqslant 1}(n+u)^{-s} \\ &= \zeta(s)\sum_{u\geqslant 0}(-1)^u(u^2+1)^{-s} + \sum_{u\geqslant 1}(-1)^u(u^2+1)^{-s}\Big[\zeta(s) - \sum_{1\leqslant k\leqslant u} k^{-s}\Big] \\ &= \zeta(s)\sum_{u\in\mathbb{Z}}(-1)^u(u^2+1)^{-s} - \sum_{1\leqslant k\leqslant u}(-1)^u(u^2+1)^{-s}k^{-s} \\ &= \zeta(s)\sum_{u\in\mathbb{Z}}(-1)^u(u^2+1)^{-s} - \sum_{\substack{k\geqslant 1\\ \ell\geqslant 0}}(-1)^{k+\ell}[(k+\ell)^2+1]^{-s}k^{-s}. \end{aligned}$$

The following facts suffice to show that Z has a simple pole at $s = 1$:

- a classical application of the residue theorem is $\sum_{u\in\mathbb{Z}}(-1)^u(u^2+1)^{-1} = \pi/\sinh(\pi)$;
- Theorem A implies that

$$s \mapsto \sum_{u\in\mathbb{Z}}(-1)^u(u^2+1)^{-s} \quad \text{and} \quad s \mapsto \sum_{\substack{k\geqslant 1\\ \ell\geqslant 0}}(-1)^{k+\ell}[(k+\ell)^2+1]^{-s}k^{-s}$$

  can be holomorphically extended to $\mathbb{C}$. □

4. Values at T-tuples of negative integers

4.1 Proof of the Exchange Lemma

The proof is a simple consequence of the following (in which the notation $Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; \cdot)$ is understood to denote the analytically continued series to $\mathbb{C}^T$).

Proposition 4.1. *Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $T_0 \in \{1, \dots, T-1\}$ for a given $T \geqslant 2$. We assume that $P_1, \dots, P_T$ satisfy HDF and that*

$$\prod_{t=1}^{T_0} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty\\ \mathbf{x}\in J^N}]{} +\infty.$$

Let $\boldsymbol{\mu} \in (\mathbb{T}\setminus\{1\})^N$ and $k_1, \dots, k_T \in \mathbb{N}$. Then

$$Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; -k_1, \dots, -k_T) = Z\bigg(Q \prod_{t=T_0+1}^{T} {P_t}^{k_t}; P_1, \dots, P_{T_0}; \boldsymbol{\mu}; -k_1, \dots, -k_{T_0}\bigg).$$

Proof. We define the holomorphic function $f\colon \mathbb{C}^{T_0} \to \mathbb{C}$ by

$$f(s_1, \dots, s_{T_0}) = Z(Q; P_1, \dots, P_T; \boldsymbol{\mu}; s_1, \dots, s_{T_0}; -k_{T_0+1}, \dots, -k_T).$$

Thanks to Proposition 3.3, there exists $\sigma_0 \in \mathbb{R}$ (depending on $(k_{T_0+1}, \dots, k_T)$) such that for any $(s_1, \dots, s_{T_0})$ with $\sigma_1, \dots, \sigma_{T_0} \geqslant \sigma_0$ we have

$$f(s_1, \dots, s_{T_0}) = \sum_{\mathbf{m}\in\mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=T_0+1}^{T} P_t(\mathbf{m})^{k_t} \prod_{t=1}^{T_0} P_t(\mathbf{m})^{-s_t}.$$

Next, define the function $g\colon \mathbb{C}^{T_0} \to \mathbb{C}$ by

$$g(s_1, \ldots, s_{T_0}) = Z\biggl(Q \prod_{t=T_0+1}^{T} P_t^{k_t}; P_1, \ldots, P_{T_0}; \boldsymbol{\mu}; s_1, \ldots, s_{T_0}\biggr).$$

That is, g is the analytic continuation of the twisted series in $(s_1, \ldots, s_{T_0})$, with the role of Q now played by $Q \prod_{t=T_0+1}^{T} P_t^{k_t}$. Theorem A also applies to this series. Thus, g is an entire function on $\mathbb{C}^{T_0}$. Proposition 3.3 therefore applies to g. As a result, there exists $\sigma_0' \in \mathbb{R}$ such that for any $(s_1, \ldots, s_{T_0})$ with $\sigma_1, \ldots, \sigma_{T_0} \geqslant \sigma_0'$ we have

$$g(s_1, \ldots, s_{T_0}) = \sum_{\mathbf{m} \in \mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=T_0+1}^{T} P_t^{k_t} \prod_{t=1}^{T_0} P_t(\mathbf{m})^{-s_t}.$$

Thus, $f(s_1, \ldots, s_{T_0}) = g(s_1, \ldots, s_{T_0})$ in the open set consisting of all $(s_1, \ldots, s_{T_0})$ such that each $\sigma_t > \max(\sigma_0, \sigma_0')$. The uniqueness of the analytic continuation then ensures that $f = g$ on $\mathbb{C}^{T_0}$. In particular, $f(-k_1, \ldots, -k_{T_0}) = g(-k_1, \ldots, -k_{T_0})$, as claimed. □

Proof of the Exchange Lemma. Proposition 4.1 tells us that both quantities are equal to

$$Z(Q; P_1, \ldots, P_T, Q_1, \ldots, Q_{T'}; \boldsymbol{\mu}; -k_1, \ldots, -k_T, -\ell_1, \ldots, -\ell_{T'}).$$ □

4.2 An application of the Exchange Lemma: the proof of Theorem B

Theorem B illustrates how one can use the Exchange Lemma. Its proof is a simple consequence of the following.

LEMMA 4.2. *Let $Q = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}} \in \mathbb{R}[X_1, \ldots, X_N]$ and $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^N$. Then*

$$Z(Q; X_1, \ldots, X_N; \boldsymbol{\mu}; 0, \ldots, 0) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \zeta_{\mu_n}(-\alpha_n).$$

Proof. Set $T = N$ and $P_t = X_t$ for each $t = 1, \ldots, T$. These polynomials evidently belong to *HDF*. Thus, if $\sigma_1, \ldots, \sigma_N$ are large enough, we have

$$\begin{aligned} Z(Q; X_1, \ldots, X_N; \boldsymbol{\mu}, \mathbf{s}) &= Z\biggl(\sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}; X_1, \ldots, X_N; \boldsymbol{\mu}; \mathbf{s}\biggr) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} Z(\mathbf{X}^{\boldsymbol{\alpha}}; X_1, \ldots, X_N; \boldsymbol{\mu}; \mathbf{s}) \\ &= \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \sum_{\mathbf{m} \in \mathbb{N}^{*N}} \boldsymbol{\mu}^{\mathbf{m}} \mathbf{m}^{\boldsymbol{\alpha}} \prod_{n=1}^{N} m_n^{-s_n} = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \sum_{m_1, \ldots, m_N \geqslant 1} \prod_{n=1}^{N} \mu_n^{m_n} m_n^{\alpha_n - s_n} \\ &= \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \sum_{m_n \geqslant 1} \mu_n^{m_n} m_n^{\alpha_n - s_n} = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \zeta_{\mu_n}(s_n - \alpha_n). \end{aligned}$$

The uniqueness of analytic continuation then implies

$$Z(Q; X_1, \ldots, X_N; \boldsymbol{\mu}, \mathbf{s}) = \sum_{\boldsymbol{\alpha} \in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \zeta_{\mu_n}(s_n - \alpha_n) \quad \forall \mathbf{s} \in \mathbb{C}^N.$$

Setting $\mathbf{s} = \mathbf{0}$ in this equality completes the proof. □

Proof of Theorem B. The argument is now very simple and goes as follows:

$$\begin{aligned}Z(Q;P_1,\ldots,P_T;\boldsymbol{\mu};-k_1,\ldots,-k_T) &= Z\Big(Q\prod_{n=1}^{N}X_n^0;P_1,\ldots,P_T;\boldsymbol{\mu};-k_1,\ldots,-k_T\Big)\\ &= Z\Big(Q\prod_{t=1}^{T}P_t^{k_t};X_1,\ldots,X_N;\boldsymbol{\mu};0,\ldots,0\Big)\\ &= \sum_{\boldsymbol{\alpha}\in S}a_{\boldsymbol{\alpha}}\prod_{n=1}^{N}\zeta_{\mu_n}(-\alpha_n).\end{aligned}$$

The Exchange Lemma implies the second equality, and Lemma 4.2 implies the third equality. □

4.3 Values at T-tuples of integers for Y

We gave the values at T-tuples of negative integers for general Y in terms of values at negative integers of the simplest Y. The proof of the following theorem follows exactly the same process as that of Theorem B.

THEOREM 4.3. *Let $Q,P_1,\ldots,P_T\in\mathbb{R}[X_1,\ldots,X_N]$ and $f_1,\ldots,f_N\in\mathcal{B}(1)$. We assume that $P_1,\ldots,$ P_T satisfy HDF and that*

$$\prod_{t=1}^{T}P_t(\mathbf{x})\xrightarrow[\mathbf{x}\in J^N]{|\mathbf{x}|\to+\infty}+\infty.$$

Let $k_1,\ldots,k_T\in\mathbb{N}$. We denote $Q\prod_{t=1}^{T}P_t^{k_t}=\sum_{\boldsymbol{\alpha}\in S}a_{\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}$. Then

$$Y(Q;P_1,\ldots,P_T;f_1,\ldots,f_N;-k_1,\ldots,-k_T)=\sum_{\boldsymbol{\alpha}\in S}a_{\boldsymbol{\alpha}}\prod_{n=1}^{N}Y(1;X;f_n;-\alpha_n).$$

Remark 4.4. For example, if f is given by $f(x)=e^{i\theta x}$, where $\theta\in\mathbb{R}^*$, then the values at negative integers of $Y(1;X;f;\cdot)$ are very easy to calculate by induction thanks to an integration by parts.

5. p-adic interpolation

The main result of this section is Theorem C. The proof is based on Theorem B and a precise description of each $\zeta_\mu(-k)$, proved in § 5.1.

5.1 A formula for the values of $\zeta_{\mu}(-k)$

The first ingredient is a classical lemma [Zag77].

LEMMA 5.1. *Let $(a_m)_{m\in\mathbb{N}^*}$ be a sequence of complex numbers and define*

$$Z(s)=\sum_{m=1}^{+\infty}\frac{a_m}{m^s}.$$

Let us assume that there exists $s\in\mathbb{C}$ such that the series converges, from which it follows that the series $f(x)=\sum_{m=1}^{+\infty}a_me^{-mx}$ converges if $x>0$.

We assume that there is a sequence $(c_k)_{k\in\mathbb{N}}$ of complex numbers such that, for all $K\in\mathbb{N}^*$, we have in a neighborhood of zero

$$f(x)=\sum_{k=0}^{K-1}c_kx^k+O(x^K).$$

Then Z can be holomorphically extended to $\mathbb{C}$ and $Z(-k)=(-1)^kk!c_k$ for all $k\in\mathbb{N}$.

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We will also need the Stirling numbers of the second kind, as well as some of their elementary properties. Let us recall the following definition.

Definition 5.2. Let $k, \ell \in \mathbb{N}$. The *Stirling number of the second kind* (associated to (k,ℓ)) is the number of partitions in ℓ parts of a set with k elements. This integer is denoted by $S(k,\ell)$.

Example 5.3. We have $S(0,0)=1$; for $k \in \mathbb{N}$, $S(k,k)=1$; if $0 \leqslant k < \ell$, then $S(k,\ell)=0$.

The proofs of the next two results can be found in [Com70].

Lemma 5.4. *For all $k \in \mathbb{N}$ and all $\ell \in \mathbb{N}^*$, $S(k+1,\ell) = \ell S(k,\ell) + S(k,\ell-1)$.*

Lemma 5.5. *For all $k, \ell \in \mathbb{N}$, we have*

$$S(k,\ell) = \frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{\ell}{j} j^k.$$

Finally, we need a general expression for each derivative of the composition of a smooth function with the exponential function.

Lemma 5.6. *Let $g\colon \mathbb{R}_+^* \to \mathbb{C}$ be smooth, and define $f = g \circ \exp$. Then, for all $k \in \mathbb{N}$, we have: $f^{(k)}(x) = \sum_{\ell=0}^{k} S(k,\ell) e^{\ell x} g^{(\ell)}(e^x)$ for all $x \in \mathbb{R}$.*

Proof. The proof is by induction on $k \in \mathbb{N}$.

- For $k=0$, the formula is true because $S(0,0)=1$.
- Assuming that the formula holds for a given k, and differentiating one more time, it follows that for all $x \in \mathbb{R}$,

$$\begin{aligned} f^{(k+1)}(x) &= \sum_{\ell=0}^{k} S(k,\ell)(\ell e^{\ell x} g^{(\ell)}(e^x) + e^{\ell x} e^x g^{(\ell+1)}(e^x)) \\ &= \sum_{\ell=0}^{k} S(k,\ell)\ell e^{\ell x} g^{(\ell)}(e^x) + \sum_{\ell=1}^{k+1} S(k,\ell-1) e^{\ell x} g^{(\ell)}(e^x). \end{aligned}$$

Since $S(k,k+1)=0$, one concludes that

$$\begin{aligned} f^{(k+1)}(x) &= \sum_{\ell=1}^{k+1}[\ell S(k,\ell) + S(k,\ell-1)] e^{\ell x} g^{(\ell)}(e^x) \\ &= \sum_{\ell=1}^{k+1} S(k+1,\ell) e^{\ell x} g^{(\ell)}(e^x) = \sum_{\ell=0}^{k+1} S(k+1,\ell) e^{\ell x} g^{(\ell)}(e^x). \end{aligned}$$

This proves the formula for $k+1$. ☐

We can now express each $\zeta_\mu(-k)$ in terms of the $S(k,\ell)$ as follows.

Lemma 5.7. *Let $\mu \in \mathbb{T} \setminus \{1\}$. Then, for all $k \in \mathbb{N}$, we have*

$$\zeta_\mu(-k) = \frac{(-1)^k \mu}{1-\mu}\sum_{\ell=0}^{k}\frac{\ell! S(k,\ell)}{(\mu-1)^\ell}.$$

Proof. For all

$$x > 0,\ \sum_{m=1}^{+\infty}\mu^m e^{-mx} = \sum_{m=1}^{+\infty}(\mu e^{-x})^m = \mu e^{-x}\frac{1}{1-\mu e^{-x}} = \frac{\mu}{e^x - \mu}.$$

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We define $f\colon \mathbb{R} \to \mathbb{C}$ by $f(x) = \mu/(e^x - \mu)$ and $g\colon \mathbb{R}_+^* \to \mathbb{C}$ by $g(y) = \mu/(y - \mu)$. Then g is smooth and $f = g \circ \exp$, so (5.6) gives: $f^{(k)}(x) = \sum_{\ell=0}^{k} S(k,\ell) e^{\ell x} g^{(\ell)}(e^x)$ for all $x \in \mathbb{R}$. Writing $g(y) = -\mu(1/(\mu - y))$, it is clear that for each ℓ, $g^{(\ell)}(y) = -\mu(\ell!/(\mu - y)^{\ell+1})$. Thus,

$$f^{(k)}(0) = \sum_{\ell=0}^{k} S(k,\ell)\left(-\mu\frac{\ell!}{(\mu-1)^{\ell+1}}\right),$$

we then apply (5.1) to finish the proof. □

5.2 Proof of Theorem C

To prove Theorem C, we need to have a formula adapted to p-adic interpolation: we want to obtain a formula similar to that appearing in the proof of Theorem 20 in [Cas82]. In the present work, such a formula is obtained during the proof of Lemma 5.9: this is the formula (7) for $Z_{\boldsymbol{\ell}}(-\mathbf{k})$.

However, for the p-adic control of $Z_{\boldsymbol{\ell}}(-\mathbf{k})$ we do not use the formula (7) but the formula (1) (cf. the proof of the Lemma 5.9), which come from Theorem B and that contain the Stirling numbers. This explains why we obtain the bound $p^{-1/p(p-1)}$, which is better than the bound 1 obtained in the work of Cassou-Noguès.

We first rewrite $\tilde{Z}$ as follows:

$$\begin{aligned}
\tilde{Z}(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; \mathbf{s}) &= \sum_{\substack{\mathbf{m}\in\mathbb{N}^{*N} \\ \forall t\in\{1,\ldots,T\},\ p\nmid P_t(\mathbf{m})}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} \\
&= \sum_{\mathbf{u}\in\{1,\ldots,p\}^N} \sum_{\substack{\mathbf{m}\in\mathbb{N}^{*N} \\ \forall t\in\{1,\ldots,T\},\ p\nmid P_t(\mathbf{m}) \\ \forall n,\ m_n\equiv u_n \mod (p)}} \boldsymbol{\mu}^{\mathbf{m}} Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t} \\
&= \sum_{\mathbf{u}\in\{1,\ldots,p\}^N} \sum_{\substack{\mathbf{m}\in\mathbb{N}^{N} \\ \forall t\in\{1,\ldots,T\},\ p\nmid P_t(\mathbf{u}+p\mathbf{m})}} \boldsymbol{\mu}^{\mathbf{u}+p\mathbf{m}} Q(\mathbf{u}+p\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{u}+p\mathbf{m})^{-s_t} \\
&= \sum_{\mathbf{u}\in\{1,\ldots,p\}^N} \sum_{\substack{\mathbf{m}\in\mathbb{N}^{N} \\ \forall t\in\{1,\ldots,T\},\ p\nmid P_t(\mathbf{u})}} \boldsymbol{\mu}^{\mathbf{u}+p\mathbf{m}} Q(\mathbf{u}+p\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{u}+p\mathbf{m})^{-s_t} \\
&= \sum_{\substack{\mathbf{u}\in\{1,\ldots,p\}^N \\ \forall t\in\{1,\ldots,T\},\ p\nmid P_t(\mathbf{u})}} \boldsymbol{\mu}^{\mathbf{u}} \tilde{Z}(Q_{\mathbf{u}}; P_{1,\mathbf{u}}, \ldots, P_{t,\mathbf{u}}; \boldsymbol{\mu}^p; \mathbf{s}),
\end{aligned}$$

where $Q_{\mathbf{u}} = Q(\mathbf{u} + p\mathbf{X})$ and $P_{t,\mathbf{u}} = P_t(\mathbf{u} + p\mathbf{X})$. Note that each $P_{t,\mathbf{u}}$ satisfies the property that $p \nmid P_{t,\mathbf{u}}(\mathbf{m})$ for all integral vectors $\mathbf{m}$, and that the twist is now determined by the vector $\boldsymbol{\mu}^p$ rather than $\boldsymbol{\mu}$.

Two lemmas are now needed to complete the proof of Theorem C.

Lemma 5.8. *Let $x \in \mathbb{C}_p$. Then $|x-1|_p > p^{-1/(p-1)} \Rightarrow |x^p - 1|_p = (|x-1|_p)^p$.*

Proof. Set $z = x - 1$. We have

$$x^p - 1 = (z+1)^p - 1 = \sum_{k=1}^{p} \binom{p}{k} z^k = z\left(\sum_{k=1}^{p-1} \binom{p}{k} z^{k-1} + z^{p-1}\right).$$

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Let $k \in \{1, \ldots, p-1\}$. We want to show that

$$\left| \binom{p}{k} z^{k-1} \right|_p < |z^{p-1}|_p.$$

Since p is prime,

$$\left| \binom{p}{k} \right|_p \leqslant p^{-1}.$$

In addition,

$$\left| \binom{p}{k} z^{k-1} \right|_p = \left| \binom{p}{k} \right|_p |z|_p^{k-1},$$

so it is enough to show that $p^{-1}|z|_p^{k-1} < |z|_p^{p-1}$. To show this, we are going to study two cases.

- *Case* $|z|_p > 1$: $p^{-1}|z|_p^{k-1} < |z|_p^{k-1} \leqslant |z|_p^{p-2} < |z|_p^{p-1}$.
- *Case* $0 < |z|_p \leqslant 1$: $p^{-1}|z|_p^{k-1} \leqslant p^{-1}$. Since $|z|_p > p^{-1/(p-1)}$, $|z|_p^{p-1} > p^{-1}$ and so we see that $p^{-1}|z|_p^{k-1} < |z|_p^{p-1}$.

From

$$\left| \binom{p}{k} z^{k-1} \right|_p < |z^{p-1}|_p \quad \text{for all } k \in \{1, \ldots, p-1\},$$

we deduce that

$$\left| \sum_{k=1}^{p-1} \binom{p}{k} z^{k-1} + z^{p-1} \right|_p = |z^{p-1}|_p.$$

The conclusion follows. □

LEMMA 5.9. *We make the same hypothesis as that in Theorem C, except that part (ii) is replaced by part (ii′): $|1-\mu_n|_p > p^{-1/(p-1)}$. However, impose the additional property that $p \nmid P_t(\mathbf{m})$ for all $\mathbf{m} \in \mathbb{N}^N$. Then for each $\mathbf{r} \in \{0, \ldots, p-2\}^T$ there exists $Z_p^{(\mathbf{r})}(Q, P_1, \ldots, P_T, \boldsymbol{\mu}, \cdot) \colon \mathbb{Z}_p{}^T \to \mathbb{C}_p$ continuous such that for all $\mathbf{k} \in \mathbb{N}^T$ satisfying $k_t \equiv r_t \mod (p-1)$ for all $t \in \{1, \ldots, T\}$, we have*

$$Z_p^{(\mathbf{r})}(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}) = Z(Q; P_1, \ldots, P_T; \boldsymbol{\mu}; -\mathbf{k}).$$

Proof. Let $\mathbf{k} \in \mathbb{N}^T$ and write $Q \prod_{t=1}^T P_t^{k_t} = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}$. Set $\mathcal{S}_{\mathbf{k}} = \{\boldsymbol{\alpha} : a_{\boldsymbol{\alpha}} \neq 0\}$. Thanks to Theorem B and Lemma 5.7, we know the following:

$$\begin{aligned}
Z(Q, P_1, \ldots, P_T, \boldsymbol{\mu}, -\mathbf{k}) &= \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[a_{\boldsymbol{\alpha}} \prod_{n=1}^N \left(\frac{(-1)^{\alpha_n} \mu_n}{1-\mu_n} \sum_{\ell_n=0}^{\alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right) \right] \\
&= \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[a_{\boldsymbol{\alpha}} (-1)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1}-\boldsymbol{\mu})^{\mathbf{1}}} \prod_{n=1}^N \left(\sum_{\ell_n=0}^{\alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right) \right] \\
&= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1}-\boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^N \sum_{\ell_n \leqslant \alpha_n} \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right] \\
&= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1}-\boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha} \in \mathcal{S}_{\mathbf{k}}} \left[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\ell} \leqslant \boldsymbol{\alpha}} \prod_{n=1}^N \frac{\ell_n! S(\alpha_n, \ell_n)}{(\mu_n - 1)^{\ell_n}} \right]
\end{aligned}$$

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$$= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1}-\boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\alpha}\in\mathcal{S}_{\mathbf{k}}} \biggl[\sum_{\boldsymbol{\ell}\leqslant\boldsymbol{\alpha}} \biggl((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu}-\mathbf{1})^{\boldsymbol{\ell}}} \prod_{n=1}^{N} S(\alpha_n,\ell_n)\biggr)\biggr]$$

$$= \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1}-\boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\ell}\in\mathbb{N}^N} Z_{\boldsymbol{\ell}}(-\mathbf{k}),$$

where, for each $\boldsymbol{\ell}\in\mathbb{N}^N$,

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) \stackrel{\text{def}}{=} \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{S}_{\mathbf{k}}\\ \boldsymbol{\ell}\leqslant\boldsymbol{\alpha}}} \biggl((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu}-\mathbf{1})^{\boldsymbol{\ell}}} \prod_{n=1}^{N} S(\alpha_n,\ell_n)\biggr).$$

The family $(Z_{\boldsymbol{\ell}}(-\mathbf{k}))_{\boldsymbol{\ell}\in\mathbb{N}^N}$ is nearly null, more precisely its support is included in $\{\boldsymbol{\ell}\in\mathbb{N}^N \mid \exists\boldsymbol{\alpha}\in S_{\mathbf{k}},\ \boldsymbol{\alpha}\geqslant\boldsymbol{\ell}\}$, which is clearly a finite subset of $\mathbb{N}^N$. Moreover, since $\ell > k \Rightarrow S(k,\ell)=0$, we see that

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) = \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu}-\mathbf{1})^{\boldsymbol{\ell}}} \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl((-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} S(\alpha_n,\ell_n)\biggr). \tag{1}$$

Finally, we note that

$$|Z_{\boldsymbol{\ell}}(-\mathbf{k})|_p \leqslant \prod_{n=1}^{N} \frac{|\ell_n!|_p}{|\mu_n-1|_p^{\ell_n}}.$$

This will be needed in the following.

By using Lemma 5.5, we manipulate the sums as follows:

$$Z_{\boldsymbol{\ell}}(-\mathbf{k}) = \frac{\boldsymbol{\ell}!}{(\boldsymbol{\mu}-\mathbf{1})^{\boldsymbol{\ell}}} \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl\{(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \biggl[\frac{1}{\ell_n!} \sum_{j_n=0}^{\ell_n} \biggl((-1)^{\ell_n-j_n}\binom{\ell_n}{j_n} j_n^{\alpha_n}\biggr)\biggr]\biggr\} \tag{2}$$

$$= (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl\{(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} \biggl[\sum_{j_n=0}^{\ell_n} \biggl((-1)^{j_n}\binom{\ell_n}{j_n} j_n^{\alpha_n}\biggr)\biggr]\biggr\} \tag{3}$$

$$= (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl\{(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \sum_{\mathbf{j}\in\prod_{n=1}^{N}\{0,\dots,\ell_n\}} \biggl[\prod_{n=1}^{N} \biggl((-1)^{j_n}\binom{\ell_n}{j_n} j_n^{\alpha_n}\biggr)\biggr]\biggr\} \tag{4}$$

$$= (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j}\in\prod_{n=1}^{N}\{0,\dots,\ell_n\}} \biggl\{\prod_{n=1}^{N} \biggl[(-1)^{j_n}\binom{\ell_n}{j_n}\biggr] \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl[(-1)^{|\boldsymbol{\alpha}|} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} j_n^{\alpha_n}\biggr]\biggr\} \tag{5}$$

$$= (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j}\in\prod_{n=1}^{N}\{0,\dots,\ell_n\}} \biggl\{\prod_{n=1}^{N} \biggl[(-1)^{j_n}\binom{\ell_n}{j_n}\biggr] \sum_{\boldsymbol{\alpha}\in S_{\mathbf{k}}} \biggl[a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} (-j_n)^{\alpha_n}\biggr]\biggr\} \tag{6}$$

$$= (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j}\in\prod_{n=1}^{N}\{0,\dots,\ell_n\}} \biggl\{(-1)^{|\mathbf{j}|}\binom{\boldsymbol{\ell}}{\mathbf{j}} Q(-\mathbf{j}) \prod_{t=1}^{T} P_t(-\mathbf{j})^{k_t}\biggr\}. \tag{7}$$

For a unit $x\in\mathbb{Z}_p$, we denote its Teichmüller representative by $w(x)$ and set $\langle x\rangle = x/w(x)$. Since each $P_t(-\mathbf{j})$ is a unit in $\mathbb{Z}_p$, it follows that if $r_t\in\{0,\dots,p-2\}$ satisfies $k_t\equiv r_t \mod (p-1)$, then $P_t(-\mathbf{j})^{k_t} = w(P_t(-\mathbf{j}))^{r_t}\langle P_t(-\mathbf{j})\rangle^{k_t}$. Setting $\mathbf{r}=(r_1,\dots,r_T)\in\{0,\dots,p-2\}^T$, we now define the function $Z_{\boldsymbol{\ell}}^{(\mathbf{r})}\colon \mathbb{Z}_p^T\to\mathbb{C}_p$ by

$$Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(s_1,\dots,s_T) = (\mathbf{1}-\boldsymbol{\mu})^{-\boldsymbol{\ell}} \sum_{\mathbf{j}\in\prod_{n=1}^{N}\{0,\dots,\ell_n\}} (-1)^{\mathbf{j}}\binom{\boldsymbol{\ell}}{\mathbf{j}} Q(-\mathbf{j}) \prod_{t=1}^{T} w\Bigl(P_t(-\mathbf{j})\Bigr)^{r_t} \langle P_t(-\mathbf{j})\rangle^{-s_t}.$$

Thus, $Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(-\mathbf{k}) = Z_{\boldsymbol{\ell}}(-\mathbf{k})$. By our previous observation, we then have the bound

$$|Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(-\mathbf{k})|_p \leqslant \prod_{n=1}^{N} \frac{|\ell_n!|_p}{|\mu_n - 1|_p^{\ell_n}}.$$

Since $Z_{\boldsymbol{\ell}}^{(\mathbf{r})}$ is continuous, and the set $\prod_{t=1}^{T}\{-r_t + (p-1)\mathbb{N}\}$ is dense in $\mathbb{Z}_p^T$, we deduce that

$$|Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(\mathbf{s})|_p \leqslant \prod_{n=1}^{N} \frac{|\ell_n!|_p}{|\mu_n - 1|_p^{\ell_n}} \quad \forall \mathbf{s} \in \mathbb{Z}_p^T.$$

To finish the argument, we define $Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \cdot)$ as an *a priori* formal series:

$$Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \mathbf{s}) = \frac{\boldsymbol{\mu}^{\mathbf{1}}}{(\mathbf{1} - \boldsymbol{\mu})^{\mathbf{1}}} \sum_{\boldsymbol{\ell} \in \mathbb{N}^N} Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(\mathbf{s}).$$

One now shows that the series converges p-adically on $\mathbb{Z}_p^T$. Using the upper bound for $Z_{\boldsymbol{\ell}}(-\mathbf{k})$ noted above, it therefore suffices to show the following for any n:

$$\frac{|\ell!|_p}{|\mu_n - 1|_p^{\ell}} \xrightarrow[\ell \to +\infty]{} 0.$$

Given $\ell \in \mathbb{N}$, we denote by $S_p(\ell)$ the sum of the digits for ℓ written in base p. It is well known that for $\ell \in \mathbb{N}$ we have

$$v_p(\ell!) = (\ell - S_p(\ell))/(p-1).$$

If c denotes the number of digits of ℓ in base p, then $S_p(\ell) \leqslant c(p-1)$ and $\ell \geqslant p^{c-1}$; from this we deduce $S_p(\ell) \ll \log \ell$. Since

$$v_p\left(\frac{\ell!}{(\mu_n - 1)^{\ell}}\right) = \frac{\ell - S_p(\ell)}{p-1} - \ell v_p(\mu_n - 1) = \left(\frac{1}{p-1} - v_p(\mu_n - 1)\right)\ell - \frac{S_p(\ell)}{p-1},$$

the two bounds $1/(p-1) - v_p(\mu_n - 1) > 0$ and $S_p(\ell) \ll \log \ell$ now imply

$$v_p(\ell!/(\mu_n - 1)^{\ell}) \xrightarrow[\ell \to +\infty]{} +\infty.$$

Thus, $\sum_{\boldsymbol{\ell}} Z_{\boldsymbol{\ell}}^{(\mathbf{r})}(\mathbf{s})$ converges p-adically on $\mathbb{Z}_p^T$. This shows that the function $Z_p^{(\mathbf{r})}(Q, P_1, \dots, P_T, \boldsymbol{\mu}, \mathbf{s})$, p-adically interpolates the function $-\mathbf{k} \mapsto Z(Q, P_1, \dots, P_T, \boldsymbol{\mu}, -\mathbf{k})$ when $\mathbf{k} \equiv \mathbf{r} \mod (p-1)$, and completes the proof of Lemma 5.9 and, therefore, the proof of Theorem C. □

6. The case of characters

Let $Q, P_1, \dots, P_T \in \mathbb{R}[X_1, \dots, X_N]$ and $\chi_1, \dots, \chi_N$ be functions from $\mathbb{N}^*$ into $\mathbb{C}$. To these data we can associate the following multivariable zeta series:

$$Z(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; s_1, \dots, s_T) = \sum_{m_1 \geqslant 1, \dots, m_N \geqslant 1} \frac{(\prod_{n=1}^{N} \chi_n(m_n))Q(m_1, \dots, m_N)}{\prod_{t=1}^{T} P_t(m_1, \dots, m_N)^{s_t}}$$

where $(s_1, \dots, s_T) \in \mathbb{C}^T$.

Thanks to the following easy lemma (proven in [Kow04, ch. I]), under a suitable hypothesis, such functions are linear combinations of functions of the type $Z(Q; P_1, \dots, P_T; \mu_1, \dots, \mu_N; \cdot)$.

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Lemma 6.1. *Let $\chi\colon \mathbb{N}^* \to \mathbb{C}$, that is D-periodic and whose mean value is null (that is, $\sum_{m=1}^{D}\chi(m) = 0$). For all $d \in \{1,\dots,D-1\}$, we set $\mu_d = \exp(2i\pi(d/D))$. Then there exists $a_1,\dots,a_{D-1}$ such that for all $m \in \mathbb{N}^*$ we have $\chi(m) = \sum_{d=1}^{D-1} a_d\mu_d^m$.*

Combining the preceding lemma and Theorem A, we obtain the following.

Theorem 6.2. *Let $Q, P_1,\dots,P_T \in \mathbb{R}[X_1,\dots,X_N]$ and $\chi_1,\dots,\chi_N\colon \mathbb{N}^* \to \mathbb{C}$ periodic of null mean value. We assume that $P_1,\dots,P_T$ satisfy HDF and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty\\ \mathbf{x}\in J^N}]{} +\infty.$$

Then $Z(Q;P_1,\dots,P_T;\chi_1,\dots,\chi_N;\cdot)$ extends to $\mathbb{C}^T$ as an entire function.

It is now very easy to copy the Exchange Lemma for the series $Z(Q;P_1,\dots,P_T;\chi_1,\dots,\chi_N;\cdot)$. Let us recall the following usual notation.

Definition 6.3. For $\chi\colon \mathbb{N}^* \to \mathbb{C}$, we set $L(s,\chi) = \sum_{m=1}^{+\infty}(\chi(m)/m^s)$.

Then, exactly as was done in § 4, using the Exchange Lemma, we obtain the following.

Theorem 6.4. *Let $Q, P_1,\dots,P_T \in \mathbb{R}[X_1,\dots,X_N]$. We assume that $P_1,\dots,P_T$ satisfy HDF and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty\\ \mathbf{x}\in J^N}]{} +\infty.$$

Let $\mathbf{k} = (k_1,\dots,k_T) \in \mathbb{N}^T$ and write

$$Q\prod_{t=1}^{T} P_t^{k_t} = \sum_{\boldsymbol{\alpha}\in S} a_{\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}.$$

Let $\chi_1,\dots,\chi_N\colon \mathbb{N}^* \to \mathbb{C}$ periodic of null mean value. Then

$$Z(Q;P_1,\dots,P_T;\chi_1,\dots,\chi_N;-\mathbf{k}) = \sum_{\boldsymbol{\alpha}\in S} a_{\boldsymbol{\alpha}} \prod_{n=1}^{N} L(-\alpha_n,\chi_n).$$

To make the p-adic interpolation, we need the following lemma (this is an exercise in [Kob77, ch. 3]).

Lemma 6.5. *We assume that $\mu \in \mathbb{C}_p$ is a primitive root of unity of order ℓ.*

(a) *If ℓ is not a power of p, then $|\mu-1|_p = 1$.*
(b) *If $\ell = p^h$, then $|\mu-1|_p = p^{-1/p^{h-1}(p-1)}$.*

Now, using the expression of the function $Z(Q;P_1,\dots,P_T;\chi_1,\dots,\chi_N;\cdot)$ in terms of functions $Z(Q;P_1,\dots,\boldsymbol{\mu};\cdot)$, Theorem C, and Lemma 6.5(a), we obtain the following.

Theorem 6.6. *Let p be a prime number. We fix a field morphism from $\mathbb{C}$ into $\mathbb{C}_p$ (left implicit in the discussion and by means of which we calculate $|x|_p$ for any $x \in \mathbb{C}$). Let $Q, P_1,\dots,P_T \in \mathbb{Z}[X_1,\dots,X_N]$ and $\chi_1,\dots,\chi_N\colon \mathbb{N}^* \to \mathbb{C}$ be periodic of null mean value. We assume that the periods are not divisible by p. We assume that $P_1,\dots,P_T$ satisfy HDF, and that*

$$\prod_{t=1}^{T} P_t(\mathbf{x}) \xrightarrow[\substack{|\mathbf{x}|\to+\infty\\ \mathbf{x}\in J^N}]{} +\infty.$$

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We set

$$\tilde{Z}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; \mathbf{s}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^{*N} \\ \forall t \in \{1,\dots,T\},\ p \nmid P_t(\mathbf{m})}} \prod_{n=1}^{N} \chi_n(m_n) Q(\mathbf{m}) \prod_{t=1}^{T} P_t(\mathbf{m})^{-s_t}.$$

Let $\mathbf{r} \in \{0, \dots, p-2\}^T$. *Then there exists* $\tilde{Z}_p^{\mathbf{r}}(Q, P_1; \dots, P_T; \chi_1, \dots, \chi_N; ; \cdot) \colon \mathbb{Z}_p{}^T \to \mathbb{C}_p$ *continuous such that for all* $\mathbf{k} \in \mathbb{N}^T$ *satisfying* $k_t \equiv r_t \mod (p-1)$ *for all* $t \in \{1, \dots, T\}$, *we have*

$$\tilde{Z}_p^{\mathbf{r}}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; -\mathbf{k}) = \tilde{Z}(Q; P_1, \dots, P_T; \chi_1, \dots, \chi_N; -\mathbf{k}).$$

Remark 6.7. If some of the periods of the χ_n are divisible by p, one needs to look at the μ_d whose coefficient a_d in Lemma 6.1 is non-zero. Depending on their p-adic absolute value (calculated in Lemma 6.5), we then may or may not be able to make the p-adic interpolation.

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