THE λ-PROPERTY IN SCHREIER'S SPACE S AND THE LORENTZ SPACE $d(a, 1)$

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0. Abstract. We add Schreier's space S and the Lorentz space $d(a, 1)$ to the list of classical Banach spaces which enjoy the λ-property, investigate the extreme point structure of S, and show that $d(a, 1)$ has a λ-function which is continuous on $S_{d(a,1)}$, though not even uniformly so.

1. Introduction. Let X be a Banach space, B_X the unit ball of X, S_X the surface of B_X, and ext B_X the set of extreme points of B_X. For points $x, y \in X$, we write $[x, y)$ for $\{\lambda x + (1-\lambda)y : 0 < \lambda \leq 1\}$.

Definition 1.1. (a) X has the *λ-property*, if for each $x \in B_X$, there exists $e \in \text{ext } B_X$, $y \in B_X$, $0 < \lambda \leq 1$ such that

$$x = \lambda e + (1-\lambda)y.$$

In this case we say that the triple (e, y, λ) is *amenable* to x, and write $(e, y, \lambda) \sim x$.

(*b*) If X has the λ-property, for each $x \in B_X$, we define

$$\lambda(x) := \sup\{\lambda : (e, y, \lambda) \sim x\}.$$

(c) If there exists $\lambda_0 > 0$ such that $\lambda(x) \geq \lambda_0$, for all $x \in B_X$, we say that X has the *uniform λ-property.*

(d) Finally, we say that X has the *convex series representation property* (C.S.R.P.), if for each $x \in B_X$, there exist $\lambda_n \geq 0$, $e_n \in \text{ext } B_X$, $(n = 1, 2, \ldots)$, such that $x = \sum_n \lambda_n e_n$ and $\sum_n \lambda_n = 1$.

These notions were developed by R. Aron and R. H. Lohman in [**1**], where (among other results) they proved: the uniform λ-property implies C.S.R.P. An easy exercise shows that C.S.R.P. implies the λ-property. Spaces that enjoy either the λ-property or the uniform λ-property are not rare [**1**], [**3**], [**4**], [**8**], and it is our belief that some strong theorems are lurking behind these concepts. In an attempt to better understand these properties we decided to investigate a couple of "exotic" sequence spaces. We begin with Schreier's space S.

2. Schreier's space S.

Definition 2.1. (a) Let $R^{(N)}$ denote the (vector) space of real sequences $x = (x(1), x(2), \ldots)$ which are finitely-non-zero (i.e., have "finite support"). A subset E of the natural numbers N is *admissible*, if $E = \{n_1, n_2, \ldots, n_k\}$, with $k \leq n_1 < n_2 \ldots < n_k$. We denote by $\mathcal{A}$ the collection of all admissible subsets of N.

(*b*) For $x \in R^{(N)}$, we define

$$\|x\|_S := \sup_{E \in \mathcal{A}} \sum_{j \in E} |x(j)|.$$

(Routine calculations show that $\|\cdot\|_S$ is a norm on $R^{(N)}$.)

(c) *Schreier's space* S is the $\|\cdot\|_S$-completion of $R^{(N)}$. (From here on, we will write "$\|\cdot\|$", for $\|\cdot\|_S$".)

The space S has been studied extensively in **[2]**, where it is shown that S is hereditarily-c_0; (hence l_1 does not embed in it). In this section we shall show that S has enough extreme points to enjoy C.S.R.P., even though we fall short of a useful characterization of ext B_S. First we note that S is not c_0 in disguise.

PROPOSITION 2.2. *S is not isomorphic to c_0.*

Proof. If we denote by $\{s_n\}_{n=1}^{\infty}$ the canonical unit vector basis for S, then for each n, l_1^n is isometric to the norm-one complemented subspace of S spanned by $\{s_i : n+1 \le i \le 2n\}$. Thus (see **[6**, p. 74**]**) S^* fails to have finite co-type. Hence S^* is not isomorphic to c_0^*, and so S is not isomorphic to c_0.

For each n, let $S_n := \text{span}\{s_i : i \le n\}$. Since S_n is finite-dimensional, ext $B_n \ne \varnothing$ (where $B_n := B_{S_n}$). In fact we shall show that

$$\text{ext } B_n \cap \text{ext } B_S \ne \varnothing.$$

The reader can easily show (by using 1-sets introduced below) that the vectors $(1, 1)$, $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $(1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are all in ext B_S (when we write $x = (x(1), x(2), \ldots, x(n))$, we mean $x(j) = 0$ when $j > n$).

DEFINITION 2.3. Let $x \in B_S$.

(a) If $E \in \mathcal{A}$, and $\sum_{j \in E} |x(j)| = 1$, we say that E is a *1-set* for x.

(b) If (in addition) $E = \{n_1 < n_2 < \ldots < n_k\}$ and $k < n_1$, we say E is a *non-maximal 1-set*, for x.

Since for $E \in \mathcal{A}$, $x \to \sum_{j \in E} |x(j)|$ is a semi-norm, we clearly have the following result.

LEMMA 2.4. *Let $x, b, c \in B_S$ with $x = \lambda b + (1-\lambda)c$ for some $0 < \lambda < 1$. Then any 1-set for x is a 1-set for b and c.*

A slight modification of the above shows that for vectors $x, b_1, b_2, \ldots$ in B_S and scalars $\lambda_1, \lambda_2, \ldots$ each >0 with $\sum_n \lambda_n = 1$ and $x = \sum_n \lambda_n b_n$, every 1-set for x is a 1-set for each b_n.

LEMMA 2.5. *Let $n \ge 1$ and $x \in$ ext B_n. If x has a non-maximal 1-set E, then $x \in$ ext B_S.*

Proof. Clearly we may assume that $\max E \le n$. Suppose $x = \lambda b + (1-\lambda)c$, for some $0 < \lambda < 1$ and some $b, c \in B_S$. If E is a non-maximal 1-set for x, then, by Lemma 2.4, E is a non-maximal 1-set for b and c. So $b(j) = 0 = c(j)$, for $j > n$, since $E \cup \{j\} \in \mathcal{A}$ for every $j > n$. But $x(j) = b(j) = c(j)$ for $j \le n$, since $x \in$ ext B_n. Thus $x = b = c$, and $x \in$ ext B_S.

We note that ext $B_n \not\subset$ ext B_S. Some calculations show that $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \text{ext } B_5 \sim \text{ext } B_S$, for instance. To show that S has C.S.R.P. we need some lemmata about certain representations.

LEMMA 2.6. *Let $x \in B_S$. Then for all $\epsilon > 0$, there exists N such that for $E \in \mathcal{A}$ with $N < \min E$, we have $\sum_{j \in E} |x(j)| < \epsilon$.*

Proof. If $x \in B_S$, then $\|x - y\| < \varepsilon$ for some finite vector $y \in B_S$.

LEMMA 2.7. *Let $x \in B_S$ have infinite support. Then there exist vectors $b, c \in B_s$ such that $x = \frac{1}{2}b + \frac{1}{2}c$, and b has finite support.*

Proof. Without loss of generality we may assume $\|x\| = 1$. Let

$$\alpha := \min\{|x(j)| : j \in E \in \mathscr{A},\ E \text{ is a 1-set for } x,\ x(j) \neq 0\},$$

and

$$\epsilon := \min\left\{1 - \sum_{j\in E} |x(j)| : E \in \mathscr{A},\ \sum_{j\in E} |x(j)| < 1\right\}.$$

Lemma 2.6 implies that $\epsilon > 0$. Choose an integer M larger than any element in any 1-set for x and larger than any element in any $E \in \mathscr{A}$ which determines ϵ. Finally, enlarge M (if needed) so that for $E \in \mathscr{A}$, $\min E \geq M$ implies

$$\sum_{j\in E} |x(j)| < \tfrac{1}{2}\min\{\alpha, \epsilon\}.$$

Now define vectors b and c by

$$\begin{cases} b(j) = x(j) = c(j), & \text{for } \ 1 \leq j < M, \\ b(j) = 0, & \text{for } \ j \geq M, \\ c(j) = 2x(j), & \text{for } \ j \geq M. \end{cases}$$

The only thing left to show is that $c \in B_S$. Towards this end, let $E \in \mathscr{A}$. If $\max E < M$, then $\sum_{j\in E} |c(j)| \leq \|x\| = 1$. If $\max E \geq M$, then

$$\sum_{j\in E} |c(j)| = \sum_{\substack{j\in E\\ j<M}} |c(j)| + \sum_{\substack{j\in E\\ j\geq M}} |c(j)| < 1 - \epsilon + 2 \cdot \frac{\epsilon}{2} = 1.$$

Note that for $x \in B_S$, applying the above Lemma recursively we obtain a representation $x = \sum_i 2^{-i} b_i$, where each $b_i \in B_S$ and each b_i has finite support. Also note that we immediately obtain the following result.

COROLLARY 2.8. *If $x \in \operatorname{ext} B_S$, then x has finite support.*

In fact, we can show more.

LEMMA 2.9. *Let $x \in B_S$ have finite support. Then x can be represented as $x = \frac{1}{2}b + \frac{1}{2}c$, for two vectors $b, c \in B_S$ each of finite support, and each having a non-maximal 1-set.*

Proof. Without loss of generality, we may assume $x \neq 0$. Let $N = \max(\text{support } x) + 1$, and let $\epsilon = \min\left\{1 - \sum_{j\in E} |x(j)| : E = \{n_1, \ldots, n_k\},\ k < n_1 < N\right\}$. (If $\epsilon = 0$, then x already has a non-maximal 1-set, and we can choose $b = c = x$.) Choose $M > N$ such that $\frac{N-2}{M} < \epsilon$. (The case where $N \leq 2$ is trivial.)

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Define vectors b and c via

$$\begin{cases} b(j) = x(j) = c(j), & \text{for} \quad 1 \le j \le M, \\ b(j) = 0 = c(j), & \text{for} \quad j > 2M, \\ b(j) = \dfrac{1}{M} = -c(j), & \text{for} \quad M+1 \le j \le 2M. \end{cases}$$

Clearly, $x = \frac{1}{2}b + \frac{1}{2}c$, and $\|b\| = \|c\|$.

To show that $\|b\| \le 1$, let $E \in \mathscr{A}$. If $\min E \ge N$, then $\sum_{j \in E} |b(j)| \le M \cdot \frac{1}{M} = 1$. If $\max E < N$, then $\sum_{j \in E} |b(j)| = \sum_{j \in E} |x(j)| \le 1$. In the only remaining case

$$\sum_{j \in E} |b(j)| = \Big(\sum_{\substack{j \in E \\ j < N}} + \sum_{\substack{j \in E \\ j \ge N}} \Big) |b(j)|$$
$$\le 1 - \varepsilon + \frac{N-2}{M} < 1.$$

So $\|b\| = \|c\| = 1$, and each has $\{M+1, M+2, \ldots, 2M\}$ for a non-maximal 1-set.

THEOREM 2.10. *Schreier's space S has C.S.R.P.*

Proof. Let $x \in B_S$. By the remark following Lemma 2.7, we may write $x = \sum_n 2^{-n} b_n$, where $\|b_n\| = 1$ and support b_n is finite, $(n = 1, 2, \ldots)$. Using Lemma 2.9 on each b_n, we can write $x = \sum_j \lambda_j c_j$, for some choices of λ_j and c_j such that $\sum_j \lambda_j = 1$, $\|c_j\| = 1$, and each vector c_j has finite support and a non-maximal 1-set. Now each vector c_j belongs to some S_n, where $n := n(j)$. Since S_n has C.S.R.P. [2], for each j we can write $c_j = \sum_i \lambda_{j,i} e_{j,i}$, a convex series where the $e_{j,i} \in \operatorname{ext} B_n$. Finally $x = \sum_{i,j} \lambda_{j,i} e_{j,i}$, and the vectors $e_{j,i}$ all belong to $\operatorname{ext} B_S$, by Lemmas 2.4 and 2.5.

This of course implies that S has the λ-property although we do not know whether it has the uniform λ-property. We mention here that the extreme points of B_S all have supports with even cardinality (we omit the proof). It is of interest to note the following result.

PROPOSITION 2.11. *$\operatorname{ext} B_S$ is countable.*

Proof. The earlier lemmas show that $\operatorname{ext} B_S \subset \bigcup_n \operatorname{ext} B_n$. We now show that each $\operatorname{ext} B_n$ is finite. Since B_n is compact, it suffices to show that for each $x \in B_n$, there is a ball (in the B_n topology) of radius $\epsilon = \epsilon(x)$ such that this ball meets $\operatorname{ext} B_n$ (at most) at the point x. Let $x \in B_n$, and assume $\|x\| = 1$. Define

$$\delta_1 = \min\{|x(j)| : x(j) \ne 0\},$$

$$\delta_2 = \min\Big\{1 - \sum_{j \in E} |x(j)| : E \in \mathscr{A} \text{ and } \sum_{j \in E} |x(j)| < 1\Big\}.$$

Let $\delta = \frac{1}{2}\min\{\delta_1, \delta_2\}$, and choose $\epsilon > 0$ so that $2n\epsilon < \delta$.

Suppose $y \in B_n$ with $\|x - y\| < \epsilon$. Note that by choice of ϵ, whenever $x(j) \ne 0$, $x(j)$

and $y(j)$ have the same sign. Now define z by

$$z(j)=\begin{cases}0, & \text{if } j>n,\\ 2y(j)-x(j), & \text{if } j\le n.\end{cases}$$

Clearly $z\in S_n$ and $y=\frac{1}{2}x+\frac{1}{2}z$. If we can show $z\in B_n$, then $y\notin \operatorname{ext} B_n$, unless $y=x$ and $x\in\operatorname{ext} B_n$. Note that ϵ was chosen small enough so that $x(j)$, $y(j)$, and $z(j)$ have the same sign as j ranges over the support of x. So for all j, we have

$$x(j)-z(j)=2(x(j)-y(j)),$$
$$|x(j)|-|z(j)|=2(|x(j)|-|y(j)|).$$

Letting $E\in\mathscr{A}$, we may assume $E\subset\{1,\ldots,n\}$. If E is not a 1-set for x, then

$$\sum_{j\in E}|z(j)|\le\sum_{j\in E}|x(j)|+2n\epsilon<1.$$

If E is a 1-set for x, then letting $E_0=\{j\in E: x(j)=0\}$, and $E_1=E\backslash E_0$, we have

$$\begin{aligned}\sum_{j\in E}|z(j)|&=\sum_{j\in E_1}|z(j)|+\sum_{j\in E_0}|z(j)|\\ &=\sum_{j\in E_1}|x(j)|-\sum_{j\in E_1}(|x(j)|-|z(j)|)+\sum_{j\in E_0}|z(j)|\\ &=\sum_{j\in E_1}|x(j)|-2\sum_{j\in E_1}(|x(j)|-|y(j)|)+2\sum_{j\in E_0}|y(j)|\\ &=2\sum_{j\in E}|y(j)|-\sum_{j\in E_1}|x(j)|\le 1.\end{aligned}$$

Thus $\|z\|\le 1$.

3. The Lorentz sequence space $d(a,1)$. We consider here Lorentz sequence spaces of type $d(a,1)$. These "weighted" versions of l_1 turn out to have the λ-property, while failing the uniform λ-property. This was demonstrated in Theorems 5 and 6 in [**8**], both of which we improve here by producing the exact form of the λ-function for norm-one vectors. This is then used to prove a continuity result. First we establish some definitions and notation.

Definition 3.1. Let $a=(a_n)\in c_0\backslash l_1$ be a positive strictly decreasing sequence with $a_1=1$. The space $d(a,1)$ consists of all real sequences $x=(x(n))\in c_0$ such that $\sup\sum|x(\pi(n))|\,a_n<\infty$, where the supremum is taken over all permutations π of the natural numbers. (If $\|x\|$ is taken to be this supremum, then $d(a,1)$ is a Banach space.)

If $x=(x(n))\in d(a,1)$, and $x\ne 0$, we write

$$M_1(x)=\|x\|_\infty,\quad\text{and}\quad F_1(x)=\{n:|x(n)|=M_1(x)\},$$
$$M_2(x)=\|x-xc_{F_1(x)}\|_\infty,\qquad F_2(x)=\{n:|x(n)|=M_2(x)\},$$

where $c_{F_1(x)}$ is the characteristic function of $F_1(x)$, etc. Then $M_k(x)\downarrow 0$, and if $M_k(x)>0$, then $M_k(x)>M_{k+1}(x)$. Also $F_k(x)$ and $F_j(x)$ are disjoint if $M_k(x)$, $M_j(x)>0$ and $k\ne j$. Let $N(x)=\{k:M_k(x)-M_{k+1}(x)>0\}$, and for $k\in N(x)$, define $n_k(x)=\operatorname{card}\left(\bigcup_{i=1}^{k}F_i(x)\right)$, and $s_k(x)=\sum_{n=1}^{n_k(x)}a_n$.

If we let $n_0(x)=0$, then we can write $\|x\|$ as

$$\|x\| = \sum_{k\in N(x)} M_k(x)\,.\,(s_k(x)-s_{k-1}(x)).$$

Importantly, for $x \in d(a, 1)$, $\|x\|$ can also be realized in another way.

PROPOSITION 3.2. *For any* $x \in d(a, 1)$,

$$\|x\| = \sum_k M_k(x)\,.\,[s_k(x)-s_{k-1}(x)] = \sum_n [M_n(x)-M_{n+1}(x)]\,.\,s_n(x).$$

Proof. It suffices to note that either sum is equal to

$$\sum_k \sum_{j\le k} (M_k(x)-M_{k+1}(x))(s_j(x)-s_{j-1}(x))$$

The extreme points of $B_{d(a,1)}$ were characterized by W. J. Davis [**10**].

PROPOSITION 3.3. $e \in \operatorname{ext} B_{d(a,1)}$ *if and only if e has the form*

$$e = \left(\sum_{n=1}^{k} a_n\right)^{-1}\left(\sum_{n=1}^{k} \epsilon_n x_{i_n}\right),$$

for some integer k, $i_1 < i_2 < \ldots < i_k$, *and signs* $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$, (*where* (x_i) *is the canonical unit vector basis of* $d(a, 1)$.)

Using this characterization, we can establish the following result.

PROPOSITION 3.4. *The space* $d(a, 1)$ *has C.S.R.P.*

Proof. Assume first that $\|x\|=1$, and that x has the form

$$x = (x(1) \ge x(2) \ge \ldots \ge x(k) > 0).$$

For any j, define $s_j = \sum_{i=1}^{j} a_i$, and denote by e_m that extreme point with non-negative coefficients and support $=\{1, 2, \ldots, \mathrm{m}\}$. Further denote by v^n that vector defined by $v^n(i)=1$, if $1 \le n$, and 0, otherwise. Then

$$\begin{aligned} x &= (x(1), x(2), \ldots, x(k), 0, \ldots)\\ &= x(k)\,.\,v^k + (x(1)-x(k), \ldots, x(k-1)-x(k), 0, \ldots)\\ &= \ldots = x(k)v^k + (x(k-1)-x(k))v^{k-1}\\ &\quad + (x(k-2)-x(k-1)v^{k-2} + \ldots (x(2)-x(3))v^2 + (x(1)+x(2))v^1\\ &= [x(k)s_k]e_k + [(x(k-1)-x(k))s_{k-1}]e_{k-1}\\ &\quad + [(x(k-2)-x(k-1))s_{k-2}]e_{k-2} + \ldots\\ &\quad + [(x(2)-x(3))s_2]e_2 + [(x(1)-x(2))s_1]e_1. \end{aligned}$$

Let $\alpha_l = (x(l)-x(l+1))s_l$, $(l = k, k-1, \ldots, 2, 1)$ and note that

$$\begin{aligned} \alpha_k + \alpha_{k-1} + \ldots + \alpha_1 &= x(k)(s_k - s_{k-2}) + x(k-1)(s_{k-1}-s_{k-2})\\ &\quad + \ldots + x(2)(s_2 - s_1) + x(1)s_1\\ &= x(k)a_k + x(k-1)a_{k-1} + \ldots + x(2)a_2 + x(1)a_1\\ &= \|x\| = 1. \end{aligned}$$

Now assume that $\|x\|=1$ and that x has the form $x=(x(1)\geq x(2)\geq\ldots>0)$. Then (using the notation above) $1=\|x\|=\lim_k \sum_{i=1}^{k} a_i x(i)=\lim_k \sum_{i=1}^{k} \alpha_i$. Arbitrary vectors x with $\|x\|=1$ are an isometry away from the two cases already considered, and if $\|x\|<1$,

$$x=\|x\|\,.\,\frac{x}{\|x\|}+\frac{1-\|x\|}{2}\,.\,e+\frac{1-\|x\|}{2}(-e)$$

(where e is *any* extreme point), leads to a convex series representation.

Proposition 3.4 implies that $d(a,1)$ has the λ-property, but we can say more. In **[8]** a lower bound is proven for the λ-function.

$$\text{If } x\in B_{d(a,1)},\ x\neq 0,\quad \text{then}\quad \lambda(x)\geq \sup_{k\in N(x)}\,[M_k(x)-M_{k+1}(x)]s_k(x). \tag{$*$}$$

In the same paper an exact formula is given for unit vectors of finite support.

If $x\in d(a,1)$ with $\|x\|=1$, and support x is finite, then

$$\lambda(x)=\max_{k\in N(x)}\,[M_k(x)-M_{k+1}(x)]s_k(x). \tag{$**$}$$

Proposition 3.2 allows us to replace the "sup" in ($*$) by a "max", and we can now remove the hypothesis about support x in ($**$).

Using the results above, we can also establish the following theorems.

THEOREM 3.5. *Assume* $x\in d(a,1)$, $\|x\|=1$. *Then*

$$\lambda(x)=\max_n\,[M_n(x)-M_{n+1}(x)]\,.\,s_n(x)$$

THEOREM 3.6. *The λ-function for $d(a,1)$ is continuous on $\{x:\|x\|=1\}$, Lipschitz-continuous on $\{x:\|x\|\leq r\}$, $(0<r<1)$, though not even uniformly continuous on $\{x:\|x\|=1\}$.*

Consideration of space forces us to omit proofs of these last two results, which will appear in **[9]**.

REMARK. R. H. Lohman **[7]** has recently shown that for Banach spaces the λ-property is equivalent to the C.S.R.P.

REFERENCES

1. R. M. Aron and R. H. Lohman, A geometric function determined by extreme points of the unit ball of a normed space, *Pacific J. Math.* **127** (1987), 209–236.

2. P. G. Casazza and T. J. Shura, *Tsirelson's space,* Lecture Notes in Mathematics No 1363 (Springer-Verlag, 1989).

3. A. S. Granero, On the Aron–Lohman's λ-property (preprint).

4. A. S. Granero, The λ-function in the spaces $(\oplus\sum X)_p$ and $L_p(\mu, X)$ (preprint).

5. P. R. Halmos, *A Hilbert space problem book,* (Van Nostrand, 1967).

6. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*; *function spaces* (Springer-Verlag, 1979).

7. R. H. Lohman, The λ-property in Banach spaces, in *Banach Space Theory, Contemporary Mathematics* **85** (1989), 345–354.

8. R. H. Lohman, T. J. Shura, Calculation of the λ-function for several classes of normed linear spaces, in *Nonlinear and convex analysis: Proceedings in honor of Ky Fan* (Marcel Dekker, 1987).

9. T. J. Shura, Dissertation, Kent State University, (to appear).

10. W. J. Davis, Positive bases in Banach spaces, *Rev. Roumaine Math. Pures Appl.*, **16** (1971), 487–492.

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