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THE FIRST COHOMOLOGY GROUP OF BANACH INVERSE SEMIGROUP ALGEBRAS WITH COEFFICIENTS IN *L*-EMBEDDED BANACH BIMODULES

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Abstract

Let S be a discrete inverse semigroup, $l^1(S)$ the Banach semigroup algebra on S and $\mathbb{X}$ a Banach $l^1(S)$-bimodule which is an L-embedded Banach space. We show that under some mild conditions $\mathcal{H}^1(l^1(S), \mathbb{X}) = 0$. We also provide an application of the main result.

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1. Introduction

Let $\mathbb{A}$ be a Banach algebra and $\mathbb{X}$ be a Banach $\mathbb{A}$-bimodule. A linear map $D : \mathbb{A} \to \mathbb{X}$ is called a *derivation* if $D(ab) = aD(b) + D(a)b$ for all $a, b \in \mathbb{A}$. For any $x \in \mathbb{X}$, the map $\mathrm{id}_x : \mathbb{A} \to \mathbb{X}$ given by $\mathrm{id}_x(a) = ax - xa$ is a continuous derivation called an *inner derivation*. We denote by $\mathcal{Z}^1(\mathbb{A}, \mathbb{X})$ the vector space of all continuous derivations from $\mathbb{A}$ into $\mathbb{X}$ and by $\mathcal{N}^1(\mathbb{A}, \mathbb{X})$ the subspace of all inner derivations from $\mathbb{A}$ into $\mathbb{X}$. The quotient space $\mathcal{H}^1(\mathbb{A}, \mathbb{X}) = \mathcal{Z}^1(\mathbb{A}, \mathbb{X})/\mathcal{N}^1(\mathbb{A}, \mathbb{X})$ is called the *first cohomology group of $\mathbb{A}$ with coefficients in $\mathbb{X}$*.

The first cohomology group of a Banach algebra with coefficients in different modules can be used to study its structure. The case $\mathcal{H}^1(\mathbb{A}, \mathbb{X}) = 0$ (that is, every continuous derivation from $\mathbb{A}$ into $\mathbb{X}$ is inner) leads to the notion of amenability of Banach algebras, introduced by Johnson [13]. Taking the coefficients in different modules leads to various types of amenability. Sakai [19] showed that every continuous derivation on a W^*-algebra is inner. Kadison [15] proved that every derivation of a C^*-algebra on a Hilbert space H is spatial (that is, of the form $a \mapsto ta - at$ for $t \in \mathbb{B}(H)$) and, in particular, every derivation on a von Neumann algebra is inner. Some results have also been obtained in the case of non-self-adjoint operator algebras. Christian [4] showed that every continuous derivation on a nest algebra on H to itself and to $\mathbb{B}(H)$ is inner, and this result was generalised in [16]. However, the cohomology is nontrivial in general. Gilfeather and Smith [10, 11] calculated the first cohomology group of some operator algebras called joins. In [7], the cohomology group of operator algebras called seminest algebras has been calculated. Forrest and Marcoux [9] found the first cohomology group of triangular Banach algebras, and various examples of triangular Banach algebras with nontrivial cohomology have been given (see [5]). The first cohomology group of the semigroup Banach algebra $l^1(S)$ on an (inverse) semigroup S with coefficients in different modules has been studied in [2, 3, 6, 8, 17].

In this paper we study the first cohomology group of $l^1(S)$ on a discrete inverse semigroup S with coefficients in a Banach $l^1(S)$-bimodule $\mathbb{X}$, which is L-embedded as a Banach space, and show that under some mild conditions $\mathcal{H}^1(l^1(S), \mathbb{X}) = 0$ (Theorem 3.4). To prove our main result we use fixed point techniques. In Lemma 3.1, which may be of independent interest, we prove that if $\mathbb{X}$, as an L-embedded Banach space, is a Banach $l^1(S)$-bimodule and $\phi : S \to \mathbb{X}$ is a bounded crossed homomorphism, then under some mild conditions ϕ is principal. As an application of our main result, we show in Corollary 3.7 that $\mathcal{H}^1(l^1(S), l^1(G_S)^{2n}) = 0$ for any $n \geq 0$, where G_S is the maximal group homomorphic image of S.

This paper is organised as follows. Section 2 is devoted to preliminaries and required tools. The main results of the paper are presented in Section 3.

2. Preliminaries

A discrete semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an *idempotent* if $e = e^* = e^2$. The set of idempotents of S is denoted by E. There is a natural order on E, defined by

$$e \leq d \Leftrightarrow ed = e \quad (e, d \in E).$$

Moreover, E is a commutative subsemigroup of S and also a semilattice [12, Theorem V.1.2]. Elements of the form ss^* are idempotents of S and in fact all elements of E are of this form. Let $\sim$ be the congruence on S defined by

$$s \sim t \quad \text{if and only if there exists } e \in E \text{ such that } es = et.$$

The quotient semigroup $S/\sim$ is then a group and is the maximal group homomorphic image of S [18]. The group $S/\sim$ is denoted by G_S.

We denote the convolution Banach semigroup algebra on S by $l^1(S)$ and the point mass measure at s by δ_s.

For the proof of the main result we use a common fixed point property for semigroups which we now recall. Let S be a (discrete) semigroup. The space of all bounded complex-valued functions on S is denoted by $\ell^\infty(S)$. It is a Banach space with the uniform supremum norm. In fact $\ell^\infty(S) = (\ell^1(S))^*$. For each $s \in S$ and each $f \in \ell^\infty(S)$, let $\ell_s f$ be the left translate of f by s, that is, $\ell_s f(t) = f(st)$, $t \in S$ (the right translate $r_s f$ is defined similarly). We recall that $f \in \ell^\infty(S)$ is *weakly almost periodic* if its left orbit $\mathcal{LO}(f) = \{\ell_s f \mid s \in S\}$ is relatively compact in the weak topology of $\ell^\infty(S)$.

We denote by $WAP(S)$ the space of all weakly almost periodic functions on S. It is a closed subspace of $\ell^\infty(S)$ containing the constant function and invariant under left and right translations.

A linear functional $m \in WAP(S)^*$ is a *mean* on $WAP(S)$ if $\|m\| = m(1) = 1$. A mean m on $WAP(S)$ is a *left invariant mean* (abbreviated *LIM*) if $m(\ell_s f) = m(f)$ for all $s \in S$ and all $f \in WAP(S)$. If S is an inverse semigroup, it is well known that $WAP(S)$ always has a *LIM* [8, Proposition 2]. Let C be a subset of a Banach space $\mathbb{X}$. We say that $\Gamma = \{T_s \mid s \in S\}$ is a *representation* of S on C if T_s is a mapping from C into C for each $s \in S$ and $T_{st}(x) = T_s(T_t(x))$ $(s, t \in S, x \in C)$. We say that $x \in C$ is a *common fixed point* for (the representation of) S if $T_s(x) = x$ for all $s \in S$.

Let $\mathbb{X}$ be a Banach space and C a nonempty subset of $\mathbb{X}$. A mapping $T : C \to C$ is called *nonexpansive* if $\|T(x) - T(y)\| \le \|x - y\|$ for all $x, y \in C$. The mapping T is called *affine* if C is convex and $T(\gamma x + \eta y) = \gamma T(x) + \eta T(y)$ for all constants $\gamma, \eta \ge 0$ with $\gamma + \eta = 1$ and $x, y \in C$. A representation Γ of a semigroup S on C acts as nonexpansive affine mappings, if each T_s $(s \in S)$ is nonexpansive and affine.

A Banach space $\mathbb{X}$ is called *L-embedded* if there is a closed subspace $\mathbb{X}_0 \subseteq \mathbb{X}^{**}$ such that $\mathbb{X}^{**} = \mathbb{X} \oplus_{\ell^1} \mathbb{X}_0$. The class of L-embedded Banach spaces includes all $L^1(\Sigma, \mu)$ (the space of of all absolutely integrable functions on a measure space (Σ, μ)), preduals of von Neumann algebras, dual spaces of M-embedded Banach spaces and the Hardy space H_1. In particular, given a locally compact group G, the space $L^1(G)$ is L-embedded, as are its even duals $L^1(G)^{(2n)}$ $(n \ge 0)$. (For more details, see [20].)

The next lemma is the common fixed point theorem for semigroups.

Lemma 2.1 [20, Theorem 2]. *Let S be a discrete semigroup and Γ a representation of S on an L-embedded Banach space $\mathbb{X}$ as nonexpansive affine mappings. Suppose that $WAP(S)$ has a LIM and suppose that there is a nonempty bounded set $B \subset \mathbb{X}$ such that $B \subseteq \overline{T_s(B)}$ for all $s \in S$. Then $\mathbb{X}$ contains a common fixed point for S.*

3. Main results

From this point on, S is a discrete inverse semigroup with the set of idempotents E. If $\mathbb{X}$ is a Banach $l^1(S)$-bimodule, we consider an associated action $S \times \mathbb{X} \to \mathbb{X}$ given by

$$(s, x) \to s.x = \delta_s x \delta_{s^*} \quad (s \in S,\ x \in \mathbb{X}).$$

Following Johnson [14], a map $\phi : S \to \mathbb{X}$ is called a *crossed homomorphism* if $\phi(st) = \phi(s) + s.\phi(t)$ for all $s, t \in S$. (This is called a *cocycle* in [1].) We say that ϕ is *bounded* if $\sup_{s \in S} \|\phi(s)\| < \infty$. A crossed homomorphism $\phi : S \to \mathbb{X}$ is called *principal* if there exists $x \in \mathbb{X}$ such that $\phi(s) = s.x - x$, for each $s \in S$.

Lemma 3.1. *Let $\mathbb{X}$ be a Banach $l^1(S)$-bimodule which is an L-embedded Banach space. Suppose that $\phi : S \to \mathbb{X}$ is a bounded crossed homomorphism such that $\delta_e \phi(s) \delta_e = \phi(s)$ for each $e \in E$ and $s \in S$. Then ϕ is principal.*

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PROOF. For any $s \in S$, define the mapping $T_s : \mathbb{X} \to \mathbb{X}$ by

$$T_s(x) = s.x + \phi(s) \quad (x \in \mathbb{X}).$$

Since ϕ is a crossed homomorphism, it follows that

$$\begin{aligned} T_{st}(x) &= (st).x + \phi(st) \\ &= s.(t.x) + s.\phi(t) + \phi(s) \\ &= s.(t.x + \phi(t)) + \phi(s) \\ &= s.T_t(x) + \phi(s) \\ &= T_s(T_t(x)), \end{aligned}$$

for all $s, t \in S$ and $x \in \mathbb{X}$. Clearly each T_s $(s \in S)$ is an affine mapping and, for every $x, y \in \mathbb{X}$ and $s \in S$,

$$\|T_s(x) - T_s(y)\| = \|s.x + \phi(s) - s.y - \phi(s)\| = \ \delta_s(x - y)\delta_{s^*}\| \leq \|x - y\|.$$

So each T_s $(s \in S)$ is nonexpansive. Hence $\Gamma = \{T_s \mid s \in S\}$ defines a representation of S on $\mathbb{X}$ which is nonexpansive and affine. Let $B = \phi(S)$. Since ϕ is bounded, it follows that B is a nonempty bounded subset of $\mathbb{X}$. From the definition of T_s, for any $s, t \in S$,

$$T_s(\phi(t)) = s.\phi(t) + \phi(s) = \phi(st).$$

Therefore $T_s(B) \subseteq B$ $(s \in S)$.

For any $e \in E$ we have $\phi(e) = \phi(e^2) = e.\phi(e) + \phi(e)$. So $e.\phi(e) = 0$ and, by the hypothesis,

$$\phi(e) = \delta_e \phi(e) \delta_e = e.\phi(e) = 0 \quad (e \in E).$$

Now from the fact that $\phi(e) = 0$ for any $e \in E$ and the hypothesis, for $x \in B$ (since $x = \phi(s)$ for some $s \in S$),

$$T_s(T_{s^*}(x)) = T_{ss^*}(x) = \delta_{ss^*} x \delta_{ss^*} + \phi(ss^*) = x \quad (s \in S).$$

Since $T_{s^*}(x) \in B$, it follows that $T_s(B) = B$ for each $s \in S$.

Here S is regarded as a discrete inverse semigroup and hence $WAP(S)$ has a *LIM*. So by Lemma 2.1, there is $z \in \mathbb{X}$ such that $T_s(z) = z$ for all $s \in S$. Therefore $s.z + \phi(s) = z$ for each $s \in S$. If we put $y = -z$, then we get $\phi(s) = s.y - y$, for all $s \in S$, that is, ϕ is principal. □

The following corollary is an immediate consequence of Lemma 3.1.

COROLLARY 3.2. *Let $\mathbb{X}$ be a Banach $l^1(S)$-bimodule which is an L-embedded Banach space. Suppose that $\delta_e x \delta_e = x$ for all $e \in E$ and $x \in \mathbb{X}$. Then any bounded crossed homomorphism $\phi : S \to \mathbb{X}$ is principal.*

Let $\mathbb{A}$ be a Banach algebra and $\mathbb{X}$ be a Banach $\mathbb{A}$-bimodule. Define the *annihilator* of $\mathbb{X}$ by $\mathrm{ann}_{\mathbb{A}}\mathbb{X} = \{a \in \mathbb{A} \mid a\mathbb{X} = \mathbb{X}a = \{0\}\}$.

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REMARK 3.3. Let $\mathbb{X}$ be a Banach $l^1(S)$-bimodule such that $\delta_e - \delta_d \in \mathrm{ann}_{l^1(S)}\mathbb{X}$ for every $e, d \in E$. Fix an element $e \in E$ and define the sets

$$\begin{aligned}
\mathbb{X}_1 &= \{\delta_e x \delta_e \mid x \in \mathbb{X}\},\\
\mathbb{X}_2 &= \{\delta_e x - \delta_e x \delta_e \mid x \in \mathbb{X}\},\\
\mathbb{X}_3 &= \{x\delta_e - \delta_e x \delta_e \mid x \in \mathbb{X}\},\\
\mathbb{X}_4 &= \{x - \delta_e x - x\delta_e + \delta_e x \delta_e \mid x \in \mathbb{X}\}.
\end{aligned}$$

By hypothesis,

$$\delta_s(\delta_e x) = \delta_s(\delta_{s^*s} x) = \delta_{ss^*}(\delta_s x) = \delta_e(\delta_s x),$$

and similarly

$$(x\delta_e)\delta_s = (x\delta_s)\delta_e = x\delta_s,$$

for all $s \in S$ and $x \in \mathbb{X}$. Hence every $\mathbb{X}_j$ for $1 \le j \le 4$ is a closed $l^1(S)$-subbimodule of $\mathbb{X}$ such that

$$\mathbb{X}_2 l^1(S) = l^1(S)\mathbb{X}_3 = \mathbb{X}_4 l^1(S) = l^1(S)\mathbb{X}_4 = \{0\}.$$

Since $\delta_d(\delta_e x) = \delta_e(\delta_e x) = \delta_e x$ and, similarly, $(x\delta_e)\delta_d = x\delta_e$ for any $d \in E$, it follows that $\delta_d x_1 = x_1\delta_d = x_1$, $\delta_d x_2 = x_2$ and $x_3\delta_d = x_3$ for all $d \in E$, $x_1 \in \mathbb{X}_1$, $x_2 \in \mathbb{X}_2$ and $x_3 \in \mathbb{X}_3$. Also $\mathbb{X} = \mathbb{X}_1 \dotplus \mathbb{X}_2 \dotplus \mathbb{X}_3 \dotplus \mathbb{X}_4$ as a sum of $l^1(S)$-bimodules.

THEOREM 3.4. *Let $\mathbb{X}$ be a Banach $l^1(S)$-bimodule which is an L-embedded Banach space and let $\delta_e - \delta_d \in \mathrm{ann}_{l^1(S)}\mathbb{X}$ for every $e, d \in E$. Then $\mathcal{H}^1(l^1(S), \mathbb{X}) = 0$.*

PROOF. Fix an element $e \in E$. As in Remark 3.3, $\mathbb{X} = \mathbb{X}_1 \dotplus \mathbb{X}_2 \dotplus \mathbb{X}_3 \dotplus \mathbb{X}_4$. From now on we use the same notation as in Remark 3.3.

Let $D : l^1(S) \to \mathbb{X}$ be a continuous derivation. So $D = D_1 + D_2 + D_3 + D_4$, where each D_j is a continuous linear map from $l^1(S)$ to $\mathbb{X}_j$. Since D is a derivation, from Remark 3.3,

$$D_1(fg) + D_2(fg) + D_3(fg) + D_4(fg) = fD_1(g) + fD_2(g) + D_1(f)g + D_3(f)g, \quad (3.1)$$

for all $f, g \in l^1(S)$.

We complete the proof by checking four steps.

Step 1. There exists an element $x_1 \in \mathbb{X}_1$ such that $D(f) = fx_1 - x_1 f$ for all $f \in l^1(S)$.

Multiply (3.1) by δ_e both on the left and on the right. By Remark 3.3, we see that D_1 is a derivation. Now we consider $\phi : S \to \mathbb{X}_1 \subseteq \mathbb{X}$ defined by

$$\phi(s) = D_1(\delta_s)\delta_{s^*} \quad (s \in S).$$

We see that

$$\begin{aligned}
\phi(st) &= D_1(\delta_s * \delta_t)\delta_{(st)^*}\\
&= (\delta_s D_1(\delta_t))\delta_{t^*} * \delta_{s^*} + (D_1(\delta_s)\delta_t)\delta_{t^*} * \delta_{s^*}\\
&= \delta_s(D_1(\delta_t)\delta_{t^*})\delta_{s^*} + (D_1(\delta_s)\delta_{tt^*})\delta_{s^*}\\
&= \delta_s(D_1(\delta_t)\delta_{t^*})\delta_{s^*} + D_1(\delta_s)\delta_{s^*}\\
&= s.\phi(t) + \phi(s),
\end{aligned}$$

for all $s, t \in S$. So ϕ is a crossed homomorphism. Since D_1 is continuous, it follows that ϕ is bounded. By Remark 3.3, for any $d \in E$ and $s \in S$,

$$\delta_d \phi(s) \delta_d = \delta_d (D_1(\delta_s)\delta_{s^*})\delta_d = \phi(s).$$

Thus we have all the requirements in Lemma 3.1 and therefore ϕ is principal, that is, there is $z \in \mathbb{X}$ such that

$$\phi(s) = s.z - z \quad (s \in S).$$

Let $x_1 = \delta_e z \delta_e$. Since $\phi(s) \in \mathbb{X}_1$ for all $s \in S$, by Remark 3.3,

$$\begin{aligned}
\phi(s) &= \delta_e \phi(s) \delta_e \\
&= \delta_e(\delta_s z \delta_{s^*})\delta_e - \delta_e z \delta_e \\
&= \delta_s x_1 \delta_{s^*} - x_1.
\end{aligned}$$

Hence

$$D_1(\delta_s) = \delta_s x_1 - x_1 \delta_s \quad (s \in S).$$

Since D_1 is continuous and functions of finite support are dense in $l^1(S)$, it follows that

$$D_1(f) = f x_1 - x_1 f,$$

for all $f \in l^1(S)$.

Step 2. There exists an element $x_2 \in \mathbb{X}_2$ such that $D(f) = f x_2$ for all $f \in l^1(S)$.

Multiply (3.1) by δ_e from the left. By Remark 3.3 and the fact that D_1 is a derivation,

$$D_2(fg) = f D_2(g) \quad (f, g \in l^1(S)).$$

So, for all $d, d' \in E$,

$$D_2(\delta_d) = \delta_{d'} D_2(\delta_d) = D_2(\delta_{d'}\delta_d) = D_2(\delta_d \delta_{d'}) = \delta_d D_2(\delta_{d'}) = D_2(\delta_{d'}).$$

Hence

$$D_2(\delta_s) = D_2(\delta_{ss^*s}) = \delta_s D_2(\delta_{s^*s}) = \delta_s D_2(\delta_e),$$

for all $s \in S$. Let $x_2 = D_2(\delta_e)$. Since D_2 is continuous and functions of finite support are dense in $l^1(S)$, it follows that

$$D_2(f) = f x_2,$$

for all $f \in l^1(S)$.

Step 3. There exists an element $x_3 \in \mathbb{X}_3$ such that $D(f) = x_3 f$ for all $f \in l^1(S)$.

This follows by Remark 3.3 using similar methods to those in Step 2.

Step 4. $D_4 = 0$.

From (3.1) and the above steps,

$$D_4(fg) = 0 \quad (f, g \in l^1(S)).$$

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Since $\overline{(l^1(S))^2} = l^1(S)$ and D_4 is continuous, it follows that $D_4 = 0$. Now define $x = x_1 + x_2 - x_3$. Since

$$\mathbb{X}_2 l^1(S) = l^1(S)\mathbb{X}_3 = \mathbb{X}_4 l^1(S) = l^1(S)\mathbb{X}_4 = \{0\},$$

by the above steps,

$$D(f) = fx_1 - x_1 f + fx_2 + x_3 f = fx - xf,$$

for all $f \in l^1(S)$. Therefore D is inner. The proof is complete. □

Remark 3.5. In the preceding theorem, instead of $\mathbb{X}$, if we assume that $\mathbb{X}_1$ is an L-embedded Banach space, the same argument shows that the theorem still holds and therefore we obtain a generalisation of Theorem 3.4.

We next give an application of Theorem 3.4. We make $l^1(G_S)$ into a Banach $l^1(S)$-bimodule in a natural manner (Remark 3.6) and then show that $\mathcal{H}^1(l^1(S), l^1(G_S)^{2n})$ vanishes for $n \geq 0$).

Remark 3.6. Suppose that $\sim$ is the equivalence relation on S given in Section 2 and let $G_S = S/\sim$ be the maximal group homomorphic image of S and $\varphi : S \to G_S$ the canonical homomorphism. The map φ extends to a continuous epimorphism of Banach spaces $\tilde{\varphi} : l^1(S) \to l^1(G_S)$ determined by $\tilde{\varphi}(\delta_s) = \delta_{\phi(s)}$ $(s \in S)$. So $I = \ker \tilde{\phi}$ is a closed subspace of $l^1(S)$. Define the mapping $\psi : l^1(S)/I \to l^1(G_S)$ by $\psi(f + I) = \tilde{\varphi}(f)$. It can be easily verified that ψ is well defined and an isomorphism of vector spaces. Let $\pi : l^1(S) \to l^1(S)/I$ be the quotient mapping so that $\psi \circ \pi = \tilde{\varphi}$. Thus $\psi \circ \pi$ is continuous and from [5, Proposition 5.2.2] we conclude that ψ is continuous. By the inverse mapping theorem, ψ is an isomorphism of Banach spaces and so $l^1(S)/I \cong l^1(G_S)$ (isomorphic as Banach spaces). We turn $l^1(S)/I$ naturally into a Banach $l^1(S)$-bimodule by the operations

$$f.(g + I) = fg \quad \text{and} \quad (g + I).f = gf \quad (f, g \in l^1(S)).$$

Now, $l^1(G_S)$ is a Banach $l^1(S)$-bimodule by the module actions

$$f.h = \psi(f.\psi^{-1}(h)) \quad \text{and} \quad h.f = \psi(\psi^{-1}(h).f) \quad (f \in l^1(S), g \in l^1(G_S)).$$

It is clear that with this definition $\psi : l^1(S)/I \to l^1(G_S)$ is a Banach $l^1(S)$-bimodule isomorphism. For any $e, d \in E$ we have $e \sim d$ and so $\varphi(e) = \varphi(d)$. Thus $\tilde{\varphi}(\delta_e) = \tilde{\varphi}(\delta_d)$ or $\delta_e - \delta_d \in I$. By noting the module action on $l^1(G_S)$,

$$\delta_e - \delta_d \in \mathrm{ann}_{l^1(S)} l^1(G_S).$$

Also, by the module action, $l^1(G_S)^{2n}$ $(n \geq 0)$ is a Banach $l^1(S)$-bimodule and

$$\delta_e - \delta_d \in \mathrm{ann}_{l^1(S)} l^1(G_S)^{2n},$$

for all $e, d \in E$.

In the following corollary we consider $l^1(G_S)^{2n}$ $(n \geq 0)$ as in the previous remark as a Banach $l^1(S)$-bimodule. On the other hand, $l^1(G_S)^{2n}$ is an L-embedded Banach space. So the corollary follows from Theorem 3.4.

Corollary 3.7. *$\mathcal{H}^1(l^1(S), l^1(G_S)^{2n}) = 0$ for all $n \geq 0$.*

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