Bipolar and Toroidal Harmonics.

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Most of the solutions of Laplace's equation in common use in mathematical physics have been expressed in the integral form given by Whittaker,* viz.

$$\int_0^{2\pi} f(x\cos t + y\sin t + iz,\ t)\, dt.$$

The solutions are well known for which f, regarded as a function of its first argument, is a power, a circular function, a Legendre function of the first or second kind, or a Bessel function. Thus this general form of solution provides a means of classifying known potential functions and of suggesting new ones. It is therefore not without interest to express outstanding solutions of Laplace's equation in this form. In this paper it will be shown that bipolar harmonics of integral order or potential functions for two spheres are obtained by taking f to be a certain rational function of its first argument. The corresponding forms for toroidal harmonics are deduced. In each case the zonal and sectorial harmonics take a simple form, while the tesseral harmonics are somewhat more complicated.

If ϖ, z, θ are cylindrical coordinates, we may define a set of orthogonal curvilinear coordinates u, v, w, by means of the relations

$$u + iv = \log\frac{\varpi + i(z+a)}{\varpi + i(z-a)},\quad w = \theta. \tag{1}$$

The surfaces corresponding to constant values of u are a set of coaxial spheres having $z = 0$ for their common radical plane. By suitably choosing the position of the origin and the value of the constant a it is possible to ensure that any two given spheres are included in this set.

* *Mathem. Analen*, 1903, 57, p. 333.

Solving (1) for ϖ and z, we have

$$\varpi = \frac{a \sin v}{\cosh u - \cos v}, \quad z = \frac{a \sinh u}{\cosh u - \cos v}. \tag{2}$$

Heine* has given a solution of Laplace's equation in these co-ordinates which may be expressed in the form

$$\sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} \,{}^{\sin}_{\cos}\, m w\, P_n^m (\cos v) \tag{3}$$

where P_n^m is the associated Legendre function.

Let us make the transformation

$$\varpi = \frac{4a^2 \xi}{\xi^2 + \eta^2}, \quad z - a = \frac{4a^2 \eta}{\xi^2 + \eta^2}, \tag{4}$$

so that $$(\varpi^2 + (z-a)^2)(\xi^2 + \eta^2) = 16a^4$$

and

$$\xi = \frac{4a^2 \varpi}{\varpi^2 + (z-a)^2}, \quad \eta = \frac{4a^2 (z-a)}{\varpi^2 + (z-a)^2}. \tag{5}$$

The following results of the transformation are readily verified:

$$\xi^2 + \eta^2 = 8a^2 (\cosh u - \cos v)\, e^u \tag{6}$$

$$\xi^2 + (\eta + 2a)^2 = 4a^2 e^{2u} \tag{7}$$

and

$$\cos v = \frac{\eta + 2a}{(\xi^2 + (\eta + 2a)^2)^{\frac{1}{2}}}, \quad \sin v = \frac{\xi}{(\xi^2 + (\eta + 2a)^2)^{\frac{1}{2}}}. \tag{8}$$

Now

$$P_n^m (\cos v) = \frac{i^m}{2\pi} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \int_0^{2\pi} (\cos v + i \sin v \cos t)^n \cos mt\, dt$$

if $R(\cos v) > 0$, the symbol R representing the real part of the expression to which it is prefixed.

Hence

$$P_n^m(\cos v) = \frac{i^m}{2\pi} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} (\xi^2 + (\eta + 2a)^2)^{-\frac{1}{2}n}$$

$$\times \int_0^{2\pi} (i \xi \cos t + \eta + 2a)^n \cos mt\, dt,$$

* *Kugelfunctionen* II., p. 270. See also a paper by the author, *Proc. Royal Soc.*, 1912, 87A, p. 109.

and from (7)

$$e^{nu} P_n^m(\cos v) = \frac{i^m}{2\pi (2a)^n} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} I$$

where

$$I = \int_0^{2\pi} (i\xi \cos t + \eta + 2a)^n \cos mt \, dt. \tag{10}$$

Expanding by the binomial theorem, we have

$$I = \int_0^{2\pi} \sum_{p=0}^{\infty} \frac{\Gamma(n+1)}{p!\,\Gamma(n-p+1)} (i\xi \cos t + \eta)^{n-p} (2a)^p \cos mt \, dt.$$

This expansion is valid if n is an integer, or otherwise if $|i\xi \cos t + \eta| > |2a|$ for $0 \leq t \leq 2\pi$ In these cases we may integrate term by term. From the two integrals of Laplace's type for the Legendre functions, we have

$$\int_0^{2\pi} (i\xi \cos t + \eta)^{n-p} \cos mt \, dt$$

$$= (-1)^m \frac{[\Gamma(n-p+1)]^2 (\xi^2+\eta^2)^{n-p+\frac{1}{2}}}{\Gamma(n-p-m+1)\,\Gamma(n-p+m+1)} \int_0^{2\pi} \frac{\cos mt \, dt}{(i\xi \cos t + \eta)^{n-p+1}}$$

Hence

$$I = (-1)^m \sum_{p=0}^{\infty} \frac{(2a)^p (\xi^2+\eta^2)^{n-p+\frac{1}{2}}}{p!} \frac{\Gamma(n+1)\,\Gamma(n-p+1)}{\Gamma(n-p-m+1)\,\Gamma(n-p+m+1)}$$

$$\times \int_0^{2\pi} \frac{\cos mt \, dt}{(i\xi \cos t + \eta)^{n-p+1}}$$

and without further difficulty we have

$$\sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} P_n^m(\cos v)$$

$$= \frac{1}{2\pi i^m} \sum_{p=0}^{\infty} \frac{\Gamma(n+m+1)\,\Gamma(n-p+1)\,(2a)^{n-p+1}}{p!\;\Gamma(n-p-m+1)\,\Gamma(n-p+m+1)}$$

$$\times \int_0^{2\pi} \frac{\cos mt \, dt}{(i\varpi \cos t + z - a)^{n-p+1}}.$$

The corresponding integral in which $\cos mt$ is replaced by $\sin mt$ clearly vanishes. Multiplying these two integrals by $\cos mw$ and

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sin mw respectively, and adding and then changing the variable of integration from t to $t - w$, we have

$$\sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} P_n^m(\cos v) \cos mw$$
$$= \frac{1}{2\sqrt{2\pi}\, i^m} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \sum_{p=0}^{\infty} \frac{(n-m)_p (n+m)_p}{p!\; n_p}$$
$$\times \int_0^{2\pi} \frac{(2a)^{n-p+1} \cos mt\, dt}{(i\,x \cos t + i\,y \sin t + z - a)^{n-p+1}} \qquad (11)$$

where $\alpha_p = \alpha(\alpha - 1)(\alpha - 2)\ldots(\alpha - p + 1)$.

If we take

$$P_n^m(\cos v) = \frac{i^{-m}}{2\pi} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \int_0^{2\pi} \frac{\cos mt\, dt}{(\cos v + i \sin v \cos t)^{n+1}}$$

we are led in the same way to the result

$$\sqrt{(\cosh u - \cos v)}\, e^{-(n+\frac{1}{2})u} P_n^m(\cos v) \cos mw$$
$$= \frac{i^m}{2\sqrt{2\pi}} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \sum_{p=0}^{\infty} \frac{(-n+m-1)_p(-n-m-1)_p}{(-n-1)_p\,(2a)^{n+p}}$$
$$\times \int_0^{2\pi} (i\,x \cos t + i\,y \sin t + z - a)^{n+p} \cos mt\, dt \qquad (12)$$

which might have been written down from (11) by putting $-n-1$ for n.

If $|\,i\xi \cos t + \eta\,| < |\,2a\,|$ for $0 \leq t \leq 2\pi$, the binomial expansion of (10) takes a different form, and in this case we arrive at the result:

$$\sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} P_n^m(\cos v) \cos mw$$
$$= \frac{i^{-m}}{2\sqrt{2\pi}} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \sum_{p=0}^{\infty} \frac{(n-m)_p (m+p)!}{p!\,(2m+p)!}$$
$$\times \int_0^{2\pi} \frac{(2a)^{m+p+1} \cos mt\, dt}{(i\,x \cos t + i\,y \sin t + z - a)^{m+p+1}} \qquad (13)$$

If n is a positive integer, the question of convergence does not arise, for both of the series (11) and (13) terminate after $n - m - 1$ terms. In this case these two forms may readily be seen to consist of the same terms taken in the reverse order. We have had to assume that $\cos v$ is positive, that is, that we are dealing with points outside a sphere centre the origin and radius a, but as both of the expressions equated in (11) are solutions of Laplace's equation

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and continuous except at the point $x = y = 0$, $z = a$, this restriction may now be removed.

If n is not an integer (this case will be of importance when we come to discuss toroidal harmonics), the series (11) and (13) will contain an infinite number of terms, and it will be necessary to examine their convergence.

In deducing (11) we have assumed that $\cos v > 0$ and $| i \xi \cos t + \eta | > 2a$. By considering a figure it is easily seen that both of these conditions are satisfied inside a sphere of radius a whose centre is $x = y = 0$, $z = 2a$. The equation (11) will remain true in any extension of this region in which the series converges uniformly. The terms of this series bear a finite ratio to the corresponding terms of the series

$$\Sigma \frac{(n-m)_p (n+m)_p}{n_p \; p!} \left\{ \frac{4a^2}{\varpi^2 + (z-a)^2} \right\}^{n-p+1}$$

which converges uniformly if $\varpi^2 + (z-a)^2 < 4a^2$, *i.e.* in the interior of a sphere of radius $2a$ whose centre is $x = y = 0$, $z = a$ or $u = +\infty$. In the same way it may be seen that (13) converges at points outside this sphere.

In some special but important cases these results take a much simpler form. First let us consider the zonal bipolar harmonics. Putting $m = 0$ in (13), we have

$$\begin{aligned} \sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} P_n (\cos v) \\ &= \frac{1}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{n_p}{p!} \int_0^{2\pi} \frac{(2a)^{p+1}\, dt}{(i\, x \cos t + i\, y \sin t + z - a)^{p+1}} \\ &= \frac{a}{\sqrt{2\pi}} \int_0^{2\pi} \frac{(i\, x \cos t + i\, y \sin t + z + a)^n}{(i\, x \cos t + i\, y \sin t + z - a)^{n+1}}\, dt, \qquad (14) \end{aligned}$$

and the restrictions imposed by considerations of convergence may now be removed. It may readily be verified that (11) leads to the same result.

The integral also takes a simple form in the case of sectorial harmonics. Putting $m = n$ in either (11) or (13), we have at once

$$\begin{aligned} \sqrt{(\cosh u - \cos v)}\, e^{(n+\frac{1}{2})u} P_n^n (\cos v) \cos nw \\ &= \frac{i^{-n} n!}{2\sqrt{2\pi}} \int_0^{2\pi} \frac{(2a)^{n+1} \cos nt\, dt}{(i\, x \cos t + i\, y \sin t + z - a)^{n+1}} \qquad (15) \end{aligned}$$

The general case of the tesseral harmonics does not admit of such simple forms. If n is a positive integer, the function f of Whittaker's formula is a rational function of its first argument; if n is not a positive integer, f may be expressed as a hypergeometric function, but the interchange of the order of the summation and integration necessary to express it in this form involves some loss to the regions of convergence.

Toroidal Harmonics.

Let us write $u = iu'$, $v = iv'$, $a = -ia'$. (1) and (2) then become

$$u' + iv' = i \log \frac{\varpi + iz - a'}{\varpi + iz + a'} \tag{1'}$$

and

$$\varpi = \frac{a' \sinh v'}{\cosh v' - \cos u'}, \quad z = \frac{a \sin u'}{\cosh v' - \cos u'} \tag{2'}$$

respectively.

Then, since from (1′)

$$v' = \tfrac{1}{2} \log \frac{(\varpi - a')^2 + z^2}{(\varpi + a')^2 + z^2}$$

the surfaces $v' = constant$ will be a series of anchor rings of circular cross-section, or "*tores.*"

The potential function (3) then becomes

$$i \sqrt{(\cosh v' - \cos u')}\, e^{i(n+\frac{1}{2})u} \begin{matrix}\sin\\ \cos\end{matrix}\, m\, w\, P_n^m (\cosh v').$$

These are the well-known toroidal harmonics introduced by Hicks.* Of the assumptions made in arriving at (11), the first, viz. that $R(\cos v) > 0$ is satisfied everywhere. The second, that $|\eta| > 2a$ may readily be seen to be satisfied at points inside a sphere whose centre is the origin and whose radius is a, and which therefore passes through the limiting circle of the tores. Hence, in place of (11), we have

$$\begin{aligned} &\sqrt{(\cosh v' - \cos u')}\, e^{i(n+\frac{1}{2})u} \cos mw\, P_n^m (\cosh v') \\ &= \frac{1}{2\sqrt{2\pi} i^{m+1}} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \sum_{p=0}^{\infty} \frac{(n-m)_p (n+m)_p}{p!\; n_p} \\ &\quad \times \int_0^{2\pi} \frac{(-2ia')^{n-p+1} \cos mt\, dt}{(i\,x \cos t + i\,y \sin t + z + ia')^{n+p+1}} \end{aligned} \tag{11'}$$

* *Phil. Trans.*, 1881, p. 609.

and similarly from (13) the same harmonic is

$$= \frac{1}{2\sqrt{2\pi}\, i^{m+1}} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \sum_{p=0}^{\infty} \frac{(n-m)_p (m+p)!}{p!\,(2m+p)!}$$

$$\times \int_0^{2\pi} \frac{(-2ia')^{m+p+1} \cos mt\, dt}{(i\,x\cos t + i\,y \sin t + z + i a')^{m+p+1}} \qquad (13')$$

Corresponding to (14) we have the zonal toroidal harmonics

$$\sqrt{(\cosh v' - \cos u')}\, e^{i(n+\frac{1}{2})u}\, P_n(\cosh v')$$

$$= \frac{a'}{\sqrt{2\pi}} \int_0^{2\pi} \frac{(i\,x\cos t + i\,y\sin t + z - ia')^n}{(i\,x\cos t + i\,y\sin t + z + ia')^{n+1}}\, dt$$

The toroidal harmonics corresponding to (15) are of little importance, for they require n to be an integer, whereas in almost all applications of toroidal harmonics n is half an odd integer.

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