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OPERATOR TOPOLOGIES AND INVARIANT OPERATOR RANGES

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ABSTRACT. The invariant operator range lattices of a wide class of uniformly closed algebras (including C^*-algebras) are stable under weak closures. There is an algebra whose invariant operator range lattice contains properly the corresponding lattice of its norm closure. An operator range transitive algebra is operator range n-transitive for all n. A normal operator is algebraic if and only if each of its invariant operator ranges is the range of some operator commuting with it.

1. Let $\mathcal{H}$ be a complex Hilbert space (not necessarily separable). A linear manifold $\mathcal{R} \subseteq \mathcal{H}$ is an operator range if there exists a T in $\mathcal{B}(\mathcal{H})$, the algebra of (bounded linear) operators on $\mathcal{H}$, such that ($T \geq 0$, cf. [1]) $T\mathcal{H} = \mathcal{R}$. For any algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we denote the lattice ([1]) of invariant operator ranges [closed subspaces] of (every element of) $\mathcal{A}$ by $\text{Lat}_{\frac{1}{2}}\, \mathcal{A}$[Lat $\mathcal{A}$]. For an operator A, $\text{Lat}_{\frac{1}{2}}\, A$(Lat A) is the lattice of invariant operator ranges (closed subspaces) of A. We adapt standard notation in [5].

The following theorem of Foiaş is very useful.

THEOREM A [2]. *If $\mathcal{A}$ is a uniformly closed algebra in $\mathcal{B}(\mathcal{H})$, and if (with $T \geq 0$) $T\mathcal{H}$ is invariant under $\mathcal{A}$, then there exists a unique bounded algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that*

$$AT = T\pi(A) \quad (A \in \mathcal{A})$$
$$\pi(A)\mathcal{H} \subseteq N(T)^{\perp}.$$

Suppose $\mathcal{A}$ contains a commutative self-adjoint algebra $\mathcal{A}_0$. Then T can be chosen in $\mathcal{A}_0'$, the commutant of $\mathcal{A}$ (and $T \geq 0$).

2. It is clear that if an operator range $\mathcal{R}$ is invariant under $\mathcal{A}$, there is no immediate reason for $\mathcal{R}$ to be invariant under the closure of $\mathcal{A}$ (in any operator topology). In fact, we shall give an example showing norm closures can reduce the invariant operator range lattices. For the same question of closing uniformly closed algebras in the weak (or strong) topology, we do not

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know the answer in general. But the invariant operator range lattices of a wide class of uniformly closed algebras are stable under this process.

PROPOSITION 1. *If $\mathcal{A}$ is a C^*-algebra, then* $\operatorname{Lat}_{\frac{1}{2}} \mathcal{A} = \operatorname{Lat}_{\frac{1}{2}} \bar{\mathcal{A}}^w$. ($\bar{\mathcal{A}}^w$ *denotes the closure of $\mathcal{A}$ in the weak operator topology.*)

Proof. Obviously, any operator range invariant under $\bar{\mathcal{A}}^w$ is invariant under $\mathcal{A}$. Let $T\mathcal{H} \in \operatorname{Lat}_{\frac{1}{2}} \mathcal{A}$. By Theorem A, T induces a bounded algebra homomorphism π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. Let $\mathcal{B} \in \bar{\mathcal{A}}^w$. By the Kaplansky density theorem there exists a net $\{A_\alpha\}$ in $\mathcal{A}$ with $\|A_\alpha\| \le \|B\|$ such that $A_\alpha \to B$ weakly. The net $\{\pi(A_\alpha)\}$ is bounded in $\mathcal{B}(\mathcal{H})$. Since uniformly closed bounded balls in $\mathcal{B}(\mathcal{H})$ are compact in the weak operator topology, by dropping down to a subnet, we may assume that $\pi(A_\alpha) \to D$ weakly in $\mathcal{B}(\mathcal{H})$. Let $x, y \in \mathcal{H}$.

$$(BTx, y) = \lim_\alpha (A_\alpha Tx, y) = \lim_\alpha (T\pi(A_\alpha)x, y) = \lim_\alpha (\pi(A_\alpha)x, T^*y)$$
$$= (Dx, T^*y) = (TDx, y).$$

$BT = TD$. $T\mathcal{H}$ is invariant under B, hence under $\bar{\mathcal{A}}^w$. This completes the proof.

The same proof goes through for any uniformly closed algebra for which the conclusion of the Kaplansky density theorem holds.

COROLLARY 2. *Let $\mathcal{A}$ be a uniformly closed algebra, whose closed unit ball $\mathcal{A}_1$ is weakly (strongly) dense in the closed unit ball $(\bar{\mathcal{A}}^w)_1$. Then* $\operatorname{Lat}_{\frac{1}{2}} \mathcal{A} = \operatorname{Lat}_{\frac{1}{2}} \bar{\mathcal{A}}^w$. [*Note (as the referee pointed out) that the uniformly closed algebra generated by the unilateral shift satisfies this hypothesis.*]

3. The following theorem is proved in [3]

THEOREM B [3]. *Every operator (with a cyclic vector) has uncountably many invariant operator ranges that are (dense and are) ranges of compact operators and each pair of them has trivial intersection.*

In view of this, the main theorem and the conjecture of [4], the question of whether singly generated uniformly closed algebras have non-trivial invariant compact operator ranges is of great interest.

PROPOSITION 3. *There exists a positive operator P that generates a uniformly closed algebra with no non-zero compact invariant operator ranges.*

Proof. Let $\mathcal{H} = L^2[0, 1]$ (with Lebesgue measure), and

$$P = M_x (M_x f(t) = tf(t), \qquad t \in [0, 1], f \in \mathcal{H}).$$

Then the uniformly closed algebra $\mathcal{A}$ generated by P is commutative (in fact, isometrically *-isomorphic to $C([0, 1])$), hence, by Theorem A every invariant operator range is the range of some (positive) operator in $\mathcal{A}'$. Since $\mathcal{A}'$ consists

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of all multiplication operators

$$M_\phi,\ \phi\in L^\infty[0,1](M_\phi f(t)=\phi(t)f(t),\ t\in[0,1],\ f\in\mathcal{H}),$$

M_ϕ is not compact if $\phi\neq 0$. The proof is thus complete.

COROLLARY 4. *Uniform closures can reduce invariant operator range lattices.*

4. If $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})$ $(=\{\{0\},\mathcal{H}\})$ and $\mathcal{A}$ is weakly closed, it is proved in [2] that $\mathcal{A}=\mathcal{B}(\mathcal{H})$. It is clear that if $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(n)}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})^{(n)}$ for all $n=1,2,\ldots$, then $\bar{\mathcal{A}}^w=\mathcal{B}(\mathcal{H})$. The former does not immediately imply the following: if $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})$ $(=\{0,\mathcal{H}\})$, then $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(n)}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})^{(n)}$; which is parallel to the Rickart-Yood Theorem.

PROPOSITION 5. *For any algebra* $\mathcal{A}\subseteq\mathcal{B}(\mathcal{H})$, $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})$ *implies* $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(n)}=\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})^{(n)}$ $(=\mathrm{Lat}\,\mathcal{B}(\mathcal{H})^{(n)})$ $n=1,2,\ldots.$

Proof. First we prove this for $n=2$. Let $\mathcal{R}$ be an operator range contained in $\mathcal{H}^{(2)}$ and in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(2)}$. Let $\mathcal{R}_1=\{0\oplus x:0\oplus x\in\mathcal{R}\}$. Then $\mathcal{R}_1$ considered in an obvious way, as a subspace of $\mathcal{H}$, is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}$, it is therefore closed (in fact it is equal to $\{0\}$ or $\mathcal{H}$). This implies that $\mathcal{R}=\mathcal{R}_1+(\mathcal{R}_1^\perp\cap\mathcal{R})$ (which is not true unless $\mathcal{R}_1$ is closed). Since $\mathcal{R}_1^\perp$ is either $\mathcal{H}\oplus\{0\}$ or $\mathcal{H}\oplus\mathcal{H}$, it is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(2)}$, and hence $\mathcal{R}_1^\perp\cap\mathcal{R}$ is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(2)}$. Furthermore, there is a linear transformation T with domain $\mathcal{D}$ such that $\mathcal{R}_1^\perp\cap\mathcal{R}=\{x\oplus Tx:x\in\mathcal{D}\}$. It is easily verified that $\mathcal{D}$ is an invariant operator range of $\mathcal{A}$; hence assume, without loss of generality, that $\mathcal{D}=\mathcal{H}$. T is everywhere defined linear transformation with graph $\mathcal{R}_1^\perp\cap\mathcal{R}$, an operator range, T is bounded by a result of Foiaş ([2], [1]). It follows that $\mathcal{R}_1^\perp\cap\mathcal{R}$ is closed. Since $\mathcal{R}_1^\perp\cap\mathcal{R}$ is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(2)}$, the closure assumption implies the invariance of $\mathcal{R}_1^\perp\cap\mathcal{R}$ under $(\bar{\mathcal{A}}^w)^{(2)}$ and hence under $\mathcal{B}(\mathcal{H})^{(2)}$. (In particular, T is a scalar multiple of the identity.) $\mathcal{R}$ is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})^{(2)}$. We now proceed by induction. Suppose the Proposition has been proved for all $k<n$. Let $\mathcal{R}$ be an element of $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(n)}$. Let

$$\mathcal{R}_1=\{0\oplus x_2\oplus\cdots\oplus xn:0\oplus x_2\oplus\cdots\oplus x_n\in\mathcal{R}\}.$$

Then $\mathcal{R}_1$, considered as a submanifold of $\mathcal{H}^{(n-1)}$ in an obvious way, is in $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{A}^{(n-1)}$). By the induction hypothesis, $\mathcal{R}_1$ is invariant under $\mathcal{B}(\mathcal{H})^{(n-1)}$ and is thus closed. Let $\mathcal{R}'=\mathcal{R}\cap\mathcal{R}_1^\perp$. Then $\mathcal{R}=\mathcal{R}_1+\mathcal{R}'$ (again, this is not true unless $\mathcal{R}_1$ is closed), and there exist linear transformations $T_1,\ldots,T_{n-1}$ such that

$$\mathcal{R}'=\{x\oplus T_1x\oplus\cdots\oplus T_{n-1}x:x\in\mathcal{D}\}$$

for some linear manifold $\mathcal{D}\subseteq\mathcal{H}$. Since $\mathcal{R}'$ is clearly invariant under $\mathcal{A}^{(n)}$, for each $j=1,2,\ldots,n-1$

$$\mathcal{G}_j=\{x\oplus T_jx:x\in\mathcal{D}\}$$

is an invariant operator range of $\mathcal{A}^{(2)}$, hence an element of $\mathrm{Lat}_{\frac{1}{2}}\,\mathcal{B}(\mathcal{H})^{(2)}$ This

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implies (assuming $\mathcal{R}$ non-trivial) that $\mathcal{D}=\mathcal{H}$ and T_j is a scalar multiple of the identity. We conclude that

$$\mathcal{R}'=\{x\oplus\lambda_1x\oplus\lambda_2x\oplus\cdots\oplus\lambda_{n-1}x:x\in\mathcal{H}\}$$

for some complex numbers $\lambda_1,\ldots,\lambda_{n-1}$; $\mathcal{R}'$ is invariant under $\mathcal{B}(\mathcal{H})^{(n)}$. It follows that $\mathcal{R}=\mathcal{R}_1+\mathcal{R}'$ is invariant under $\mathcal{B}(\mathcal{H})^{(n)}$. This completes the proof.

5. For any operator A, if T commutes with A, then $T\mathcal{H}\in \mathrm{Lat}_{\frac{1}{2}} A$. For general operators we cannot say any more about the relationship between $\{A\}'$ and $\mathrm{Lat}_{\frac{1}{2}} A$. For normal operators we do have the following.

Proposition 6. *If A is a non-algebraic normal operator, then*

$$\mathrm{Lat}_{\frac{1}{2}} A \supsetneq \{T\mathcal{H}:T\in\{A\}'\}.$$

Proof. We consider two cases. Case (1): *A has a cyclic vector x*, that is $\bigvee_{n=0}^{\infty}\{A^nx\}=\mathcal{H}$. Then the von Neumann algebra generated by A is maximal abelian, hence $\{A\}'$ is commutative, and each pair of dense ranges of operators $\{A\}'$ has a dense intersection; while $\mathrm{Lat}_{\frac{1}{2}} A$ contains uncountably many elements dense in $\mathcal{H}$ each pair of which has intersection $\{0\}$ (Theorem B). Therefore

$$\mathrm{Lat}_{\frac{1}{2}} A \supsetneq \{T\mathcal{H}:T\in\{A\}'\}.$$

Case (2): A has no cyclic vectors. Since A is assumed to be non-algebraic, there exists, by a theorem of Kaplansky ([5]), an $x\in\mathcal{H}$ such that the space

$$\mathcal{M}=\bigvee\{A^nx:n=0,1,2,\ldots\}$$

is of infinite dimension. Let $E_{\mathcal{M}}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If $E_{\mathcal{M}}\notin\{A\}'$, then, since $\{A\}'$, is a von Neumann algebra which contains the range projection of any operator it contains, $\mathcal{M}$ is not the range of any operator in $\{A\}'$. We can, therefore, assume that $E_{\mathcal{M}}\in\{A\}'$. Then $A_1=A\mid\mathcal{M}$ is normal with a cyclic vector. It follows from case (1) that there exists $T_1\in\mathcal{B}(\mathcal{M})$ such that $T_1\mathcal{M}\in\mathrm{Lat}_{\frac{1}{2}} A_1$ and $T_1\mathcal{M}$ is not the range of any operator in $\{A_1\}'$. Let $T\in\mathcal{B}(\mathcal{H})$ be the operator $T_1\oplus 0$ relative to the decomposition $\mathcal{M}\oplus\mathcal{M}^{\perp}$ of $\mathcal{H}$. Then $T\mathcal{H}$ is not the range of any operator in $\{A\}'$. For if it were the range of some operator S in $\{A\}'$, then, by self-adjointness of $\{A\}'$, we would be able to assume S to be positive. The operator range $S\mathcal{H}$ is contained in $\mathcal{M}$. It would, then, follow that $\mathcal{M}$ reduces S. Let $S_1=S\mid\mathcal{M}$. Then $S_1\in\{A_1\}'$ and $S_1\mathcal{M}=T_1\mathcal{M}$. This contradicts the choice of T. Hence $T\mathcal{H}$ is invariant under A and is not the range of any operator in $\{A\}'$. This completes the proof.

Corollary 7. *Let A be a normal operator. Then A is algebraic if and only if*

$$\mathrm{Lat}_{\frac{1}{2}} A=\{S\mathcal{H}:S\in\{A\}'\}.$$

Proof. If A is algebraic and normal, then the C^*-algebra $\mathcal{A}$ generated by A

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is commutative and finite dimensional. Therefore $\mathrm{Lat}_{\frac{1}{2}}\, \mathscr{A} = \mathrm{Lat}_{\frac{1}{2}}\, A = \{S\mathscr{H} : S \in \{A\}'\}$ by Theorem A. Proposition 6 shows the converse.

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