STABILITY OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH RESPECT TO A CLOSED SET

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1. Introduction. The stability of the solutions of an ordinary differential equation will be discussed here. The purpose of this note is to compare the stability results which are valid with respect to a compact set and the stability results valid with respect to an unbounded set. The stability of sets is a generalization of stability in the sense of Liapunov and has been discussed by LaSalle (**5**; **6**), LaSalle and Lefschetz (**7**, p. 58), and Yoshizawa (**8**; **9**; **10**).

The following examples indicate the importance of considering the stability of a given set as opposed to the stability of the origin in the sense of Liapunov. Consider the system

$$\text{(a)} \qquad \dot{x} = (x - y)(1 - x^2 - y^2), \qquad \dot{y} = (x + y)(1 - x^2 - y^2),$$

where the dot denotes the derivative with respect to t. The circle $x^2 + y^2 = 1$ is not a limit cycle of the system (a); however, the phase curves do approach this circle asymptotically. (This example may be found in (**1**, p. 18).) It is noted that in this example the asymptotic curve is compact.

For an example in a non-compact case, consider the differential equation

$$\text{(b)} \qquad y'' + g_1(t)y' + g_2(t)y = h(t) \qquad (y' = dy/dt),$$

where $h(t) = bt^m + R(t)$ with $b \neq 0$, $m > -1$, and $R(t) = o(t^m)$ as $t \to \infty$. Furthermore, let $g_i(t)$ and $h(t)$ be continuous and $\lim_{t\to\infty} t^{p_i} g_i(t) = c_i \neq 0$ with $p_i > 2 - i$, $i = 0, 1$. Under these hypotheses $y(t)$ has the asymptotic behaviour $y^{(i)}(t)/t^{m+2-i} a_i \neq 0$, $i = 0, 1$; see (**3**). However, it is clear that $y = a_0 t^{m+2}$ is not necessarily a solution of the differential equation (b).

As indicated in the above examples, a family of solutions of a differential equation may approach asymptotically a curve which need not be a solution of the differential equation. This fact motivates the study of the stability of sets.

LaSalle (**5**) has indicated that the classical theorems on the stability theory of equilibrium points may be easily extended to the study of the stability of compact sets; however, for an unbounded manifold this generalization is not immediate. Also, a physical problem in control theory is mentioned in (**5**), in which it is required to determine the stability of the solutions of a system of

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differential equations with respect to a non-compact manifold. A more specialized problem of stability arises in the synthesis problem of control theory. (See, for example, (**4**, Chapter 10).)

In § 2 of this paper we indicate the extension of the theorems of Liapunov and Ĉetaev to compact sets. Section 3 will be devoted to some companion theorems which are valid for arbitrary closed sets; in particular, we note that the case where the given set is unbounded is included. It should be observed that should the given set be an integral curve of the differential equation, then known stability theorems may be applied.

The definitions which will be required in this note are given below. Consider

$$\dot{x} = f(t, x) \qquad (\dot{x} \equiv dx/dt), \tag{1}$$

where x is a vector defined in an open set $\Omega \subseteq R^n$, n-dimensional Euclidean space. Suppose $f(t, x)$ is a continuous function whose domain is $I \times \Omega$, $I = [0, \infty)$. The distance between the point p and the set A will be denoted by $\rho(p, A)$, i.e., $\rho(p, A) = \inf_{a \in A} ||p - a||$; where $||\cdot||$ is the Euclidean norm. The set of points x in R^n such that $\rho(x, A) < \epsilon$ will be called an ϵ-neighbourhood of the set A. A solution $x(t)$ of (1) such that $x(t_0) = x_0$ will be denoted by $x(t; t_0, x_0)$.

Definition 1. Let Γ be a closed subset of Ω. The differential equation (1) *is stable with respect to* Γ if for any $\epsilon > 0$ and t_0 in I, there exists a $\delta(t_0, \epsilon) > 0$ such that if $x_0 \in \Omega$ and $\rho(x_0, \Gamma) < \delta$, then $x(t; t_0, x_0)$ remains in Ω and $\rho(x(t; t_0, x_0), \Gamma) < \epsilon$ for all $t \geqslant t_0$.

Definition 2. Differential equation (1) *is unstable with respect to* Γ if (1) is not stable with respect to Γ.

Definition 3. Differential equation (1) *is asymptotically stable with respect to* Γ if (1) is stable with respect to Γ and $\lim_{t\to\infty} \rho(x(t; t_0, x_0), \Gamma) = 0$ if $x_0 \in \Omega$.

2. The compact case. First, LaSalle's remark concerning the extension of known stability theorems to a compact set will be confirmed in the case of stability and in other cases the statements of the generalizations will be given. We shall consider here only the simple case where the Liapunov function depends upon x only. See (**7**) for related theorems valid in the conventional case.

Remark 1 (A generalization of Liapunov's theorem). Let Γ be a compact point set contained in Ω. Let there exist a function $V(x)$ of class C^1 in Ω such that

(i) $V(x) \geqslant 0$ if $x \in \Omega$;
(ii) $V(x) = 0$ if and only if $x \in \Gamma$;
(iii) along any trajectory $x = x(t; t_0, x_0)$ of (1)

$$dV(x(t; t_0, x_0)/dt \leqslant 0, \qquad x \in \Omega,\ t \geqslant t_0.$$

Subject to the above hypotheses, (1) is stable with respect to Γ.

Proof of Remark 1. Let $\epsilon > 0$ be given. Without loss of generality, we assume that R_ϵ, the ϵ-neighbourhood of Γ, is contained in Ω. Since Γ is compact, $\bar{R}_\epsilon$, the closure of R_ϵ, is compact. It is sufficient to prove that some level surface $V(x) = c$, c a non-zero constant, lies entirely in R_ϵ since δ may be taken to be any radius of a neighbourhood of Γ contained in $V(x) = c$. Suppose that no such level surface exists; then, let $\{x_k\}$ denote the sequence of points which are situated on the intersection of the boundary ∂R_ϵ, of R_ϵ, with the level surfaces $V(x) = 1/k$, $k = 1, 2, 3, \ldots$. Since ∂R_ϵ is compact, this sequence has a limit point, $\hat{x}$, on ∂R_ϵ. By the continuity of $V(x)$, we obtain

$$V(\hat{x}) = \lim_{k\to\infty} V(x_k) = 0.$$

However, this implies that $\hat{x}$ lies on Γ, which is not the case, and completes the proof of the theorem.

Remark 2 (Asymptotic stability theorem). Let the conditions (i) and (ii) of Remark 1 be imposed. Furthermore, let

(iii*) along any trajectory $x = x(t; t_0, x_0)$ of (1)

$$dV(x(t; t_0, x_0))/dt < 0 \qquad \text{if } x \in \Omega - \Gamma,\ t \geqslant t_0,$$

be satisfied. Under these hypotheses, (1) is asymptotically stable with respect to Γ.

The proof follows the classical case (**7**, p. 38) and will be omitted. This pattern will continue throughout the next two remarks.

Remark 3 (The first instability theorem of Liapunov). Let

$$dV(x(t; t_0, x_0))/dt > 0$$

along any trajectory $x = x(t; t_0, x_0)$ of (1), $x \in \Omega - \Gamma$, $t \geqslant t_0$. Suppose that $V(x)$ assumes positive values in every sufficiently small neighbourhood of Γ. Then the differential equation (1) is unstable with respect to Γ.

Remark 4 (Ĉetaev instability theorem). Let Γ be a compact set contained in Ω. Let there exist a function $V(x)$, a region Γ_ϵ contained in R_ϵ, an ϵ-neighbourhood of Γ, with the following properties:

(i) $V(x)$ is of the class C^1 in Γ_ϵ;

(ii) $V(x)$ and dV/dt are positive in Γ_ϵ;

(iii) $V(x) = 0$ if x is on that part of the boundary of Γ_ϵ which is contained in R_ϵ;

(iv) Γ is contained in the boundary of Γ_ϵ.

Under these conditions, system (1) is unstable with respect to Γ.

We now present some general comments. First, note that our assumption regarding the Liapunov function $V(x)$, i.e. $V(x) > 0$ if $x \notin \Gamma$, $V(x) = 0$ if $x \in \Gamma$, and $dV/dt \leqslant 0$, makes Γ an invariant set for the differential equation (1). However, if Γ contains no trajectory, then no trajectory may intersect Γ either. Also, we observe that if $V(x) \geqslant 0$, there can be no trajectory

$x = x(t; t_0, x_0)$ such that $dV(x(t; t_0, x_0))/dt < c < 0$ (c a constant) along the entire trajectory for $t \geqslant t_0$. This is an immediate consequence of the mean-value theorem.

3. The unbounded case. The case where Γ is an unbounded set will now be considered. If Γ is an unbounded set, then the obvious generalizations of the classical stability theorems, as given in the preceding section, are incorrect. The following examples verify this statement.

Example 1. Consider the system

$$\text{(c)} \qquad \dot{x} = x, \qquad \dot{y} = y,$$

where the dot denotes the derivative with respect to t ($t \geqslant 0$). Let

$$\Omega = \{(x, y) \colon y > 1\}$$

and let Γ be the y axis. Consider as the Liapunov function

$$V(x, y) = (x/y)^2 + \begin{cases} ce^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where c is a non-negative constant. Along any trajectory on which $x \not\equiv 0$, $\dot{V}(x, y)$ is non-positive, and on Γ, $\dot{V}(x, y) = 0$. We observe that when $c=0$ the Liapunov function is $V(x, y) = (x/y)^2$; in this case, $\dot{V} \equiv 0$. The Liapunov function satisfies the hypotheses of Remark 1; however, the solutions of (c),

$$x = c_1e^t, \qquad y = c_2e^t$$

are not stable with respect to Γ. Thus, if Γ is an unbounded set, then the Liapunov stability theorem is not necessarily true.

Example 2. In this example the system

$$\text{(d)} \qquad \dot{x} = -x, \qquad \dot{y} = -y$$

will be considered. Let $\Omega = \{(x, y) | \, x \geqslant 0, y > x\} \cup \{(x, y) | \, x < 0, y > -x\}$ and Γ be the y axis; and select as the Liapunov function

$$V(x, y) = (x/y)^2 + \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

With V chosen as above, the hypotheses of Remark 3 are satisfied; however, system (d) is stable with respect to Γ. This example also serves as a counterexample to a "generalization" of the Ĉetaev instability theorem as given in Remark 4 above. To observe this fact, choose as the set Γ_ϵ of that remark the set $x < 0$ in Ω.

Some theorems which will be valid for unbounded sets will now be given. The additional hypotheses required involve the control of the growth of the Liapunov function by suitable functions of the distance. The given closed set will be denoted by Γ as usual, and we require that $\Gamma \subset \Omega$. Related theorems

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in the traditional case may be found in Halanay **(2)**. It will also be necessary to assume that all solutions of (1) exist in the future.

THEOREM 1. *Let there exist a function* $V(t, x)$ *of class* C^1 *in* Ω *satisfying the following conditions*:

(i) $V(t, x) \equiv 0$ *if* $x \in \Gamma$, $t \geqslant 0$;

(ii) $V(t, x) \geqslant a(r(x))$, *where* $r(x) \equiv \rho(x, \Gamma)$ *and* $a(r)$ *is a continuous monotone increasing function of* r *such that* $a(0) = 0$;

(iii) *along any trajectory* $x = x(t; t_0, x_0)$ *of* (1) *where* $x_0 \in \Omega$

$$dV(t, x(t; t_0, x_0))/dt \leqslant 0.$$

Subject to these hypotheses, system (1) *is stable with respect to the set* Γ.

Proof. Let $\epsilon > 0$ be given where ϵ is sufficiently small so that the ϵ-neighbourhood of Γ is contained in Ω. Since $V(t, x)$ is continuous, a point $x_0' \in \Omega$ may be selected such that $V(t_0, x_0') < a(\epsilon)$. The set of all x such that $r(x) < r(x_0')$ constitutes a neighbourhood, Γ_δ, of Γ. Suppose x_0 is a point of $\Gamma_\delta \cap \Omega$, $x_0 \notin \Gamma$. Then, along the trajectory $x = x(t; t_0, x_0)$, $V(t, x(t; t_0, x_0))$ is a non-increasing function. Therefore

$$a(r(x(t; t_0, x_0))) \leqslant V(t, x(t; t_0, x_0)) < a(\epsilon), \qquad t \geqslant t_0.$$

Since $a(r)$ is a monotone increasing function of r, we have $r(x(t; t_0, x_0)) < \epsilon$ for every $t \geqslant t_0$. Thus, system (1) is stable with respect to Γ.

Remark 5. The structure of the set Γ together with the existence of a Liapunov function satisfying conditions (i), (ii), and (iii) of Theorem 1 can determine the existence of the solutions in the future; and, therefore, in some circumstances this hypothesis can be removed. For example, if Γ is compact, the existence of such a Liapunov function implies that all solutions of (1) exist in the future. If Γ is not bounded and there exists a Liapunov function having the above properties, then the solutions need not exist in the future. An example which illustrates this fact is the following.

Example 3. The differential equation is

$$x' = \begin{cases} -x, & x \geqslant 0, \\ -x^2, & x < 0. \end{cases} \tag{e}$$

Let Γ be the set $x < 0$; and consider the Liapunov function

$$V(x) = \begin{cases} x^2, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

This Liapunov function satisfies conditions (i), (ii), and (iii), but the solutions of (e) do not exist in the future. The authors would like to thank Professor Taro Yoshizawa for this example.

If some solution $x(t; t_0, x_0)$ of equation (1) does not exist in the future, say $x(t; t_0, x_0)$ exists on the interval $[t_0, T)$, then

$$\limsup_{t \to T^-} \rho(x(t; t_0, x_0), \Gamma) < \infty.$$

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Suppose that this limit is infinite; then there exists a $t = t_1$ close to T, $t_1 < T$, such that

$$V(t_0, x_0) < a(r(x(t_1; t_0, x_0))) \leqslant V(t_1, x(t_1; t_0, x_0)).$$

However, since V satisfies (iii), then

$$V(t_0, x_0) \geqslant V(t, x(t; t_0, x_0))$$

for all t such that $x(t; t_0, x_0)$ exists, and we are led to a contradiction.

THEOREM 2. *The system* (1) *is unstable with respect to* $\Gamma \subset \Omega$ *if there exists a function* $V(t, x)$ *defined in* $[0, \infty) \times \Omega$ *with the following properties*:

(i) $V(t, x) \leqslant b(r(x))$, *where* $r(x) \equiv \rho(x, \Gamma)$ *and* $b(r)$ *is a continuous monotone increasing function of the distance* $r(x)$;

(ii) *for every* $\delta > 0$ *and every* $t_0 > 0$ *there exists an* $x_0 \in \Omega$ *such that* $\rho(x_0, \Gamma) < \delta$ *and* $V(t_0, x_0) > 0$;

(iii) $\dot{V}(t, x(t; t_0, x_0) \geqslant c(r(x))$, *where* $c(r)$ *is a continuous monotone increasing function of* $r(x)$ *with* $c(0) = 0$.

Proof. Suppose that system (1) is stable with respect to Γ. Then, for every $\epsilon > 0$ and $t_0 > 0$ there exists a $\delta(\epsilon, t_0) > 0$ such that $\rho(x_0, \Gamma) < \delta$ implies that $\rho(x(t; t_0, x_0), \Gamma) < \epsilon$ for $t \geqslant t_0$. Choose x_0 such that $\rho(x_0, \Gamma) < \delta$ and $V(t_0, x_0) > 0$; the existence of such a choice is guaranteed by hypothesis (ii).

For $t \geqslant t_0$, we have

$$V(t, x(t; t_0, x_0)) \leqslant b(r(x(t; t_0, x_0))) \leqslant b(\epsilon). \tag{2}$$

From the fact that $V(t(x(t; t_0, x_0))$ is monotone increasing, we obtain

$$b(r(x(t; t_0, x_0))) \geqslant V(t_0, x_0).$$

Hence,

$$r(x(t; t_0, x_0)) \geqslant b^{-1}[V(t_0, x_0)]. \tag{3}$$

Integration of the inequality in (iii) along the trajectory $x(t; t_0, x_0)$ and (3) above lead to

$$\begin{aligned} V(t, x(t; t_0, x_0)) &\geqslant V(t_0, x_0) + \int_{t_0}^{t} c[r(x(s; t_0, x_0))]ds \\ &\geqslant V(t_0, x_0) + c[b^{-1}(V(t_0, x_0))](t - t_0). \end{aligned}$$

Using the above inequality, it is clear that if t is sufficiently large,

$$V(t, x(t; t_0, x_0)) > b(\epsilon),$$

contradicting (2). This completes the proof of the theorem.

THEOREM 3. *Let there exist a continuous function* $V(t, x)$ *defined for* $t \geqslant 0$, $x \in \Omega$ *with the following properties*:

(i) $V(t, x) \equiv 0$ *if* $x \in \Gamma$;

(ii) $V(t, x) \geqslant a(r(x))$, *where* $a(r)$ *is a continuous monotone increasing function of* r, *with* $r \equiv \rho(x, \Gamma)$ *and* $a(0) = 0$;

(iii) $\dot{V}(t, x(t; t_0, x_0)) \leqslant -C[V(t; x(t; t_0, x_0)]$ *along any trajectory* $x(t; t_0, x_0)$

of (1), *where* $C(V)$ *is a continuous monotone increasing function of* V *with* $C(0) = 0$.

Subject to the above hypotheses, system (1) *is asymptotically stable with respect to* Γ.

Proof. System (1) is stable with respect to Γ by virtue of Theorem 1. Along any trajectory $x(t; t_0, x_0)$ of (1) $V(t, x(t; t_0, x_0))$ is a decreasing function; hence the limit

$$\lim_{t\to\infty} V(t, x(t; t_0, x_0)) = V_0$$

exists. If $V_0 \neq 0$, then $C(V_0) \neq 0$ and we have

$$-C[V(t, x(t; t_0, x_0))] < -C(V_0). \tag{4}$$

Therefore, from (iii) and (4) we obtain

$$V(t, x(t; t_0, x_0)) - V(t_0, x_0) \leqslant -C(V_0)(t - t_0). \tag{5}$$

Letting t approach infinity in (5) leads to $\lim_{t\to\infty} V(t, x(t; t_0, x_0)) = -\infty$, which contradicts (ii).

Thus, the assumption that $V_0 \neq 0$ was incorrect and

$$\lim_{t\to\infty} V(t, x(t; t_0, x_0)) = 0.$$

However, this implies that $\lim_{t\to\infty} a(r(x(t; t_0, x_0))) = 0$; and since $a(r)$ is continuous, we obtain $\lim_{t\to\infty} r(x(t; t_0, x_0)) = 0$. This shows that system (1) is asymptotically stable with respect to Γ and completes the proof of the theorem.

The above theorems have counterparts where the signs of the functions involved are reversed, as is the case in the traditional stability theorems.

It should be mentioned that information regarding the converse of the above theorems may be found in **(10)**.

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