

BANACH LATTICES OF POSITIVELY HOMOGENEOUS FUNCTIONS INDUCED BY A BANACH SPACE

NIELS JAKOB LAUSTSEN¹ AND PEDRO TRADACETE² 

¹*School of Mathematical Sciences, Fylde College, Lancaster University, Lancaster LA1 4YF, UK*

²*Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13–15, Campus de Cantoblanco UAM, 28049 Madrid, Spain*

Corresponding author: Pedro Tradacete, email: pedro.tradacete@icmat.es

(Received 17 June 2024)

Abstract Motivated by the construction of the free Banach lattice generated by a Banach space, we introduce and study several vector and Banach lattices of positively homogeneous functions defined on the dual of a Banach space E . The relations between these lattices allow us to give multiple characterizations of when the underlying Banach space E is finite-dimensional and when it is reflexive. Furthermore, we show that lattice homomorphisms between free Banach lattices are always composition operators, and study how these operators behave on the scale of lattices of positively homogeneous functions.

Keywords: Banach lattice; Banach space; free Banach lattice; lattice homomorphism

2010 Mathematics subject classification: 46B42; 46E05; 46B50

1. Introduction

From the point of view of Banach space theory, Banach lattices constitute an important class which includes many of the classical function spaces arising in analysis. The order structure of a Banach lattice is deeply tied to the geometry of its norm, and much research has been carried out to elucidate this relationship.

A common approach in the literature is to use a Banach space E to construct, or “induce”, a Banach lattice. We shall study a collection of Banach and vector lattices obtained in this way, consisting of positively homogeneous functions defined on the dual space of E , with the aim of understanding how key geometric properties of E are expressed in the order structure of and relationship between the Banach lattices it induces.

Our motivation comes from the study of “free Banach lattices”. This notion arose in the work of de Pagter and Wickstead [14], who defined the free Banach lattice over a set.

© The Author(s), 2025. Published by Cambridge University Press on Behalf of The Edinburgh Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<http://creativecommons.org/licenses/by/4.0>), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Subsequently, Avilés, Rodríguez and Tradacete [6] defined the free Banach lattice $\text{FBL}[E]$ generated by a Banach space E and observed that this object generalizes that of de Pagter and Wickstead in the sense that, for a set A , $\text{FBL}[\ell_1(A)]$ is the free Banach lattice over A .

As shown in [3, 4, 14], free Banach lattices are closely related to the notion of projectivity. They also played a key role in the recent construction of push-outs in the category of Banach lattices [7], and they have been instrumental in resolving a number of open problems about the structure of Banach lattices, including the following:

- A Banach lattice may be lattice weakly compactly generated, but not embed into any weakly compactly generated Banach space [6, Corollary 5.5]; this answered a question of Diestel.
- A lattice homomorphism between Banach lattices need not attain its operator norm [12].
- A closed, infinite-dimensional subspace of a Banach lattice need not contain any bibasic sequences [29, Theorem 7.5].

Other recent results concern the relationship between a Banach space E and the free Banach lattice $\text{FBL}[E]$ it generates. Let us briefly mention a few of these:

- E is separable if and only if $\text{FBL}[E]$ is separable [6, Theorem 3.2], if and only if $\text{FBL}[E]$ has a quasi-interior point [29, Proposition 9.4].
- $\text{FBL}[E]^*$ is order continuous precisely when E does not contain any complemented copies of ℓ_1 [29, Theorem 9.20].
- $\text{FBL}[E]$ satisfies an upper p -estimate if and only if the identity operator on E^* is $(p^*, 1)$ -summing, where p^* is the conjugate exponent of p [29, Theorem 9.21].
- $\text{FBL}[E]$ always satisfies the countable chain condition [5].

Formally speaking, the *free Banach lattice generated by a Banach space E* is a Banach lattice $\text{FBL}[E]$ together with a linear isometry $\delta^E: E \rightarrow \text{FBL}[E]$ such that, for every Banach lattice X and every bounded linear operator $T: E \rightarrow X$, there exists a unique lattice homomorphism $\hat{T}: \text{FBL}[E] \rightarrow X$ such that the diagram

$$\begin{array}{ccc} & \text{FBL}[E] & \\ \delta^E \uparrow & \searrow \hat{T} & \\ E & \xrightarrow{T} & X \end{array}$$

is commutative, and $\|\hat{T}\| = \|T\|$. This construction defines a covariant functor from the category of Banach spaces and bounded linear operators into the category of Banach lattices and lattice homomorphisms.

Standard arguments show that if the free Banach lattice generated by a Banach space E exists, then it is unique up to isometric lattice isomorphism, so the significance of [6] is the proof that $\text{FBL}[E]$ exists. Constructing free vector lattices is quite straightforward (see [8, 10]), but the method does not immediately carry over to Banach lattices (although it is possible to adapt it, as shown in [33]). Instead, the authors of [6] took a different route, constructing $\text{FBL}[E]$ explicitly as a Banach lattice of functions. This additional feature

has turned out to be very useful and is a cornerstone of our work. Let us therefore begin with a review of the construction.

Take a Banach space E , and denote by $H[E]$ the linear subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions, where we recall that a function $f: E^* \rightarrow \mathbb{R}$ is positively homogeneous if $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in E^*$ and $\lambda \in [0, \infty)$. For $f \in H[E]$, define

$$\|f\|_{\text{FBL}[E]} = \sup \left\{ \sum_{j=1}^n |f(x_j^*)| : n \in \mathbb{N}, (x_j^*)_{j=1}^n \subset E^*, \sup_{\|x\| \leq 1} \sum_{j=1}^n |x_j^*(x)| \leq 1 \right\}. \quad (1.1)$$

Then $H^1[E] = \{f \in H[E] : \|f\|_{\text{FBL}[E]} < \infty\}$ is a Banach lattice with respect to the norm $\|\cdot\|_{\text{FBL}[E]}$ and the pointwise vector lattice operations. Now, for $x \in E$, the map $\delta_x: E^* \rightarrow \mathbb{R}$ defined by $\delta_x(x^*) = x^*(x)$ belongs to $H^1[E]$ with $\|\delta_x\|_{\text{FBL}[E]} = \|x\|_E$, and [6, Theorem 2.5] shows that the free Banach lattice $\text{FBL}[E]$ can be realized as the closed sublattice of $H^1[E]$ generated by $\{\delta_x : x \in E\}$, together with the linear isometry $\delta^E: E \rightarrow \text{FBL}[E]$ given by $\delta^E(x) = \delta_x$ for $x \in E$.

This construction was generalized in [25] to produce the *free p -convex Banach lattice* generated by a Banach space E for $1 \leq p \leq \infty$. To state the definition, set

$$\|f\|_{\text{FBL}^p[E]} = \sup \left\{ \left(\sum_{j=1}^n |f(x_j^*)|^p \right)^{\frac{1}{p}} : n \in \mathbb{N}, (x_j^*)_{j=1}^n \subset E^*, \|(x_j^*)_{j=1}^n\|_{p, \text{weak}} \leq 1 \right\} \quad (1.2)$$

for $f \in H[E]$, where

$$\|(x_j^*)_{j=1}^n\|_{p, \text{weak}} = \sup_{\|x\| \leq 1} \left(\sum_{j=1}^n |x_j^*(x)|^p \right)^{\frac{1}{p}} \quad (1.3)$$

denotes the weak p -summing norm of the n -tuple $(x_j^*)_{j=1}^n$ in E^* . (As usual, expressions of the form $\left(\sum_{j=1}^n |t_j|^p \right)^{\frac{1}{p}}$ should be interpreted as $\max_{1 \leq j \leq n} |t_j|$ for $p = \infty$.) Note that for $p = 1$, (1.3) can equivalently be written as

$$\|(x_j^*)_{j=1}^n\|_{1, \text{weak}} = \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j^* \right\|. \quad (1.4)$$

Then $H^p[E] = \{f \in H[E] : \|f\|_{\text{FBL}^p[E]} < \infty\}$ is a Banach lattice as before, and [25, Theorem 6.1] shows that the closed sublattice of $H^p[E]$ generated by the set $\{\delta_x : x \in E\}$ together with the linear isometry $\delta^E: E \rightarrow \text{FBL}^p[E]$ given by $\delta^E(x) = \delta_x$ for $x \in E$ is the free p -convex Banach lattice generated by E .

We observe that $\text{FBL}[E] = \text{FBL}^1[E]$ because (1.2) reduces to (1.1) for $p = 1$. A less obvious result, shown in [29, Proposition 2.2], is that $\text{FBL}^\infty[E]$ can be identified with the space of positively homogeneous weak*-continuous functions defined on the dual unit ball.

The role of positive homogeneity in these constructions is not coincidental, but a consequence of the fact that the functions δ_x for $x \in E$ are our building blocks. They are linear, and hence lattice combinations of them will always be positively homogeneous. We remark that earlier work on Banach lattices of positively homogeneous functions can be found in [19, 30, 31, 34].

The purpose of the present paper is to investigate $H^p[E]$ for $1 \leq p < \infty$ in its own right, together with a number of sublattices and ideals lying between $\text{FBL}^p[E]$ and $H^p[E]$. This will in particular lead to characterizations of the cases where the underlying Banach space E is finite-dimensional (theorem 3.5) and reflexive (theorem 5.5).

In section 4, we shall see that in the infinite-dimensional situation, one should be able to provide examples of weak*-continuous functions in $H^p[E]$ that are not in $\text{FBL}^p[E]$. Surprisingly, our proof requires the existence of a separable quotient of the underlying Banach space E . Although this might be just an artefact of the proof, in any case it provides a fairly general condition that allows us to distinguish $\text{FBL}^p[E]$ within the weak* continuous functions of $H^p[E]$.

Finally, in Section 6 we study lattice homomorphisms between free Banach lattices. Our main result (theorem 6.4) states that every lattice homomorphism $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$, where $1 \leq p < \infty$ and E and F are Banach spaces, can be expressed as a composition operator

$$Tf = f \circ \Phi_T \quad (f \in \text{FBL}^p[E])$$

for a unique map $\Phi_T: F^* \rightarrow E^*$, which is positively homogeneous and maps weakly summable sequences in F^* to weakly summable sequences in E^* , and whose restriction to the closed unit ball of F^* is weak*-continuous. We explore the consequences and applications of this result, including the impact of the map Φ_T on the other Banach lattices under investigation. We refer the reader to [1] and [2] for unexplained terms from the classical theory of Banach spaces and Banach lattices.

2. Banach lattices of positively homogeneous functions defined on a dual Banach space

Take $p \in [1, \infty]$, and let E be a Banach space, always over the real field. (We refer to [13] for the construction and study of free complex Banach lattices over a complex Banach space.) The Banach lattice $(H^p[E], \|\cdot\|_{\text{FBL}^p[E]})$ plays a central role in the definition of $\text{FBL}^p[E]$, being the ambient lattice that $\text{FBL}^p[E]$ is defined as a sublattice of.

We shall consider the following vector lattices and ideals, several of which were already mentioned in the Introduction; for ease of reference, we include them here:

$$\begin{aligned} H[E] &= \{f \in \mathbb{R}^{E^*} : f \text{ is positively homogeneous}\} \\ \text{lat}\delta[E] &= \text{the sublattice of } H[E] \text{ generated by } \{\delta_x : x \in E\} \\ I[E] &= \text{the ideal of } H[E] \text{ generated by } \{\delta_x : x \in E\} \\ H^p[E] &= \{f \in H[E] : \|f\|_{\text{FBL}^p[E]} < \infty\} \end{aligned}$$

$$\begin{aligned}
\text{FBL}^p[E] &= \text{the closed sublattice of } H^p[E] \text{ generated by } \{\delta_x : x \in E\} \\
H_{w^*}[E] &= \{f \in H[E] : f|_{B_{E^*}} \text{ is weak}^*\text{-continuous}\} \\
I_{w^*}[E] &= I[E] \cap H_{w^*}[E] \\
H_{w^*}^p[E] &= H^p[E] \cap H_{w^*}[E] \\
J_{w^*}^p[E] &= \text{the ideal of } H^p[E] \text{ generated by } H_{w^*}^p[E] \\
M_{w^*,0}^p[E] &= \{f \in H^p[E] : f|_{B_{E^*}} \text{ is weak}^*\text{-continuous at } 0\},
\end{aligned}$$

where B_{E^*} denotes the closed unit ball of E^* . Note the change of terminology here: what we call $H^1[E]$ was originally denoted $H_0[E]$ in [6]. As already stated in the Introduction, $H^p[E]$ is a sublattice of $H[E]$ and a Banach lattice with respect to the norm $\|\cdot\|_{\text{FBL}^p[E]}$ defined by (1.2).

For $f \in H[E]$, we write

$$\|f|_{B_{E^*}}\|_\infty = \sup\{|f(x^*)| : x^* \in B_{E^*}\} \in [0, \infty] \quad (2.1)$$

for the uniform norm of the restriction of f to B_{E^*} . Then [29, Proposition 2.2] shows that $(\text{FBL}^\infty[E], \|\cdot\|_{\text{FBL}^\infty[E]})$ is isometrically lattice isomorphic to $(H_{w^*}[E], \|\cdot\|_\infty)$, as previously mentioned. For that reason, we shall restrict our attention to the case $p < \infty$ in the remainder of this paper.

We begin with some basic general observations.

Lemma 2.1. *Let E be a Banach space, and take $1 \leq p < \infty$. Then:*

- (i) $I[E] = \{f \in H[E] : |f| \leq \sum_{j=1}^n |\delta_{x_j}| \text{ for some } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in E\}$
 $= \{f \in H[E] : |f| \leq \bigvee_{j=1}^n |\delta_{x_j}| \text{ for some } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in E\}.$
- (ii) $H_{w^*}[E]$ is a sublattice of $H[E]$.
- (iii) The norm (1.2) dominates the norm (2.1); that is, $\|f\|_{\text{FBL}^p[E]} \geq \|f|_{B_{E^*}}\|_\infty$ for every $f \in H^p[E]$.
- (iv) $H_{w^*}[E]$ is closed in the norm $\|\cdot\|_\infty$ defined by (2.1) in the following precise sense: Suppose that (f_n) is a sequence in $H_{w^*}[E]$ which converges uniformly on B_{E^*} to a function $f \in H[E]$. Then $f \in H_{w^*}[E]$.
- (v) $H_{w^*}^p[E]$ is a closed sublattice of the Banach lattice $(H^p[E], \|\cdot\|_{\text{FBL}^p[E]})$.
- (vi) $\text{FBL}^p[E] \subseteq H_{w^*}[E]$.
- (vii) $I[E] \subseteq J_{w^*}^p[E]$.
- (viii) $\overline{I_{w^*}[E]} \subseteq H_{w^*}^p[E]$, where the closure is taken in $(H^p[E], \|\cdot\|_{\text{FBL}^p[E]})$.
- (ix) $M_{w^*,0}^p[E]$ is a closed ideal of $H^p[E]$ containing $\overline{J_{w^*}^p[E]}$, where the closure is again taken with respect to the norm $\|\cdot\|_{\text{FBL}^p[E]}$.

Proof. (i) The first equality follows from the fact that the set on the right-hand side is the smallest solid subspace of $H[E]$ which contains δ_x for every $x \in E$. The second equality is immediate from the inequalities

$$\bigvee_{j=1}^n |\delta_{x_j}| \leq \sum_{j=1}^n |\delta_{x_j}| \leq \bigvee_{j=1}^n |\delta_{nx_j}| \quad (n \in \mathbb{N}, x_1, \dots, x_n \in E).$$

(ii) This is a consequence of the fact that the lattice operations in $H[E]$ are defined pointwise.

(iii) This is clear because $\|x^*\|_{p,\text{weak}} = \|x^*\|$ for each $x^* \in E^*$.

(iv) This follows from the general result that the uniform limit of a sequence of continuous functions defined on a compact Hausdorff space is continuous.

(v) Clause (ii) implies that $H_{w^*}^p[E]$ is a sublattice of $H^p[E]$. To see that it is closed, suppose that (f_n) is a sequence in $H_{w^*}^p[E]$ which converges to $f \in H^p[E]$ in the norm $\|\cdot\|_{\text{FBL}^p[E]}$. Then (f_n) converges uniformly on B_{E^*} to f by (iii), so (iv) implies that $f \in H_{w^*}^p[E]$.

(vi) This is proved for $p = 1$ in [6, Lemma 4.10], and the argument given therein carries over verbatim to $p > 1$. Indeed, (v) implies that $S := \text{FBL}^p[E] \cap H_{w^*}^p[E]$ is a closed sublattice of $(\text{FBL}^p[E], \|\cdot\|_{\text{FBL}^p[E]})$. Clearly $\delta_x \in S$ for every $x \in E$, so $S = \text{FBL}^p[E]$; that is, $\text{FBL}^p[E] \subseteq H_{w^*}^p[E]$.

(vii) We begin by showing that $I[E] \subseteq H^p[E]$. Take $f \in I[E]$. By (i), we can find $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ such that $|f| \leq \sum_{k=1}^n |\delta_{x_k}|$. Suppose that $m \in \mathbb{N}$ and $x_1^*, \dots, x_m^* \in E^*$ with $\|(x_j^*)_{j=1}^m\|_{p,\text{weak}} \leq 1$. Then $\sum_{j=1}^m |x_j^*(x_k)|^p \leq \|x_k\|^p$ for each $k \in \{1, \dots, n\}$, and consequently

$$\left(\sum_{j=1}^m |f(x_j^*)|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^m \left(\sum_{k=1}^n |x_j^*(x_k)| \right)^p \right)^{\frac{1}{p}} \leq \sum_{k=1}^n \left(\sum_{j=1}^m |x_j^*(x_k)|^p \right)^{\frac{1}{p}} \leq \sum_{k=1}^n \|x_k\|,$$

where the second inequality is simply the statement that the norm of the sum of n vectors in ℓ_p^m is dominated by the sum of their norms. We conclude that $\|f\|_{\text{FBL}^p[E]} \leq \sum_{k=1}^n \|x_k\| < \infty$, and therefore $f \in H^p[E]$, as desired.

This implies that $I[E]$ is the ideal of $H^p[E]$ generated by δ_x for $x \in E$. These generators belong to $H_{w^*}^p[E]$ (as we already observed in (vi) above), and hence $I[E] \subseteq J_{w^*}^p[E]$.

(viii) We have just seen that $I[E] \subseteq H^p[E]$, so $I_{w^*}[E] \subseteq H_{w^*}^p[E]$. Now the conclusion follows from the fact that $H_{w^*}^p[E]$ is closed by (v).

(ix) Suppose that f belongs to the closure of the ideal of $H^p[E]$ generated by $M_{w^*,0}^p[E]$, and take a weak*-null net (x_α^*) in B_{E^*} . The set $M_{w^*,0}^p[E]$ is a sublattice of $H[E]$ because the vector lattice operations are defined pointwise. Therefore, for each $\varepsilon > 0$, we can find $g \in H^p[E]$ and $h \in M_{w^*,0}^p[E]$ such that $\|f - g\|_{\text{FBL}^p[E]} \leq \varepsilon$ and $|g| \leq |h|$. Since $h(0) = 0$ by positive homogeneity, we can choose α_0 such that $|h(x_\alpha^*)| \leq \varepsilon$ for every $\alpha \geq \alpha_0$. Then we have

$$\begin{aligned} |f(x_\alpha^*) - f(0)| &= |f(x_\alpha^*)| \leq |f(x_\alpha^*) - g(x_\alpha^*)| + |g(x_\alpha^*)| \\ &\leq \|f - g\|_{\text{FBL}^p[E]} + |h(x_\alpha^*)| \leq 2\varepsilon \quad (\alpha \geq \alpha_0), \end{aligned}$$

which shows that $f \in M_{w^*,0}^p[E]$, and therefore $M_{w^*,0}^p[E]$ is a closed ideal of $H^p[E]$.

This implies that $\overline{J_{w^*}^p[E]} \subseteq M_{w^*,0}^p[E]$ because $H_{w^*}^p[E] \subseteq M_{w^*,0}^p[E]$. $\square$

The following two chains of inclusions summarize the relationship among the sets defined above:

$$\begin{aligned} \text{lat}\delta[E] &\subseteq \text{FBL}^p[E] \subseteq \overline{I_{w^*}[E]} \subseteq H_{w^*}^p[E] \\ &\subseteq J_{w^*}^p[E] \subseteq \overline{J_{w^*}^p[E]} \subseteq M_{w^*,0}^p[E] \subseteq H^p[E], \end{aligned} \quad (2.2)$$

where $\overline{S}$ denotes the closure of a set S with respect to the norm $\|\cdot\|_{\text{FBL}^p[E]}$, and

$$\text{lat}\delta[E] \subseteq I_{w^*}[E] \subseteq I[E] \subseteq J_{w^*}^p[E]. \quad (2.3)$$

In the next three sections we shall examine when two (or more) of these lattices are equal.

3. Characterizations of finite-dimensionality

In this section we study the chains of lattices (2.2) and (2.3) when the Banach space E is finite-dimensional. It turns out that in this case the lattices will effectively reduce to two options, either $C(S_{E^*})$ or $\ell_\infty(S_{E^*})$, where $S_{E^*} = \{x^* \in E^* : \|x^*\| = 1\}$ denotes the dual unit sphere; see proposition 3.2 and corollary 3.6 for details.

We begin by showing that in dimensions 0 and 1, even more is true: all the lattices defined at the beginning of section 2 are equal, and this happens only in these two dimensions. Perhaps more surprisingly, this is equivalent to the second inclusion $I_{w^*}[E] \subseteq I[E]$ in (2.3) being an equality.

Lemma 3.1. *Let E be a Banach space, and take $1 \leq p < \infty$. Then the following conditions are equivalent:*

- (a) $\dim E \leq 1$;
- (b) $\text{lat}\delta[E] = H[E]$;
- (c) $H^p[E] \subseteq H_{w^*}[E]$;
- (d) $H_{w^*}^p[E] = J_{w^*}^p[E]$;
- (e) $I_{w^*}[E] = I[E]$.

Proof. (a) $\Rightarrow$ (b). We have $\text{lat}\delta[E] = H[E]$ for $E = \{0\}$ because $H[E] = \{0\}$ in this case.

Now suppose that $E = \mathbb{R}$, in which case $E^* = \mathbb{R}$, and the duality is given by $\delta_x(y) = \langle x, y \rangle = xy$ for $x, y \in \mathbb{R}$. Take $f \in H[\mathbb{R}]$, and suppose that $f(1) \geq -f(-1)$. Then, for each $y \in \mathbb{R}$, we have

$$\begin{aligned} (\delta_{f(1)} \vee \delta_{-f(-1)})(y) &= \delta_{f(1)}(y) \vee \delta_{-f(-1)}(y) = f(1)y \vee (-f(-1)y) \\ &= \begin{cases} f(1)y & \text{for } y \geq 0 \\ -f(-1)y & \text{for } y < 0 \end{cases} = f(y) \end{aligned}$$

by positive homogeneity. Hence $f = \delta_{f(1)} \vee \delta_{-f(-1)}$. A similar argument shows that $f = \delta_{f(1)} \wedge \delta_{-f(-1)}$ when $f(1) \leq -f(-1)$, and so $f \in \text{lat}\delta[\mathbb{R}]$ in both cases.

We have (b) $\Rightarrow$ (c) because the inclusions $H^p[E] \subseteq H[E]$ and $\text{lat}\delta[E] \subseteq H_{w^*}[E]$ are always true.

(c) $\Rightarrow$ (d). Suppose that $H^p[E] \subseteq H_{w^*}[E]$. Then we have $H_{w^*}^p[E] = H^p[E]$, from which (d) follows.

(d) $\Rightarrow$ (e). Suppose that $H_{w^*}^p[E] = J_{w^*}^p[E]$. Then lemma 2.1(vii) implies that $I[E] \subseteq H_{w^*}^p[E] \subseteq H_{w^*}[E]$.

Finally, we prove that (e) $\Rightarrow$ (a) by contraposition. Suppose that $\dim E \geq 2$, and choose two linearly independent functionals $x_0^*, y_0^* \in E^*$, each having norm $\frac{1}{2}$. Then $(x_0^* + \frac{1}{n}y_0^*)_{n \in \mathbb{N}}$ is a sequence in B_{E^*} which converges to x_0^* in norm and therefore also in the weak* topology. Define $f: E^* \rightarrow \mathbb{R}$ by

$$f(x^*) = \begin{cases} t & \text{if } x^* = tx_0^* \text{ for some } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f is positively homogeneous. Choose $x_0 \in E$ such that $x_0^*(x_0) = 1$. Then $|f| \leq |\delta_{x_0}|$, so $f \in I[E]$. On the other hand, $f(x_0^* + \frac{1}{n}y_0^*) = 0$ for each $n \in \mathbb{N}$ by definition, but $f(x_0^*) = 1$, so $f|_{B_{E^*}}$ is discontinuous at x_0^* , and therefore $f \notin H_{w^*}[E]$. $\square$

Let E be a non-zero Banach space. The restriction mapping $R: f \mapsto f|_{S_{E^*}}$ defines a lattice isomorphism of $H[E]$ onto the vector lattice of functions $S_{E^*} \rightarrow \mathbb{R}$ because every function $S_{E^*} \rightarrow \mathbb{R}$ extends uniquely to a positively homogeneous function $E^* \rightarrow \mathbb{R}$. When E is finite-dimensional, we can use this lattice isomorphism R to identify the Banach lattices $\text{FBL}^p[E]$ and $H^p[E]$ explicitly.

Proposition 3.2. *Let E be a non-zero, finite-dimensional Banach space. Then, for every $1 \leq p < \infty$,*

$$R(\text{FBL}^p[E]) = R(H_{w^*}[E]) = C(S_{E^*}) \quad \text{and} \quad R(J_{w^*}^p[E]) = R(H^p[E]) = \ell_\infty(S_{E^*}),$$

where $\ell_\infty(S_{E^*})$ denotes the Banach lattice of bounded functions $S_{E^*} \rightarrow \mathbb{R}$.

Proof. Lemma 2.1(vi) implies that $R(\text{FBL}^p[E]) \subseteq R(H_{w^*}[E])$, while the inclusion $R(H_{w^*}[E]) \subseteq C(S_{E^*})$ follows from the fact that the weak* and norm topologies on E^* coincide. Therefore, to prove the left-hand identity, it only remains to verify that $C(S_{E^*}) \subseteq R(\text{FBL}^p[E])$, which can be done by arguing as in the proof of [14, Proposition 5.3]; see also [6, Corollary 2.9(iii)] for the connection between de Pagter and Wickstead's notion of the free Banach lattice over a set studied in [14] and the free Banach lattice over a Banach space.

To prove the right-hand identity, we begin by observing that $R(J_{w^*}^p[E]) \subseteq R(H^p[E])$ because $J_{w^*}^p[E] \subseteq H^p[E]$ by definition, and $R(H^p[E]) \subseteq \ell_\infty(S_{E^*})$ because $|f(x^*)| \leq \|f\|_{\text{FBL}^p[E]}$ for every $x^* \in B_{E^*}$ and $f \in H^p[E]$. Finally, to verify that $\ell_\infty(S_{E^*}) \subseteq R(J_{w^*}^p[E])$, take $f_0 \in \ell_\infty(S_{E^*})$. In view of the remarks we made before the statement of the result, we can find $f \in H[E]$ such that $Rf = f_0$. Let $\nu \in H[E]$ be the “norm function” defined by $\nu(x^*) = \|x^*\|$ for every $x^* \in E^*$. Then $R\nu$ is the constant function 1, which obviously belongs to $C(S_{E^*})$, so $\nu \in \text{FBL}^p[E] \subseteq H_{w^*}^p[E]$ by the first part of the

proof and lemma 2.1(vi). Since f is positively homogeneous, we have $|f| \leq \|f \upharpoonright_{S_{E^*}}\|_\infty \nu$, which shows that $f \in J_{w^*}^p[E]$, and the conclusion follows. $\square$

Remark 3.3. The equality signs in proposition 3.2 mean “equal as vector lattices”, with only equivalence of the norms. In fact, one can check that the constants in the equivalence tend to ∞ as the dimension of the underlying Banach space E grows.

The following general observation will be useful for us.

Lemma 3.4. *Let (x_n) be a sequence in a Banach space E with $\|x_n\| \xrightarrow{n \rightarrow \infty} 0$. Then*

$$\bigvee_{n \in \mathbb{N}} |\delta_{x_n}| \in H_{w^*}[E].$$

Proof. Set $f = \bigvee_{n \in \mathbb{N}} |\delta_{x_n}|$, which is clearly a positively homogeneous function, and consider the functions $f_m = \bigvee_{n=1}^m |\delta_{x_n}| \in H_{w^*}[E]$ for $m \in \mathbb{N}$. They satisfy

$$\begin{aligned} \|(f - f_m) \upharpoonright_{B_{E^*}}\|_\infty &= \sup_{x^* \in B_{E^*}} \left| \bigvee_{n=1}^\infty |x^*(x_n)| - \bigvee_{n=1}^m |x^*(x_n)| \right| \\ &\leq \sup_{x^* \in B_{E^*}} \bigvee_{n=m+1}^\infty |x^*(x_n)| = \bigvee_{n=m+1}^\infty \|x_n\| \xrightarrow{m \rightarrow \infty} 0, \end{aligned}$$

so $f \in H_{w^*}[E]$ by lemma 2.1(iv). $\square$

Theorem 3.5. *Let E be a Banach space, and take $1 \leq p < \infty$. Then the following conditions are equivalent:*

- (a) E is finite-dimensional;
- (b) $\text{FBL}^p[E] = H_{w^*}[E]$;
- (c) $H_{w^*}[E] \subseteq H^p[E]$;
- (d) $I[E] = H^p[E]$;
- (e) $I[E] = J_{w^*}^p[E]$;
- (f) $I[E]$ is closed in $H^p[E]$;
- (g) $\text{FBL}^p[E] \subseteq I[E]$;
- (h) $I_{w^*}[E] = H_{w^*}[E]$;
- (i) $I_{w^*}[E]$ is closed in $H^p[E]$;
- (j) $\text{FBL}^p[E] \subseteq I_{w^*}[E]$.

Proof. The proof has three parts: first we show that conditions (a)–(c) are equivalent, then we show that conditions (a) and (d)–(g) are equivalent, and finally we show that conditions (h)–(j) are equivalent to the other conditions.

(a) $\Rightarrow$ (b). This follows from the first identity in proposition 3.2.

(b) $\Rightarrow$ (c). This is obvious because $\text{FBL}^p[E] \subseteq H^p[E]$ by definition.

(c) $\Rightarrow$ (a). To prove the contrapositive, suppose that E is infinite-dimensional. Then, by the weak Dvoretzky–Rogers theorem [15, Theorem 2.18], E^* contains a sequence (x_n^*) which is weakly p -summable, but not strongly p -summable. By scaling this sequence by a suitable constant, we may suppose that

$$\sup_{x \in B_E} \sum_{n=1}^{\infty} |x_n^*(x)|^p \leq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \|x_n^*\|^p = \infty.$$

For each $n \in \mathbb{N}$, choose $x_n \in B_E$ such that $x_n^*(x_n) \geq \frac{\|x_n^*\|}{2}$, and choose a decreasing sequence (s_n) in $(0, 1)$ such that

$$\lim_{n \rightarrow \infty} s_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} s_n^p \|x_n^*\|^p = \infty.$$

Lemma 3.4 shows that the function $f = \bigvee_{n \in \mathbb{N}} |\delta_{s_n x_n}|$ belongs to $H_{w^*}[E]$. However, $f \notin H^p[E]$ because the fact that $\|(x_k^*)_{k=1}^m\|_{p, \text{weak}} \leq 1$ for every $m \in \mathbb{N}$ implies that

$$\begin{aligned} \|f\|_{\text{FBL}^p[E]}^p &\geq \sum_{k=1}^m |f(x_k^*)|^p = \sum_{k=1}^m \left(\bigvee_{n \in \mathbb{N}} s_n |x_k^*(x_n)| \right)^p \\ &\geq \sum_{k=1}^m s_k^p |x_k^*(x_k)|^p \geq \frac{1}{2^p} \sum_{k=1}^m s_k^p \|x_k^*\|^p \xrightarrow{m \rightarrow \infty} \infty. \end{aligned}$$

Consequently $H_{w^*}[E] \not\subseteq H^p[E]$.

(a) $\Rightarrow$ (d). Suppose that E has dimension $n \in \mathbb{N}$, and let $(b_j)_{j=1}^n$ be a basis for E with coordinate functionals $(b_j^*)_{j=1}^n$, which form a basis for E^* . We may suppose that $\|b_j^*\| = 1$ for each $j \in \{1, \dots, n\}$. Given $f \in H^p[E]$, the number $t = \|f|_{B_{E^*}}\|_{\infty}$ is finite by lemma 2.1(iii), and for each non-zero $x^* \in E^*$, we have

$$|f(x^*)| = \|x^*\| \left| f\left(\frac{x^*}{\|x^*\|}\right) \right| \leq \|x^*\| \cdot t = \left\| \sum_{j=1}^n x^*(tb_j) b_j^* \right\| \leq \sum_{j=1}^n |x^*(tb_j)|.$$

This inequality is trivially true for $x^* = 0$, so $|f| \leq \sum_{j=1}^n |\delta_{tb_j}|$, and therefore $f \in I[E]$ by lemma 2.1(i). This shows that $H^p[E] \subseteq I[E]$, while the opposite inclusion follows from lemma 2.1(vii).

It is clear that (d) $\Rightarrow$ (f) $\Rightarrow$ (g). We also have (d) $\Rightarrow$ (e) $\Rightarrow$ (g) because lemma 2.1(vii) and (vi) show that the following inclusions always hold:

$$I[E] \subseteq J_{w^*}^p[E] \subseteq H^p[E] \quad \text{and} \quad \text{FBL}^p[E] \subseteq H_{w^*}^p[E] \subseteq J_{w^*}^p[E].$$

We prove that (g) $\Rightarrow$ (a) by contraposition. Suppose that E is infinite-dimensional, and take a normalized basic sequence $(b_n)_{n \in \mathbb{N}}$ in E . Let $b_n^* \in E^*$ be a Hahn–Banach extension of the n^{th} coordinate functional of this basic sequence for each $n \in \mathbb{N}$, and define $f = \sum_{n=1}^{\infty} |\delta_{b_n}|/2^n$, which belongs to $\text{FBL}^p[E]$ because the series converges absolutely in the norm $\|\cdot\|_{\text{FBL}^p[E]}$.

We claim that $f \notin I[E]$. By lemma 2.1(i), this amounts to showing that for every $m \in \mathbb{N}$ and $x_1, \dots, x_m \in E$, we can find $x^* \in E^*$ such that $f(x^*) > \sum_{j=1}^m |x^*(x_j)|$. Since $m+1 > m$, the system of m homogeneous linear equations

$$\sum_{k=1}^{m+1} b_k^*(x_j) t_k = 0 \quad (j = 1, \dots, m)$$

in the $m+1$ real variables $t_1, \dots, t_{m+1}$ has a non-trivial solution, which we denote $(t_1, \dots, t_{m+1}) \in \mathbb{R}^{m+1} \setminus \{(0, \dots, 0)\}$. Then the functional $x^* = \sum_{k=1}^{m+1} t_k b_k^* \in E^*$ satisfies $x^*(x_j) = 0$ for each $j = 1, \dots, m$, and therefore

$$f(x^*) = \sum_{n=1}^{m+1} \frac{|t_n|}{2^n} > 0 = \sum_{j=1}^m |x^*(x_j)|.$$

We have thus shown that conditions (a)–(g) are equivalent. Combining conditions (c) and (d), we see that they imply condition (h).

(h) $\Rightarrow$ (i). Suppose that $I_{w^*}[E] = H_{w^*}[E]$. Then $I_{w^*}[E] = H_{w^*}^p[E]$, which is closed by lemma 2.1(v).

(i) $\Rightarrow$ (j). This follows from the general fact that $\text{FBL}^p[E] \subseteq \overline{I_{w^*}[E]}$.

Finally, we can complete the proof by observing that the implication (j) $\Rightarrow$ (g) is trivial. $\square$

As a consequence, we obtain the following simplified version of the chains of inclusions exhibited in (2.2) and (2.3) when the Banach space E is finite-dimensional.

Corollary 3.6. *Let E be a Banach space of finite dimension at least 2, and take $1 \leq p < \infty$. Then*

$$\text{FBL}^p[E] = I_{w^*}[E] = H_{w^*}[E] \subsetneq I[E] = J_{w^*}^p[E] = M_{w^*,0}^p[E] = H^p[E].$$

Proof. The equalities follow from theorem 3.5 and lemma 2.1(ix), while proposition 3.2 implies that the inclusion in the middle is proper. $\square$

4. $H_{w^*}^p$ vs. FBL^p for infinite-dimensional spaces

Theorem 3.5 shows that when the Banach space E is infinite-dimensional, we should replace $I[E]$ and $I_{w^*}[E]$ with their closures $\overline{I[E]}$ and $\overline{I_{w^*}[E]}$ in the norm $\|\cdot\|_{\text{FBL}^p[E]}$, for $1 \leq p < \infty$, before we compare them with the other Banach lattices we study. In this section we shall focus on the inclusions

$$\text{FBL}^p[E] \subseteq \overline{I_{w^*}[E]} \subseteq H_{w^*}^p[E].$$

We note that all three sets are closed sublattices of $H^p[E]$, and theorem 3.5 shows that $\text{FBL}^p[E] = H_{w^*}^p[E]$ when E is finite-dimensional. However, at this point we do not know whether one or both of these inclusions could be an equality for some infinite-dimensional Banach space E , raising the following two questions:

Question 4.1. For which Banach spaces E is it true that $\text{FBL}^p[E] = \overline{I_{w^*}[E]}$ for some/all $1 \leq p < \infty$?

Question 4.2. For which Banach spaces E is it true that $\overline{I_{w^*}[E]} = H_w^p[E]$ for some/all $1 \leq p < \infty$?

We begin by addressing the first of these questions. Our aim is to prove the following result.

Theorem 4.3. *Let E be a Banach space which admits an infinite-dimensional, separable quotient space, and let $1 \leq p < \infty$. Then $I_{w^*}[E] \not\subseteq \text{FBL}^p[E]$, and hence*

$$\text{FBL}^p[E] \subsetneq \overline{I_{w^*}[E]}.$$

We remark that no infinite-dimensional Banach space E which fails to admit an infinite-dimensional, separable quotient space is known. In view of this, we conjecture that the condition that $\text{FBL}^p[E] = \overline{I_{w^*}[E]}$ for some (or all) $1 \leq p < \infty$ is equivalent to the Banach space E being finite-dimensional.

The following lemma will play a key role in the proof of theorem 4.3.

Lemma 4.4. *Let E be a Banach space. The following conditions are equivalent:*

- (a) *For every $(\lambda_n) \in \ell_2^+$, E^* contains a weakly 1-summable sequence (x_n^*) such that $\|x_n^*\| = \lambda_n$ for each $n \in \mathbb{N}$ and the subspace $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^\infty \ker x_n^*$ is dense in E .*
- (b) *For every $(\lambda_n) \in c_0^+$, E^* contains a weakly 2-summable sequence (x_n^*) such that $\|x_n^*\| = \lambda_n$ for each $n \in \mathbb{N}$ and the subspace $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^\infty \ker x_n^*$ is dense in E .*
- (c) *E^* contains a sequence (x_n^*) of linearly independent functionals such that the subspace $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^\infty \ker x_n^*$ is dense in E .*
- (d) *E^* contains a sequence (x_n^*) of non-zero functionals such that the subspace $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^\infty \ker x_n^*$ is dense in E .*
- (e) *E admits an infinite-dimensional, separable quotient space.*

Proof. We begin by showing that conditions (c), (d) and (e) are equivalent. It is clear that (c) implies (d).

(d) $\Rightarrow$ (e). Suppose that (x_n^*) is a sequence in $E^* \setminus \{0\}$ such that $\bigcup_{m \in \mathbb{N}} W_m$ is dense in E , where $W_m = \bigcap_{n=m}^\infty \ker x_n^*$. Clearly $W_m \subseteq W_{m+1}$ for each $m \in \mathbb{N}$, and either $W_m = W_{m+1}$ or W_m has codimension one in W_{m+1} . We cannot have $W_m = W_{m+1}$ for all but finitely many $m \in \mathbb{N}$ because if we did, we would have $\bigcup_{m \in \mathbb{N}} W_m = W_{m_0}$ for some $m_0 \in \mathbb{N}$, so $E = W_{m_0}$ because $\bigcup_{m \in \mathbb{N}} W_m$ is dense in E and W_{m_0} is closed. This would imply that $x_n^* = 0$ for each $n \geq m_0$, which contradicts our assumption. Hence, by passing to a subsequence, we may suppose that $W_m \subsetneq W_{m+1}$ for each $m \in \mathbb{N}$. According to a theorem of Mujica (see [24, page 199]), this implies that (e) is satisfied.

Alternatively, we can easily verify (e) directly, showing that the quotient space E/W_1 is separable and infinite-dimensional. To this end, take $w_m \in W_{m+1} \setminus W_m$ such that $W_{m+1} = W_m + \mathbb{R}w_m$ for each $m \in \mathbb{N}$. Then the subspace

$$W_1 + \text{span}\{w_m : m \in \mathbb{N}\}$$

is dense in E , and therefore the subspace $\text{span}\{Qw_m : m \in \mathbb{N}\}$ is dense in E/W_1 , where $Q: E \rightarrow E/W_1$ denotes the quotient map. Hence E/W_1 is separable. Furthermore, E/W_1

is infinite-dimensional because the sequence $(Qw_m)_{m \in \mathbb{N}}$ is linearly independent. Indeed, assuming the contrary, we can write $Qw_{m+1} = \sum_{j=1}^m s_j Qw_j$ for some $m \in \mathbb{N}$ and some $s_1, \dots, s_m \in \mathbb{R}$. Consequently $w_{m+1} - \sum_{j=1}^m s_j w_j \in W_1$, so $w_{m+1} \in W_{m+1}$, which is a contradiction.

(e) $\Rightarrow$ (c). Suppose that W is a closed subspace of E such that E/W is infinite-dimensional and separable. Then E/W has a Markushevich basis (see for instance [18, Theorem 4.59]), say $(Qx_m)_{m \in \mathbb{N}}$, where $x_m \in E$ and $Q: E \rightarrow E/W$ denotes the quotient map. Set $x_n^* = Q^*(y_n^*) \in E^*$ for each $n \in \mathbb{N}$, where $y_n^* \in (E/W)^*$ is the n^{th} biorthogonal functional of $(Qx_m)_{m \in \mathbb{N}}$, and observe that $(x_n^*)_{n \in \mathbb{N}}$ is linearly independent because Q^* is injective. We claim that $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^{\infty} \ker x_n^*$ is dense in E . Indeed, given $x \in E$ and $\varepsilon > 0$, we can find $m \in \mathbb{N}$, $s_1, \dots, s_m \in \mathbb{R}$ and $w \in W$ such that $\|x - \sum_{j=1}^m s_j x_j - w\| < \varepsilon$ because $\text{span}\{Qx_n : n \in \mathbb{N}\}$ is dense in E/W . Now the conclusion follows because

$$\left\langle \sum_{j=1}^m s_j x_j + w, x_n^* \right\rangle = \sum_{j=1}^m s_j \langle Qx_j, y_n^* \rangle = 0$$

for each $n > m$, so $\sum_{j=1}^m s_j x_j + w \in \bigcap_{n=m+1}^{\infty} \ker x_n^*$.

Clearly (a) implies (d) and (b) implies (d). We shall now complete the proof by showing that (c) implies (a) and (c) implies (b). This is the harder part of the proof, but it is actually quite simple, as the former consists of making appropriate adjustments to the proof of the Dvoretzky–Rogers theorem given in [15, Theorem 1.2], and the latter is a minor variation, using ideas from [23, Theorem 2]. We include the details for the convenience of the reader.

(c) $\Rightarrow$ (a). Let $(\lambda_n) \in \ell_2^+$, and set $n_0 = 0$. As in the proof of [15, Theorem 1.2], we choose integers $1 \leq n_1 < n_2 < \dots$ such that

$$\sum_{j=n_k+1}^{\infty} \lambda_j^2 \leq 2^{-2k} \quad (k \in \mathbb{N}). \quad (4.1)$$

Suppose that (y_n^*) is a linearly independent sequence in E^* such that the subspace $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^{\infty} \ker y_n^*$ is dense in E . For each $k \in \mathbb{N}$, we can apply [15, Lemma 1.3] to the $2(n_k - n_{k-1})$ -dimensional space $\text{span}\{y_j^* : 2n_{k-1} < j \leq 2n_k\}$ to find functionals

$$z_{n_{k-1}+1}^*, z_{n_{k-1}+2}^*, \dots, z_{n_k}^* \in \text{span}\{y_j^* : 2n_{k-1} < j \leq 2n_k\},$$

each having norm between $1/2$ and 1 , such that

$$\left\| \sum_{j=n_{k-1}+1}^{n_k} \alpha_j z_j^* \right\| \leq \left(\sum_{j=n_{k-1}+1}^{n_k} \alpha_j^2 \right)^{1/2} \quad (\alpha_{n_{k-1}+1}, \dots, \alpha_{n_k} \in \mathbb{R}). \quad (4.2)$$

Define $x_j^* = \lambda_j z_j^* / \|z_j^*\| \in E^*$ for every $j \in \{n_{k-1} + 1, \dots, n_k\}$, so that $\|x_j^*\| = \lambda_j$, and observe that

$$\bigcap_{i=2n_{k-1}+1}^{2n_k} \ker y_i^* \subseteq \ker z_j^* \subseteq \ker x_j^* \quad (j \in \{n_{k-1} + 1, \dots, n_k\}). \quad (4.3)$$

The argument given in [15] shows that the sequence (x_j^*) is weakly 1-summable.

Hence it only remains to show that $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^{\infty} \ker x_n^*$ is dense in E . Given $x \in E$ and $\varepsilon > 0$, choose $m \in \mathbb{N}$ and $y \in \bigcap_{n=m}^{\infty} \ker y_n^*$ such that $\|x - y\| < \varepsilon$. Then, choosing $\ell \in \mathbb{N}_0$ such that $2n_\ell + 1 \geq m$, we obtain

$$y \in \bigcap_{i=2n_\ell+1}^{\infty} \ker y_i^* = \bigcap_{k=\ell}^{\infty} \left(\bigcap_{i=2n_k+1}^{2n_{k+1}} \ker y_i^* \right) \subseteq \bigcap_{k=\ell}^{\infty} \left(\bigcap_{j=n_k+1}^{n_{k+1}} \ker x_j^* \right) = \bigcap_{j=n_\ell+1}^{\infty} \ker x_j^*,$$

where the inclusion in the middle follows from (4.3).

(c) $\Rightarrow$ (b). As already stated, this proof is a minor variation of the argument we have just given to show that (c) implies (a), so we shall focus on explaining the differences. It suffices to consider $(\lambda_n) \in c_0^+$ with norm at most 1. We can no longer ensure that (4.1) is satisfied; instead, we choose integers $0 = n_0 < n_1 < n_2 < \dots$ such that

$$\sup_{j > n_k} \lambda_j \leq 2^{-k} \quad (k \in \mathbb{N}). \quad (4.4)$$

Our assumption that $\|(\lambda_n)\|_{\infty} \leq 1$ implies that this estimate is also valid for $k = 0$. Now, for (y_n^*) chosen as before, we can find the sequence (z_j^*) and define (x_j^*) as above. Then the proof that $\bigcup_{m \in \mathbb{N}} \bigcap_{n=m}^{\infty} \ker x_n^*$ is dense in E carries over verbatim, so we only need to verify that (x_j^*) is weakly 2-summable. For every $(a_j) \in B_{\ell_2}$ and $m \in \mathbb{N}$, we can apply (4.2), (4.4) and the fact that $\|z_j^*\| \geq \frac{1}{2}$ to obtain

$$\begin{aligned} \left\| \sum_{j=1}^{n_m} a_j x_j^* \right\| &\leq \sum_{k=1}^m \left\| \sum_{j=n_{k-1}+1}^{n_k} a_j x_j^* \right\| = \sum_{k=1}^m \left\| \sum_{j=n_{k-1}+1}^{n_k} \frac{a_j \lambda_j}{\|z_j^*\|} z_j^* \right\| \\ &\leq \sum_{k=1}^m \left(\sum_{j=n_{k-1}+1}^{n_k} \frac{a_j^2 \lambda_j^2}{\|z_j^*\|^2} \right)^{\frac{1}{2}} \leq \sum_{k=1}^m 2^{2-k} \left(\sum_{j=n_{k-1}+1}^{n_k} a_j^2 \right)^{\frac{1}{2}} \leq 4. \end{aligned}$$

This shows that (x_j^*) is weakly 2-summable because $n_m \rightarrow \infty$ as $m \rightarrow \infty$. $\square$

Proof of theorem 4.3. This proof is a more subtle variant of the proof of the implication (c) $\Rightarrow$ (a) in theorem 3.5.

Recall that we seek a function $f \in I_{w^*}[E] \setminus \text{FBL}^p[E]$. It turns out to be convenient to decompose E as $E = F \oplus \mathbb{R}x_1$ and then carry out the main technical part of the construction of f within the subspace F . Consequently we choose $x_1 \in E$ and $x_1^* \in E^*$ such that $\|x_1\| = \|x_1^*\| = 1 = x_1^*(x_1)$, and we then define $F = \ker x_1^*$.

Since E admits an infinite-dimensional, separable quotient space, so does F ; that is, F satisfies one, and hence all, of the equivalent conditions in lemma 4.4. Using this, we shall

prove that, for a suitably chosen real number $q > p$, F^* contains a weakly p -summable sequence (y_n^*) such that

$$\|y_n^*\| = n^{-\frac{1}{q}} \quad (n \in \mathbb{N}) \quad \text{and} \quad \overline{\bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \ker y_n^*} = F. \quad (4.5)$$

We split the proof of this claim in two cases:

- If $p < 2$, we choose $q \in (p, 2)$ and observe that $(n^{-\frac{1}{q}})_{n \in \mathbb{N}} \in \ell_2^+$. Therefore lemma 4.4(a) implies that F^* contains a weakly 1-summable sequence (y_n^*) which satisfies (4.5). Since $p \geq 1$, it follows that (y_n^*) is weakly p -summable.
- Otherwise $p \geq 2$, in which case we choose $q \in (p, \infty)$. Since $(n^{-\frac{1}{q}})_{n \in \mathbb{N}} \in c_0^+$, lemma 4.4(b) shows that F^* contains a weakly 2-summable sequence (y_n^*) which satisfies (4.5). The fact that $p \geq 2$ ensures that (y_n^*) is weakly p -summable.

Let P be the projection of E onto F given by $Px = x - x_1^*(x)x_1$ for $x \in E$. For each $n \in \mathbb{N}$, set $x_{n+1}^* = P^*y_n^* \in E^*$, and take a unit vector $x_{n+1} \in F$ such that $y_n^*(x_{n+1}) \geq \|y_n^*\|/2$. Then we have:

- $x_{n+1}^*(x_1) = 0$ and $x_{n+1}^*(x_{n+1}) \geq \|y_n^*\|/2$ for every $n \in \mathbb{N}$;
- the subspace $W = \bigcup_{m \in \mathbb{N}} W_m$ is dense in E , where $W_m = \bigcap_{n=m}^\infty \ker x_n^*$;
- the sequence (x_n^*) is weakly p -summable, so

$$K := \sup_{x \in B_E} \left(\sum_{n=1}^\infty |x_n^*(x)|^p \right)^{\frac{1}{p}} < \infty. \quad (4.6)$$

Set $s_n = n^{\frac{1}{q} - \frac{1}{p}} \in (0, 1]$ for each $n \in \mathbb{N}$. Then $s_n \rightarrow 0$ as $n \rightarrow \infty$, so lemma 3.4 shows that the function $g = \bigvee_{n=1}^\infty |\delta_{s_n x_{n+1}}|$ belongs to $H_{w^*}[E]$, and therefore $f = |\delta_{x_1}| \wedge g$ belongs to $I_{w^*}[E]$ by lemma 2.1(ii). We shall now complete the proof by showing that $f \notin \text{FBL}^p[E]$, using (x_n^*) to produce a suitable “witness sequence”.

The sublattice L of $H[E]$ generated by $\{\delta_w : w \in W\}$ is dense in $\text{FBL}^p[E]$ because W is dense in E . (This follows easily from the construction of $\text{FBL}^p[E]$ in [25, Theorem 6.1].) Therefore it will suffice to show that

$$\inf \left\{ \|f - h\|_{\text{FBL}^p[E]} : h \in L \right\} > 0. \quad (4.7)$$

Since W is a subspace of E , every function $h \in L$ can be expressed in the form $h = \bigvee_{i=1}^k \delta_{v_i} - \bigvee_{i=1}^k \delta_{w_i}$ for some $k \in \mathbb{N}$ and some $v_1, \dots, v_k, w_1, \dots, w_k \in W$. As W is the union of the increasing sequence (W_m) of subspaces, we can find $m \in \mathbb{N}$ such that $v_1, \dots, v_k, w_1, \dots, w_k \in W_m$. Then, for each $j \in \{1, \dots, m\}$, we see that the functional $z_j^* = m^{-\frac{1}{p}} x_1^* + x_{m+j}^* \in E^*$ satisfies

$$\begin{aligned} h(z_j^*) &= \bigvee_{i=1}^k \left(m^{-\frac{1}{p}} x_1^*(v_i) + x_{m+j}^*(v_i) \right) - \bigvee_{i=1}^k \left(m^{-\frac{1}{p}} x_1^*(w_i) + x_{m+j}^*(w_i) \right) \\ &= \bigvee_{i=1}^k m^{-\frac{1}{p}} x_1^*(v_i) - \bigvee_{i=1}^k m^{-\frac{1}{p}} x_1^*(w_i) = m^{-\frac{1}{p}} h(x_1^*) \end{aligned} \quad (4.8)$$

because $v_1, \dots, v_k, w_1, \dots, w_k \in W_m \subseteq \ker x_n^*$ for every $n \geq m$.

It follows from (4.6) that $\|(x_j^*)_{j=m+1}^{2m}\|_{p,\text{weak}} \leq K$. Combining this estimate with the easy observation that

$$\left\| \underbrace{(m^{-\frac{1}{p}} x_1^*, \dots, m^{-\frac{1}{p}} x_1^*)}_m \right\|_{p,\text{weak}} = \|x_1^*\| = 1 \quad (4.9)$$

and the subadditivity of $\|\cdot\|_{p,\text{weak}}$, we see that $\|(z_j^*)_{j=1}^m\|_{p,\text{weak}} \leq K + 1$, so

$$(K + 1) \|f - h\|_{\text{FBL}^p[E]} \geq \left(\sum_{j=1}^m |f(z_j^*) - h(z_j^*)|^p \right)^{\frac{1}{p}}. \quad (4.10)$$

Another application of (4.9) shows that

$$\|f - h\|_{\text{FBL}^p[E]} \geq \left(\sum_{j=1}^m |m^{-\frac{1}{p}} (f(x_1^*) - h(x_1^*))|^p \right)^{\frac{1}{p}}. \quad (4.11)$$

Now we combine (4.10) and (4.11) with the subadditivity of the norm on ℓ_p^m to obtain

$$\begin{aligned} (K + 2) \|f - h\|_{\text{FBL}^p[E]} &\geq \left(\sum_{j=1}^m |f(z_j^*) - h(z_j^*) - m^{-\frac{1}{p}} (f(x_1^*) - h(x_1^*))|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{j=1}^m |f(z_j^*) - m^{-\frac{1}{p}} f(x_1^*)|^p \right)^{\frac{1}{p}}, \end{aligned} \quad (4.12)$$

where the final equality follows from (4.8).

To find a lower bound on the right-hand side of (4.12), we recall that $f = |\delta_{x_1}| \wedge g$, where $g(x_1^*) = 0$ because $x_{n+1} \in F = \ker x_1^*$ for every $n \in \mathbb{N}$, so $f(x_1^*) = 0$, while

$$\begin{aligned} f(z_j^*) &= |m^{-\frac{1}{p}} x_1^*(x_1) + x_{m+j}^*(x_1)| \wedge \bigvee_{n=1}^{\infty} s_n |m^{-\frac{1}{p}} x_1^*(x_{n+1}) + x_{m+j}^*(x_{n+1})| \\ &= m^{-\frac{1}{p}} \wedge \bigvee_{n=1}^{\infty} s_n |x_{m+j}^*(x_{n+1})| \geq m^{-\frac{1}{p}} \wedge s_{m+j-1} |x_{m+j}^*(x_{m+j})| \\ &\geq m^{-\frac{1}{p}} \wedge \frac{s_{m+j-1} \|y_{m+j-1}^*\|}{2} = m^{-\frac{1}{p}} \wedge \frac{1}{2} (m + j - 1)^{-\frac{1}{p}} \geq \frac{1}{2} (2m)^{-\frac{1}{p}} \end{aligned}$$

for each $j \in \{1, \dots, m\}$. Substituting these estimates into (4.12), we conclude that

$$(K + 2) \|f - h\|_{\text{FBL}^p[E]} \geq 2^{-\frac{p+1}{p}},$$

from which (4.7) follows because the constant K is independent of h .

This completes the proof that $f \in I_{w^*}[E] \setminus \text{FBL}^p[E]$. The final clause follows because $\text{FBL}^p[E] \subseteq \overline{I_{w^*}[E]}$. $\square$

Moving on to Question 4.2, we shall only give one result which shows that this inclusion need not be proper, and finding a general answer may not be easy. We first need to introduce some machinery.

Let E be a Banach space, and equip $B_{E^{**}}$ with the relative weak* topology. Then, for $1 \leq p < \infty$ and a positive Radon measure μ on $B_{E^{**}}$ (so in other words $\mu \in C(B_{E^{**}})_+^*$), we can define a function $f_\mu^p: E^* \rightarrow \mathbb{R}_+$ by

$$f_\mu^p(x^*) = \left(\int_{B_{E^{**}}} |x^{**}(x^*)|^p d\mu(x^{**}) \right)^{\frac{1}{p}} \quad (x^* \in E^*). \quad (4.13)$$

This construction is useful for our purposes because [25, Propositions 7.6 and 7.7] show that, on the one hand, $f_\mu^p \in H^p[E]$ with $\|f_\mu^p\|_{\text{FBL}^p[E]} \leq \|\mu\|$ and on the other, for every $f \in H^p[E]$, we can find a probability measure $\mu \in C(B_{E^{**}})_+^*$ such that

$$|f(x^*)| \leq \|f\|_{\text{FBL}^p[E]} f_\mu^p(x^*) \quad (x^* \in E^*). \quad (4.14)$$

Proposition 4.5. For $1 \leq p < \infty$ and any set A ,

$$\overline{I_{w^*}[\ell_1(A)]} = H_{w^*}^p[\ell_1(A)],$$

where the closure is taken with respect to the norm $\|\cdot\|_{\text{FBL}^p[\ell_1(A)]}$.

Proof. Set $E = \ell_1(A)$, and let $R: E^* \rightarrow C(B_{E^{**}})$ be the bounded linear operator defined by $(Rx^*)(x^{**}) = x^{**}(x^*)$ for $x^* \in E^*$ and $x^{**} \in B_{E^{**}}$; that is, Rx^* is the restriction of the evaluation map δ_{x^*} to $B_{E^{**}}$.

For $f \in H_{w^*}^p[E]$, we can choose a probability measure $\mu \in C(B_{E^{**}})_+^*$ such that (4.14) is satisfied. Let $J_\infty: C(B_{E^{**}}) \rightarrow L_\infty(B_{E^{**}}, \mu)$ and $J_p: L_\infty(B_{E^{**}}, \mu) \rightarrow L_p(B_{E^{**}}, \mu)$ be the formal inclusion maps. Then, by [11, Theorem I.8] or [15, page 97], the composite operator $T = J_p J_\infty R: E^* \rightarrow L_p(B_{E^{**}}, \mu)$ is strictly p -integral, and we can restate the inequality (4.14) as

$$|f(x^*)| \leq \|f\|_{\text{FBL}^p[E]} \|Tx^*\| \quad (x^* \in E^*). \quad (4.15)$$

Let $J_0: c_0(A) \rightarrow \ell_\infty(A) = E^*$ be the inclusion map. Its domain is an Asplund space, so [11, Theorem II.3] shows that the operator TJ_0 is p -nuclear; therefore, using [11, Definition I.1] or [15, Proposition 5.23], we can find sequences $(y_j)_{j \in \mathbb{N}}$ in $E = c_0(A)^*$ and $(z_j)_{j \in \mathbb{N}}$ in $L_p(B_{E^{**}}, \mu)$ such that

- (i) $TJ_0 x^* = \sum_{j=1}^\infty \langle y_j, x^* \rangle z_j$ for every $x^* \in c_0(A)$;
- (ii) $(y_j)_{j \in \mathbb{N}}$ is strongly p -summable; that is, $\sum_{j=1}^\infty \|y_j\|^p < \infty$;

- (iii) $(z_j)_{j \in \mathbb{N}}$ is weakly p^* -summable, where $p^* \in (1, \infty]$ is the conjugate exponent of p ; that is, $\sup_{j \in \mathbb{N}} \|z_j\| < \infty$ if $p = 1$, and otherwise

$$\sup \left\{ \sum_{j=1}^{\infty} |\langle z^*, z_j \rangle|^{p^*} : z^* \in L_{p^*}(B_{E^{**}}, \mu), \|z^*\| \leq 1 \right\} < \infty,$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$. By rescaling, we may suppose that these suprema equal 1.

Condition (ii) implies that we can define a bounded linear operator $U: E^* \rightarrow \ell_p$ by $Ux^* = (\langle y_j, x^* \rangle)_{j \in \mathbb{N}}$. Applying (i) and (iii), as well as Hölder's inequality if $p > 1$, we obtain

$$\begin{aligned} \|TJ_0x^*\| &= \sup_{\|z^*\| \leq 1} |\langle z^*, TJ_0x^* \rangle| = \sup_{\|z^*\| \leq 1} \left| \sum_{j=1}^{\infty} \langle y_j, x^* \rangle \langle z^*, z_j \rangle \right| \\ &\leq \left(\sum_{j=1}^{\infty} |\langle y_j, x^* \rangle|^p \right)^{\frac{1}{p}} = \|Ux^*\| \quad (x^* \in c_0(A)). \end{aligned} \quad (4.16)$$

We now claim that the map $g \in H[E]$ defined by $g(x^*) = \|Ux^*\|$ for $x^* \in E^*$ belongs to $\text{FBL}^p[E]$. For $\varepsilon > 0$, take $m \in \mathbb{N}$ such that $\sum_{j=m+1}^{\infty} \|y_j\|^p \leq \varepsilon^p$, and set $g_m = \left(\sum_{j=1}^m |\delta_{y_j}|^p \right)^{\frac{1}{p}} \in \text{FBL}^p[E]$. We observe that $g_m(x^*) = \left(\sum_{j=1}^m |\langle y_j, x^* \rangle|^p \right)^{\frac{1}{p}} = \|P_m Ux^*\|$ for $x^* \in E^*$, where $P_m: \ell_p \rightarrow \ell_p$ denotes the m^{th} basis projection. Hence, for $n \in \mathbb{N}$ and $(x_k^*)_{k=1}^n \subset E^*$ with $\|(x_k^*)_{k=1}^n\|_{p, \text{weak}} \leq 1$, we have

$$\begin{aligned} \sum_{k=1}^n |(g - g_m)(x_k^*)|^p &= \sum_{k=1}^n \left| \|Ux_k^*\| - \|P_m Ux_k^*\| \right|^p \leq \sum_{k=1}^n \|(I_{\ell_p} - P_m)Ux_k^*\|^p \\ &= \sum_{k=1}^n \sum_{j=m+1}^{\infty} |\langle y_j, x_k^* \rangle|^p = \sum_{j=m+1}^{\infty} \sum_{k=1}^n |\langle y_j, x_k^* \rangle|^p \leq \sum_{j=m+1}^{\infty} \|y_j\|^p \leq \varepsilon^p. \end{aligned}$$

This proves that $\|g - g_m\|_{\text{FBL}^p[E]} \leq \varepsilon$, and the claim follows.

Combining the inequalities (4.15)–(4.16) with the definition of g , we see that $|f(x^*)| \leq \|f\|_{\text{FBL}^p[E]} g(x^*)$ for $x^* \in c_0(A)$. Goldstine's theorem allows us to extend this inequality to $x^* \in E^*$ because the restrictions of f and g to B_{E^*} are weak*-continuous. This shows that $f \in \overline{I_{w^*}[E]}$, and therefore $H_w^p[E] \subseteq \overline{I_{w^*}[E]}$.

The opposite inclusion follows from lemma 2.1(viii). $\square$

The above proof follows an approach due to T. Oikhberg [28], and fixes a gap in an earlier version.

5. Characterizations of reflexivity

To understand the case where the underlying Banach space E is reflexive, we begin with the observation that $E^{**} \subseteq H[E]$ because every functional $x^{**} \in E^{**}$ is positively

homogeneous. Under this inclusion, the canonical image of E inside E^{**} is identified with $\{\delta_x : x \in E\}$. The following lemma clarifies where the elements of E^{**} sit in the hierarchy of sublattices (2.2)–(2.3) of $H[E]$ that we study.

Lemma 5.1. *Let E be a Banach space, and take $1 \leq p < \infty$. Then:*

- (i) $E^{**} \subseteq H^p[E]$ with $\|x^{**}\|_{\text{FBL}^p[E]} = \|x^{**}\|$ for every $x^{**} \in E^{**}$.
- (ii) $E^{**} \cap I[E] = E^{**} \cap H_{w^*}[E] = E^{**} \cap M_{w^*,0}^p[E] = \{\delta_x : x \in E\}$.

Proof. (i) Since

$$\left(\sum_{j=1}^m |x^{**}(x_j^*)|^p \right)^{\frac{1}{p}} \leq \|x^{**}\| \cdot \|(x_j^*)_{j=1}^m\|_{p,\text{weak}}$$

for $x^{**} \in E^{**}$, $m \in \mathbb{N}$ and $x_1^*, \dots, x_m^* \in E^*$, we have $\|x^{**}\|_{\text{FBL}^p[E]} \leq \|x^{**}\|$. The opposite inequality follows from the fact that $\|x^*\|_{p,\text{weak}} = \|x^*\|$ for every $x^* \in E^*$.

(ii) It is clear that $\delta_x \in E^{**} \cap I[E] \cap H_{w^*}[E]$ for every $x \in E$, and lemma 2.1(vii) and (ix) show that $I[E] \subseteq M_{w^*,0}^p[E]$. Hence we can complete the proof by showing that if $x^{**} \in E^{**}$ belongs to $H_{w^*}[E]$ or $M_{w^*,0}^p[E]$, then $x^{**} = \delta_x$ for some $x \in E$. In both cases $x^{**}|_{B_{E^*}}$ is weak*-continuous at 0, so $x^{**}|_{B_{E^*}}^{-1}(\{0\}) = \ker x^{**} \cap B_{E^*}$ is weak*-closed. This implies that $x^{**} = \delta_x$ for some $x \in E$ by an application of the Krein–Smulian theorem (see, e.g., [18, Corollary 3.94]). $\square$

Corollary 5.2. *Let E be a non-reflexive Banach space and $1 \leq p < \infty$. Then*

$$H^p[E] \not\subseteq M_{w^*,0}^p[E] \cup H_{w^*}[E].$$

Proof. Lemma 5.1 shows that $E^{**} \setminus \{\delta_x : x \in E\} \subseteq H^p[E] \setminus (M_{w^*,0}^p[E] \cup H_{w^*}[E])$. $\square$

Before stating our next result, we recall the following classical characterization of weak sequential convergence in $C(K)$ -spaces (see [16, Theorem 1.1] or [17, Corollary IV.6.4]), which will be required in the proof.

Theorem 5.3. *Let K be a compact Hausdorff space. A sequence $(f_n)_{n \in \mathbb{N}}$ in $C(K)$ converges weakly to a function $f \in C(K)$ if and only if $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$ and $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for every $x \in K$.*

Lemma 5.4. *Let $1 \leq p < \infty$, let E be a reflexive Banach space, and equip its unit ball B_E with the weak topology. Then the map $\psi_p : B_{E^*} \rightarrow C(B_E)$ defined by*

$$\psi_p(x^*)(x) = |x^*(x)|^p \quad (x^* \in B_{E^*}, x \in B_E)$$

is continuous with respect to the weak topologies on B_{E^} and $C(B_E)$, respectively.*

Proof. We begin by showing that ψ_p is weakly sequentially continuous. Take a sequence $(x_n^*)_{n \in \mathbb{N}}$ in B_{E^*} which converges weakly to some $x^* \in B_{E^*}$. Then $\sup_{n \in \mathbb{N}} \|\psi_p(x_n^*)\|_\infty \leq 1$ and

$$\psi_p(x_n^*)(x) = |x_n^*(x)|^p \xrightarrow{n \rightarrow \infty} |x^*(x)|^p = \psi_p(x^*) \quad (x \in B_E),$$

so theorem 5.3 shows that $(\psi_p(x_n^*))_{n \in \mathbb{N}}$ converges weakly to $\psi_p(x^*)$.

Now, in order to complete the proof, take a weakly closed subset C of $C(B_E)$, and let us prove that $\psi_p^{-1}(C)$ is weakly closed in B_{E^*} . By the reflexivity of E and the Eberlein–Smulian theorem, this is equivalent to showing that $\psi_p^{-1}(C)$ is weakly sequentially compact. To verify this, take a sequence $(x_n^*)_{n \in \mathbb{N}}$ in $\psi_p^{-1}(C)$. Due to the weak sequential compactness of B_{E^*} , $(x_n^*)_{n \in \mathbb{N}}$ has a subsequence $(x_{n_j}^*)_{j \in \mathbb{N}}$ which converges weakly to some $x^* \in B_{E^*}$. The first part of the proof implies that $(\psi_p(x_{n_j}^*))_{j \in \mathbb{N}}$ converges weakly to $\psi_p(x^*)$. Since $\psi_p(x_{n_j}^*)$ belongs to the weakly closed set C for each $j \in \mathbb{N}$, we see that $\psi_p(x^*) \in C$. In conclusion, we have shown that $(x_n^*)_{n \in \mathbb{N}}$ has a subsequence which converges weakly to $x^* \in \psi_p^{-1}(C)$, and the result follows. $\square$

Recall that for a Banach space E , $1 \leq p < \infty$ and $\mu \in C(B_{E^{**}})_+^*$ (where $B_{E^{**}}$ is equipped with the relative weak* topology), we have defined a function $f_\mu^p \in H^p[E]$ by (4.13).

Theorem 5.5. *Let $1 \leq p < \infty$, and let E be a Banach space. The following conditions are equivalent:*

- (a) E is reflexive.
- (b) $f_\mu^p \in H_{w^*}[E]$ for every $\mu \in C(B_{E^{**}})_+^*$.
- (c) $H^p[E] = J_{w^*}^p[E]$.
- (d) $J_{w^*}^p[E]$ is dense in $H^p[E]$ in the $\|\cdot\|_{FBL^p[E]}$ -norm.
- (e) $f_\mu^p \in M_{w^*,0}^p[E]$ for every $\mu \in C(B_{E^{**}})_+^*$.

Proof. (a) $\Rightarrow$ (b). Suppose that E is reflexive, so that $E^{**} = E$ with the weak* and weak topologies coinciding, and take $\mu \in C(B_E)_+^*$. For $x^* \in B_{E^*}$, we have

$$f_\mu^p(x^*)^p = \int_{B_E} |x^*(x)|^p d\mu(x) = \langle \psi_p(x^*), \mu \rangle = (\mu \circ \psi_p)(x^*),$$

so that $(f_\mu^p \upharpoonright_{B_{E^*}})^p = \mu \circ \psi_p$, where ψ_p is the map defined in lemma 5.4. Take a net (x_α^*) in B_{E^*} which weak*-converges to x^* . Then, by reflexivity, (x_α^*) converges weakly to x^* , so lemma 5.4 implies that $(\psi_p(x_\alpha^*))$ converges weakly to $\psi_p(x^*)$, which means that $(\langle \psi_p(x_\alpha^*), \mu \rangle) = (f_\mu^p(x_\alpha^*)^p)$ converges to $\langle \psi_p(x^*), \mu \rangle = f_\mu^p(x^*)^p$. This shows that $f_\mu^p \in H_{w^*}[E]$.

(b) $\Rightarrow$ (c). Suppose that $f_\mu^p \in H_{w^*}[E]$ for every $\mu \in C(B_{E^{**}})_+^*$. Then $f_\mu^p \in H_{w^*}^p[E]$ because f_μ^p always belongs to $H^p[E]$. As mentioned above, for every $f \in H^p[E]$, we can find a probability measure $\mu \in C(B_{E^{**}})_+^*$ such that (4.14) is satisfied. This implies that $f \in J_{w^*}^p[E]$, and consequently $H^p[E] = J_{w^*}^p[E]$.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (e). This follows from lemma 2.1(ix) because $f_\mu^p \in H^p[E]$ for every $\mu \in C(B_{E^{**}})_+^*$.

Finally, we prove the implication (e) $\Rightarrow$ (a) by contraposition. Suppose that E is non-reflexive. Then B_{E^*} contains a net (x_α^*) which weak*-converges to 0, but does not

converge weakly to 0. Take $x_0^{**} \in E^{**}$ such that $(x_0^{**}(x_\alpha^*))$ does not converge to 0, and consider the Dirac measure $\mu = \delta_{x_0^{**}} \in C(B_{E^{**}})_+^*$; it satisfies

$$f_\mu^p(x_\alpha^*) = \left(\int_{B_{E^{**}}} |x^{**}(x_\alpha^*)|^p d\mu(x^{**}) \right)^{\frac{1}{p}} = |x_0^{**}(x_\alpha^*)| \not\rightarrow 0 = f_\mu^p(0).$$

This shows that $f_\mu^p|_{B_{E^*}}$ fails to be weak*-continuous at 0, so $f_\mu^p \notin M_{w^*,0}^p[E]$. $\square$

6. Lattice homomorphisms

It is well known that for compact Hausdorff spaces K and L , every lattice homomorphism $T: C(K) \rightarrow C(L)$ arises by combining a composition operator with pointwise multiplication by a positive continuous function. (A precise statement of this result can for instance be found in [27, Theorem 3.2.10].) The aim of this section is to obtain an analogous representation for lattice homomorphisms between free Banach lattices and related lattices of positively homogeneous functions, and then study the properties and consequences of this representation.

The natural setting in which to develop this theory is a vector-valued version of the Banach space $H^p[E]$ defined as follows. For $1 \leq p < \infty$ and Banach spaces E and F , let $\text{PH}^p[E, F]$ denote the set of positively homogeneous maps $E \rightarrow F$ that map weakly p -summable sequences in E to weakly p -summable sequences in F . This is a linear subspace of the vector space of functions $E \rightarrow F$, and a Banach space with respect to the norm

$$\|\Phi\|_{\text{PH}^p[E, F]} = \sup \left\{ \left\| \left(\Phi(x_j) \right)_{j=1}^m \right\|_{p, \text{weak}} : m \in \mathbb{N}, \right. \\ \left. (x_j)_{j=1}^m \subset E, \|(x_j)_{j=1}^m\|_{p, \text{weak}} \leq 1 \right\} \quad (\Phi \in \text{PH}^p[E, F]).$$

We shall often use the elementary observation that a positively homogeneous map $\Phi: E \rightarrow F$ belongs to $\text{PH}^p[E, F]$ if and only if the quantity $\|\Phi\|_{\text{PH}^p[E, F]}$ just defined is finite. Furthermore, we see that $H^p[E] = \text{PH}^p[E^*, \mathbb{R}]$ with equality of norms, which justifies our claim that the Banach space $\text{PH}^p[E, F]$ (or more precisely $\text{PH}^p[E^*, F]$) is a vector-valued generalization of $H^p[E]$. However, $\text{PH}^p[E, F]$ generally does not have a natural lattice structure.

We shall only consider positively homogeneous maps between dual Banach spaces; therefore, the Banach space $\text{PH}^p[F^*, E^*]$ will suffice for our purposes. We begin with an easy result which establishes its connection, via composition operators, to the vector and Banach lattices that we are investigating.

Lemma 6.1. *Let $\Phi: F^* \rightarrow E^*$ be a positively homogeneous map for some Banach spaces E and F .*

(i) *The composition operator*

$$C_\Phi: f \mapsto f \circ \Phi$$

defines a lattice homomorphism $C_\Phi: H[E] \rightarrow H[F]$.

(ii) Suppose that $\Phi \in \text{PHP}[F^*, E^*]$ for some $1 \leq p < \infty$. Then

$$\Phi(B_{F^*}) \subseteq \|\Phi\|_{\text{PHP}[F^*, E^*]} B_{E^*} \quad \text{and} \quad C_\Phi(H^p[E]) \subseteq H^p[F],$$

and C_Φ is bounded with norm $\|C_\Phi\| = \|\Phi\|_{\text{PHP}[F^*, E^*]}$ when viewed as a linear operator $(H^p[E], \|\cdot\|_{\text{FBL}^p[E]}) \rightarrow (H^p[F], \|\cdot\|_{\text{FBL}^p[F]})$.

Suppose in addition that $\Phi|_{B_{F^*}}$ is weak*-to-weak* continuous. Then

$$\begin{aligned} C_\Phi(H_{w^*}[E]) &\subseteq H_{w^*}[F], & C_\Phi(H_{w^*}^p[E]) &\subseteq H_{w^*}^p[F], \\ C_\Phi(J_{w^*}^p[E]) &\subseteq J_{w^*}^p[F], & C_\Phi(\overline{J_{w^*}^p[E]}) &\subseteq \overline{J_{w^*}^p[F]}, \\ C_\Phi(M_{w^*,0}^p[E]) &\subseteq M_{w^*,0}^p[F]. \end{aligned}$$

(iii) The map $p \mapsto \|\Phi\|_{\text{PHP}[F^*, E^*]}$ is decreasing; that is,

$$\|\Phi\|_{\text{PHP}[F^*, E^*]} \leq \|\Phi\|_{\text{PHP}[F^*, E^*]} \quad (1 \leq p < q < \infty).$$

Proof. (i) The composite map $f \circ \Phi$ is clearly positively homogeneous when f and Φ are, so $C_\Phi(H[E]) \subseteq H[F]$, and C_Φ is a lattice homomorphism because the vector lattice operations are defined coordinatewise.

(ii) Suppose that $\Phi \in \text{PHP}[F^*, E^*]$. Then $\|\Phi(y^*)\| \leq \|\Phi\|_{\text{PHP}[F^*, E^*]}$ for every $y^* \in B_{F^*}$ because $\|z^*\|_{p, \text{weak}} = \|z^*\|$ for every functional z^* (in either E^* or F^*), and therefore $\Phi(B_{F^*}) \subseteq \|\Phi\|_{\text{PHP}[F^*, E^*]} B_{E^*}$. Furthermore, working straight from the definitions of the norms, we obtain

$$\begin{aligned} \|C_\Phi(f)\|_{\text{FBL}^p[F]} &= \sup \left\{ \left(\sum_{j=1}^m |f(\Phi y_j^*)|^p \right)^{\frac{1}{p}} : m \in \mathbb{N}, (y_j^*)_{j=1}^m \subset F^*, \right. \\ &\quad \left. \|(y_j^*)_{j=1}^m\|_{p, \text{weak}} \leq 1 \right\} \leq \|\Phi\|_{\text{PHP}[F^*, E^*]} \|f\|_{\text{FBL}^p[E]} \quad (f \in H^p[E]). \end{aligned}$$

This shows that C_Φ maps $H^p[E]$ into $H^p[F]$ and $\|C_\Phi\| \leq \|\Phi\|_{\text{PHP}[F^*, E^*]}$.

On the other hand, given $x \in B_E$ and $(y_j^*)_{j=1}^m \subset F^*$ for some $m \in \mathbb{N}$, we have

$$\begin{aligned} \left(\sum_{j=1}^m |(\Phi y_j^*)(x)|^p \right)^{\frac{1}{p}} &= \left(\sum_{j=1}^m |\delta_x(\Phi y_j^*)|^p \right)^{\frac{1}{p}} = \left(\sum_{j=1}^m |(C_\Phi \delta_x)(y_j^*)|^p \right)^{\frac{1}{p}} \\ &\leq \|C_\Phi \delta_x\|_{\text{FBL}^p[F]} \|(y_j^*)_{j=1}^m\|_{p, \text{weak}} \leq \|C_\Phi\| \|(y_j^*)_{j=1}^m\|_{p, \text{weak}}, \end{aligned}$$

which shows that $\|\Phi\|_{\text{PHP}[F^*, E^*]} \leq \|C_\Phi\|$, and therefore $\|C_\Phi\| = \|\Phi\|_{\text{PHP}[F^*, E^*]}$.

Now suppose that the restriction of Φ to B_{F^*} is weak*-to-weak* continuous (as well as $\Phi \in \text{PHP}[F^*, E^*]$). Since $\Phi(B_{F^*}) \subseteq \|\Phi\|_{\text{PHP}[F^*, E^*]} B_{E^*}$ and the functions in $H_{w^*}[E]$ are positively homogeneous, we see that C_Φ maps $H_{w^*}[E]$ into $H_{w^*}[F]$ and $M_{w^*,0}^p[E]$ into $M_{w^*,0}^p[F]$. The inclusion $C_\Phi(H_{w^*}^p[E]) \subseteq H_{w^*}^p[F]$ follows by combining the two inclusions $C_\Phi(H_{w^*}[E]) \subseteq H_{w^*}[F]$ and $C_\Phi(H^p[E]) \subseteq H^p[F]$.

Finally, for each $f \in J_{w^*}^p[E]$, we can find $g \in H_{w^*}^p[E]$ such that $|f| \leq g$. Then we have $|C_\Phi(f)| \leq C_\Phi(g) \in H_{w^*}^p[F]$ by the previous inclusion, and consequently $C_\Phi(f) \in J_{w^*}^p[F]$.

This implies that $C_\Phi(\overline{J_{w^*}^p[E]}) \subseteq \overline{J_{w^*}^p[F]}$ because C_Φ is continuous with respect to the FBL^p -norms, as we have shown above.

(iii) This proof is similar to the proof of [25, Proposition 7.3]. We may suppose that $\|\Phi\|_{\text{PH}^p[F^*, E^*]} < \infty$, as otherwise there is nothing to prove. Take $x \in B_E$ and $(y_j^*)_{j=1}^m \subset F^*$ with $\| (y_j^*)_{j=1}^m \|_{q, \text{weak}} \leq 1$ for some $m \in \mathbb{N}$, and set $\lambda_j = |\langle x, \Phi(y_j^*) \rangle|^{q/r} \geq 0$ for $j \in \{1, \dots, m\}$, where $r = pq/(q-p) \in (p, \infty)$. Then $(1+r/q)p = r$, which together with the positive homogeneity of Φ implies that

$$\begin{aligned} \sum_{j=1}^m \lambda_j^r &= \sum_{j=1}^m \left(\lambda_j |\langle x, \Phi(y_j^*) \rangle| \right)^p = \sum_{j=1}^m |\langle x, \Phi(\lambda_j y_j^*) \rangle|^p \\ &\leq \|(\Phi(\lambda_j y_j^*))_{j=1}^m\|_{p, \text{weak}}^p \leq \|\Phi\|_{\text{PH}^p[F^*, E^*]}^p \|(\lambda_j y_j^*)_{j=1}^m\|_{p, \text{weak}}^p. \end{aligned} \quad (6.1)$$

The choice of r means that the conjugate exponent of $r/p \in (1, \infty)$ is q/p . Hence, for $y \in B_F$, Hölder's inequality gives

$$\sum_{j=1}^m |\langle y, \lambda_j y_j^* \rangle|^p = \sum_{j=1}^m \lambda_j^p |\langle y, y_j^* \rangle|^p \leq \left(\sum_{j=1}^m \lambda_j^r \right)^{\frac{p}{r}} \left(\sum_{j=1}^m |\langle y, y_j^* \rangle|^q \right)^{\frac{p}{q}} \leq \left(\sum_{j=1}^m \lambda_j^r \right)^{\frac{p}{r}}.$$

This shows that $\|(\lambda_j y_j^*)_{j=1}^m\|_{p, \text{weak}}^p \leq \left(\sum_{j=1}^m \lambda_j^r \right)^{p/r}$, so by (6.1), we have

$$\sum_{j=1}^m \lambda_j^r \leq \|\Phi\|_{\text{PH}^p[F^*, E^*]}^p \left(\sum_{j=1}^m \lambda_j^r \right)^{\frac{p}{r}}.$$

Rearranging this inequality and using once more that $1 - p/r = p/q$, we obtain

$$\|\Phi\|_{\text{PH}^p[F^*, E^*]}^p \geq \left(\sum_{j=1}^m \lambda_j^r \right)^{\frac{p}{q}} = \left(\sum_{j=1}^m |\langle x, \Phi(y_j^*) \rangle|^q \right)^{\frac{p}{q}}.$$

Now the conclusion follows by taking the p th roots and the supremum over x and $(y_j^*)_{j=1}^m$. $\square$

We can rephrase lemma 6.1(iii) as saying that for $1 \leq p < q < \infty$, we have a formal inclusion map $\text{PH}^p[F^*, E^*] \rightarrow \text{PH}^q[F^*, E^*]$ of norm 1.

Example 6.2. Identify ℓ_∞ with ℓ_1^* , as usual. By [29, Remark 10.6], there is a map $\Phi \in \text{PH}^1[\ell_\infty, \ell_\infty]$ for which $\Phi|_{B_{\ell_\infty}}$ is weak*-to-weak* continuous (so that Φ satisfies both hypotheses of lemma 6.1(ii) for $p = 1$), but the corresponding composition operator C_Φ does not map $\text{FBL}^1[\ell_1]$ into itself.

We can now provide a much more general version of this result. Take $1 \leq p < \infty$, and let E and F be Banach spaces, where E is non-zero and F admits an infinite-dimensional, separable quotient space. By theorem 4.3, we can find a function $f \in H_{w^*}^p[F] \setminus \text{FBL}^p[F]$. (In fact, we can take $f \in I_{w^*}[F] \setminus \text{FBL}^p[F]$, but this will not help the following argument.) Choose $x_0^* \in E^* \setminus \{0\}$, and define a map $\Phi: F^* \rightarrow E^*$ by $\Phi(y^*) = f(y^*)x_0^*$ for $y^* \in$

F^* . Since $f \in H_{w^*}^p[F]$, it is straightforward to check that Φ is positively homogeneous, $\Phi|_{B_{F^*}}$ is weak*-to-weak* continuous, and $\|\Phi\|_{\text{PH}^p[F^*, E^*]} = \|f\|_{\text{FBL}^p[F]} \|x_0^*\| < \infty$, so $\Phi \in \text{PH}^p[F^*, E^*]$. However, if we pick $x_0 \in E$ such that $x_0^*(x_0) = 1$, then

$$C_\Phi(\delta_{x_0})(y^*) = (\delta_{x_0} \circ \Phi)(y^*) = \Phi(y^*)(x_0) = f(y^*) \quad (y^* \in F^*),$$

so $C_\Phi(\delta_{x_0}) = f \notin \text{FBL}^p[F]$. Hence C_Φ does not map $\text{FBL}^p[E]$ into $\text{FBL}^p[F]$.

The following question arises naturally by comparing the list of inclusions established in lemma 6.1(ii) with example 6.2. We have been unable to answer it, partly due to the lack of progress on question 4.2.

Question 6.3. Let $\Phi \in \text{PH}^p[F^*, E^*]$ for some $1 \leq p < \infty$ and infinite-dimensional Banach spaces E and F , and suppose that $\Phi|_{B_{F^*}}$ is weak*-to-weak* continuous. Is

$$C_\Phi(\overline{I_{w^*}[E]}) \subseteq \overline{I_{w^*}[F]}?$$

We shall now present our main representation theorem for lattice homomorphisms between free Banach lattices.

Theorem 6.4. *Let $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ be a lattice homomorphism for some $1 \leq p < \infty$ and some Banach spaces E and F . Then there is a unique map $\Phi_T: F^* \rightarrow E^*$ such that*

$$Tf = f \circ \Phi_T \quad (f \in \text{FBL}^p[E]). \quad (6.2)$$

This map Φ_T is positively homogeneous and satisfies

- (i) $\Phi_T \in \text{PH}^p[F^*, E^*]$ with $\|\Phi_T\|_{\text{PH}^p[F^*, E^*]} = \|T\|$;
- (ii) $\Phi_T|_{B_{F^*}}$ is weak*-to-weak* continuous.

Proof. We begin with the uniqueness. Suppose that $\Phi_T: F^* \rightarrow E^*$ is a map which satisfies (6.2). Substituting $f = \delta_x$ for some $x \in E$ into this identity and evaluating it at some $y^* \in F^*$, we obtain

$$(T\delta_x)(y^*) = (\delta_x \circ \Phi_T)(y^*) = \Phi_T(y^*)(x). \quad (6.3)$$

Reading this equation from the right to the left, we conclude that there is only one possible way to define Φ_T . Taking this as our definition, we must check that $\Phi_T(y^*) \in E^*$, which is easy: Linearity follows from the fact that $\delta_{\lambda w + x} = \lambda \delta_w + \delta_x$ for $\lambda \in \mathbb{R}$ and $w, x \in E$, and continuity from the estimate $|(T\delta_x)(y^*)| \leq \|T\| \|x\| \|y^*\|$. This estimate also shows that Φ_T maps B_{F^*} into $\|T\| B_{E^*}$.

It is equally easy to see that Φ_T is positively homogeneous because

$$\Phi_T(\lambda y^*)(x) = (T\delta_x)(\lambda y^*) = \lambda (T\delta_x)(y^*) = \lambda \Phi_T(y^*)(x) \quad (\lambda \geq 0, y^* \in F^*, x \in E).$$

Next, we shall prove (i)–(ii), beginning with the latter. Suppose that (y_α^*) is a net in B_{F^*} which weak*-converges to some $y^* \in B_{F^*}$, and take $x \in E$. Lemma 2.1(vi) implies that

the net $((T\delta_x)(y_\alpha^*))$ converges to $(T\delta_x)(y^*)$, which in view of the definition (6.3) means that the net $(\Phi_T(y_\alpha^*)(x))$ converges to $\Phi_T(y^*)(x)$.

To verify (i), we observe that the uniqueness part of the universal property of $\text{FBL}^p[E]$ implies that T is the lattice homomorphism induced by the bounded linear operator $T \circ \delta^E: E \rightarrow \text{FBL}^p[F]$; that is, $T = \overline{T \circ \delta^E}$, and therefore

$$\begin{aligned} \|T\| &= \|T \circ \delta^E\| = \sup \left\{ \|T\delta_x\|_{\text{FBL}^p[F]} : x \in B_E \right\} = \sup \left\{ \left(\sum_{j=1}^m |(T\delta_x)(y_j^*)|^p \right)^{\frac{1}{p}} : \right. \\ &\quad \left. x \in B_E, m \in \mathbb{N}, (y_j^*)_{j=1}^m \subset F^*, \|(y_j^*)_{j=1}^m\|_{p, \text{weak}} \leq 1 \right\} \\ &= \sup \left\{ \|(\Phi_T(y_j^*))_{j=1}^m\|_{p, \text{weak}} : m \in \mathbb{N}, (y_j^*)_{j=1}^m \subset F^*, \|(y_j^*)_{j=1}^m\|_{p, \text{weak}} \leq 1 \right\} \\ &= \|\Phi_T\|_{\text{PH}^p[F^*, E^*]}. \end{aligned}$$

It remains to verify that the map Φ_T defined by (6.3) satisfies (6.2). We have already shown that $\Phi_T \in \text{PH}^p[F^*, E^*]$, so lemma 6.1(ii) implies that it induces a lattice homomorphism $C_{\Phi_T}: H^p[E] \rightarrow H^p[F]$ by composition; that is, $C_{\Phi_T}(f) = f \circ \Phi_T$ for $f \in H^p[E]$. The fact that

$$C_{\Phi_T}(\delta_x)(y^*) = \delta_x(\Phi_T(y^*)) = \Phi_T(y^*)(x) = (T\delta_x)(y^*) \quad (x \in E, y^* \in F^*)$$

shows that the restriction of C_{Φ_T} to $\text{FBL}^p[E]$ satisfies $C_{\Phi_T}|_{\text{FBL}^p[E]} \circ \delta^E = \iota \circ T \circ \delta^E$, where $\iota: \text{FBL}^p[F] \rightarrow H^p[F]$ denotes the inclusion map. This in turn implies that $C_{\Phi_T}|_{\text{FBL}^p[E]} = \iota \circ T$ by [25, Corollary 3.5(ii)], bearing in mind that the map δ^E is denoted ϕ_E^D in [25]. Now the conclusion follows because

$$Tf = C_{\Phi_T}(f) = f \circ \Phi_T \quad (f \in \text{FBL}^p[E]).$$

□

Definition 6.5. Given a lattice homomorphism $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$, we call the unique map $\Phi_T: F^* \rightarrow E^*$ satisfying (6.2) the map induced by T .

Remark 6.6. Suppose that $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ is a lattice homomorphism for some $1 \leq p < \infty$ and some Banach spaces E and F , and let $\Phi_T: F^* \rightarrow E^*$ be the induced map. Theorem 6.4 shows that $\Phi_T \in \text{PH}^p[F^*, E^*]$ and $\Phi_T|_{B_{F^*}}$ is weak*-to-weak* continuous, so lemma 6.1 implies that the composition operator $C_{\Phi_T}: H[E] \rightarrow H[F]$ induces a lattice homomorphism between each of the following four pairs of Banach lattices, for every $q \in [p, \infty)$:

$$\begin{aligned} H^q[E] &\rightarrow H^q[F], & H_{w^*}^q[E] &\rightarrow H_{w^*}^q[F], \\ \overline{J_{w^*}^q[E]} &\rightarrow \overline{J_{w^*}^q[F]}, & M_{w^*,0}^q[E] &\rightarrow M_{w^*,0}^q[F]. \end{aligned} \quad (6.4)$$

In fact, we already used the first of these identities for $q = p$ in the proof of theorem 6.4. As we saw towards the end of that proof, we can interpret (6.2) as the statement that the lattice homomorphism C_{Φ_T} is an extension of T because

$$C_{\Phi_T}(f) = f \circ \Phi_T = Tf \quad (f \in \text{FBL}^p[E]). \quad (6.5)$$

We shall now complement (6.4) by showing that the answer to Question 6.3 is positive in the case where $\Phi = \Phi_T$ is induced by a lattice homomorphism $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$. Furthermore, using lemma 6.1(iii), we can show that FBL^q is C_{Φ_T} -invariant for every $q > p$.

Lemma 6.7. *Let $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ be a lattice homomorphism for some $1 \leq p < \infty$ and some Banach spaces E and F . Then:*

- (i) $C_{\Phi_T}(\overline{I_{w^*}[E]}) \subseteq \overline{I_{w^*}[F]}$;
- (ii) $C_{\Phi_T}(\text{FBL}^q[E]) \subseteq \text{FBL}^q[F]$ for every $q \in (p, \infty)$.

Proof. (i) It suffices to show that $C_{\Phi_T}(f) \in \overline{I_{w^*}[F]}$ for every $f \in I_{w^*}[E]_+$ because C_{Φ_T} is a lattice homomorphism (in particular continuous). Take $m \in \mathbb{N}$ and $x_1, \dots, x_m \in E$ such that $f \leq \sum_{j=1}^m |\delta_{x_j}|$. Using that C_{Φ_T} is a lattice homomorphism together with (6.5), we obtain

$$0 \leq C_{\Phi_T}(f) \leq C_{\Phi_T}\left(\sum_{j=1}^m |\delta_{x_j}|\right) = \sum_{j=1}^m |C_{\Phi_T}(\delta_{x_j})| = \sum_{j=1}^m |T\delta_{x_j}| \in \text{FBL}^p[F] \subseteq \overline{I_{w^*}[F]}.$$

Since $\overline{I_{w^*}[F]}$ is an ideal of $H_{w^*}^p[F]$ and $C_{\Phi_T}(f) \in H_{w^*}^p[F]$ by (6.4), we conclude that $C_{\Phi_T}(f) \in \overline{I_{w^*}[F]}$, as desired.

(ii) Suppose that $1 \leq p < q < \infty$. Then $\|f\|_{\text{FBL}^q[F]} \leq \|f\|_{\text{FBL}^p[F]}$ for every $f \in H[F]$ by [25, Proposition 7.3] (or lemma 6.1(iii) in the special case $\Phi = f: F^* \rightarrow \mathbb{R}$). This implies that $\text{FBL}^p[F] \subseteq \text{FBL}^q[F]$ and therefore, using (6.5), we obtain

$$C_{\Phi_T}(\delta_x) = T\delta_x \in \text{FBL}^p[F] \subseteq \text{FBL}^q[F] \quad (x \in E).$$

Now the conclusion follows by viewing C_{Φ_T} as a lattice homomorphism from $H^q[E]$ to $H^q[F]$, as shown in (6.4). $\square$

Corollary 6.8. *Let E and F be Banach spaces for which $\text{FBL}^p[E]$ and $\text{FBL}^p[F]$ are lattice isomorphic for some $1 \leq p < \infty$. Then so are each of the following six pairs for every $q \in [p, \infty)$ (where the closures are taken with respect to the FBL^q -norms):*

$$\begin{aligned} \text{FBL}^q[E] &\cong \text{FBL}^q[F]; & \overline{I_{w^*}[E]} &\cong \overline{I_{w^*}[F]}; & H_{w^*}^q[E] &\cong H_{w^*}^q[F] \\ \overline{J_{w^*}^q[E]} &\cong \overline{J_{w^*}^q[F]}; & M_{w^*,0}^q[E] &\cong M_{w^*,0}^q[F]; & H^q[E] &\cong H^q[F]. \end{aligned}$$

Proof. Suppose that $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ is a lattice isomorphism with inverse $T^{-1}: \text{FBL}^p[F] \rightarrow \text{FBL}^p[E]$. Then, for $x \in E$ and $x^* \in E^*$, we have

$$x^*(x) = (T^{-1}(T\delta_x))(x^*) = (\delta_x \circ \Phi_T \circ \Phi_{T^{-1}})(x^*) = (\Phi_T \circ \Phi_{T^{-1}})(x^*)(x),$$

with a similar calculation showing that $\Phi_{T^{-1}} \circ \Phi_T = I_{F^*}$; that is, the induced maps $\Phi_T: F^* \rightarrow E^*$ and $\Phi_{T^{-1}}: E^* \rightarrow F^*$ are inverses of each other, and therefore the composition operators C_{Φ_T} and $C_{\Phi_{T^{-1}}}$ are also inverses of each other. As we saw in (6.4) and lemma 6.7, they restrict to lattice homomorphisms between each of specified pairs, from which the conclusions follow. (In the case of the FBL^q -closure of the ideal I_{w^*} for $q > p$, a little extra care is required; before invoking lemma 6.7(i), we use the second part of the same result to deduce that C_{Φ_T} is a lattice isomorphism between $\text{FBL}^q[E]$ and $\text{FBL}^q[F]$ with inverse $C_{\Phi_{T^{-1}}}$.) $\square$

Example 6.9. In general, one cannot reverse the implication

$$\text{FBL}^p[E] \cong \text{FBL}^p[F] \implies H^p[E] \cong H^p[F]$$

proved in corollary 6.8. To see this, take Banach spaces E and F whose dual spaces E^* and F^* are linearly isomorphic. Then it is clear that $H^p[E]$ and $H^p[F]$ are lattice isomorphic. In particular, $H^1[\ell_1]$ is lattice isomorphic to $H^1[L_1[0, 1]]$, but $\text{FBL}^1[\ell_1]$ and $\text{FBL}^1[L_1[0, 1]]$ are not lattice isomorphic by [6, Theorems 4.11 and 4.13].

Proposition 3.2 provides another example, valid for any $1 \leq p < \infty$; it shows that $H^p[E]$ and $H^p[F]$ are lattice isomorphic for every pair E and F of finite-dimensional Banach spaces of dimension at least 2, but $\text{FBL}^p[E]$ and $\text{FBL}^p[F]$ are only lattice isomorphic if $\dim E = \dim F$.

Our next result uses lemma 6.1 to obtain an infinite-dimensional generalization of the above observation that $H^p[E]$ and $H^p[F]$ are lattice isomorphic when E and F have finite dimension at least 2.

Proposition 6.10. *Let F be a closed subspace of finite codimension in a Banach space E , and suppose that F has dimension at least 2. Then the Banach lattices $H^p[E]$ and $H^p[F]$ are lattice isomorphic for every $1 \leq p < \infty$.*

Proof. Since $H^p[G_1]$ and $H^p[G_2]$ are lattice isomorphic whenever G_1 and G_2 are linearly isomorphic Banach spaces, it suffices to show that $H^p[E]$ and $H^p[F]$ are lattice isomorphic for $E = \ell_1^m \oplus G$ and $F = \ell_1^n \oplus G$, where G is a Banach space and $m, n \in \mathbb{N} \setminus \{1\}$. Take a bijection $\theta: S_{\ell_\infty^n} \rightarrow S_{\ell_\infty^m}$, where $S_{\ell_\infty^k}$ denotes the unit sphere of $\ell_\infty^k = (\ell_1^k)^*$, as usual. (Such a bijection exists because both sets have cardinality $\mathfrak{c}$.) Let $\Theta: \ell_\infty^n \rightarrow \ell_\infty^m$ be the positively homogeneous extension of θ , that is, $\Theta(0) = 0$ and

$$\Theta(y^*) = \|y^*\|_\infty \theta\left(\frac{y^*}{\|y^*\|_\infty}\right) = \bigvee_{i=1}^n |\delta_{e_i}(y^*)| \theta\left(\frac{y^*}{\|y^*\|_\infty}\right) \quad (y^* \in \ell_\infty^n \setminus \{0\}),$$

where $(e_i)_{i=1}^n$ denotes the unit vector basis for ℓ_1^n .

To verify that $\Theta \in \text{PHP}[\ell_\infty^n, \ell_\infty^m]$, take $(y_j^*)_{j=1}^k \subset \ell_\infty^n$ with $\|(y_j^*)_{j=1}^k\|_{p, \text{weak}} \leq 1$ for some $k \in \mathbb{N}$. We may suppose that $y_j^* \neq 0$ for each $j \in \{1, \dots, k\}$, and find

$$\begin{aligned} \left\| \left(\Theta(y_j^*) \right)_{j=1}^k \right\|_{p, \text{weak}} &= \sup_{x \in B_{\ell_1^m}} \left(\sum_{j=1}^k \left(\bigvee_{i=1}^n \|\delta_{e_i}(y_j^*)\| \right)^p \left| \left\langle \theta\left(\frac{y_j^*}{\|y_j^*\|_\infty}\right), x \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{j=1}^k \left(\bigvee_{i=1}^n \|\delta_{e_i}(y_j^*)\| \right)^p \right)^{\frac{1}{p}} \leq \left\| \bigvee_{i=1}^n |\delta_{e_i}| \right\|_{\text{FBL}^p[\ell_1^n]}, \end{aligned}$$

where the first inequality follows from the fact that $\|\theta(y_j^*/\|y_j^*\|_\infty)\|_\infty = 1$. Hence $\|\Theta\|_{\text{PHP}[\ell_\infty^n, \ell_\infty^m]} \leq \left\| \bigvee_{i=1}^n |\delta_{e_i}| \right\|_{\text{FBL}^p[\ell_1^n]} < \infty$, so $\Theta \in \text{PHP}[\ell_\infty^n, \ell_\infty^m]$.

We can obviously extend Θ to a positively homogeneous map $\Phi: F^* \rightarrow E^*$ by defining $\Phi(y^*, z^*) = (\Theta(y^*), z^*)$ for $y^* \in \ell_\infty^n$ and $z^* \in G^*$. This extension belongs to $\text{PHP}[F^*, E^*]$ with $\|\Phi\|_{\text{PHP}[F^*, E^*]} \leq \|\Theta\|_{\text{PHP}[\ell_\infty^n, \ell_\infty^m]} + 1$ because

$$\begin{aligned} \|(\Phi(y_j^*, z_j^*))_{j=1}^k\|_{p, \text{weak}} &= \|(\Theta(y_j^*), z_j^*)_{j=1}^k\|_{p, \text{weak}} \\ &\leq \|(\Theta(y_j^*))_{j=1}^k\|_{p, \text{weak}} + \|(z_j^*)_{j=1}^k\|_{p, \text{weak}} \leq \|\Theta\|_{\text{PHP}[\ell_\infty^n, \ell_\infty^m]} + 1 \end{aligned}$$

for every $k \in \mathbb{N}$ and $(y_j^*, z_j^*)_{j=1}^k \subset F^*$ with $\|(y_j^*, z_j^*)_{j=1}^k\|_{p, \text{weak}} \leq 1$.

On the other hand, by repeating the above arguments for the inverse map $\theta^{-1}: S_{\ell_\infty^m} \rightarrow S_{\ell_\infty^n}$ instead of θ , we obtain a map $\Phi^{-1} \in \text{PHP}[E^*, F^*]$ which is an inverse of Φ (as our choice of notation indicates). Therefore, lemma 6.1(ii) implies that we may regard the restrictions of C_Φ and $C_{\Phi^{-1}}$ as lattice homomorphisms $C_\Phi: H^p[E] \rightarrow H^p[F]$ and $C_{\Phi^{-1}}: H^p[F] \rightarrow H^p[E]$, respectively. They are clearly inverses of each other, from which the conclusion follows. $\square$

Example 6.11. Gowers [20] and Gowers and Maurey [21] have shown that there are infinite-dimensional Banach spaces E which are not isomorphic to their hyperplanes. However, proposition 6.10 shows that $H^p[E]$ and $H^p[F]$ are lattice isomorphic whenever F is a closed subspace of finite codimension in E and $1 \leq p < \infty$.

More generally, for every $k \in \{0, 2, 3, \dots\}$, Gowers and Maurey [22] have constructed a Banach space E_k such that E_k is isomorphic to a closed subspace $F \subset E_k$ if and only if F has finite codimension in E_k , and this codimension is a multiple of k . As above, $H^p[E_k]$ and $H^p[F]$ are lattice isomorphic for every $1 \leq p < \infty$ and every closed subspace F of finite codimension in E_k , irrespective of the value of this codimension.

Remark 6.12. Examples 6.9 and 6.11 demonstrate that there are non-isomorphic Banach spaces E and F for which $H^p[E]$ and $H^p[F]$ are lattice isomorphic for every $1 \leq p < \infty$. As the proof of proposition 6.10 shows, the lattice isomorphisms in example 6.11

are induced by composition with a bijection $\Phi \in \text{PH}^p[F^*, E^*]$ that lacks any weak* continuity properties.

In general, exhibiting weak* homeomorphisms between the unit balls of dual Banach spaces is not difficult. For instance, Keller's theorem (see, *e.g.*, [18, section 12.3]) implies that the dual unit ball B_{E^*} is weak*-homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ for every separable, infinite-dimensional Banach space E . Under some technical assumptions, this weak* homeomorphism can be chosen to be positively homogeneous, as shown in [29, Theorem 10.24].

Unfortunately, none of the above arguments provide any information about the corresponding free Banach lattices. It remains a major open problem whether there exist non-isomorphic Banach spaces E and F such that $\text{FBL}^p[E]$ is lattice isomorphic to $\text{FBL}^p[F]$ for some $1 \leq p < \infty$.

Our next lemma relates the properties of a lattice homomorphism between free Banach lattices to the map it induces.

Lemma 6.13. *Let $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ be a lattice homomorphism for some $1 \leq p < \infty$ and some Banach spaces E and F . Then:*

- (i) *T has dense range if and only if the sublattice generated by the range of the operator $T \circ \delta^E$ is dense in $\text{FBL}^p[F]$.*
- (ii) *Suppose that one, and hence both, conditions in (i) are satisfied. Then the induced map $\Phi_T: F^* \rightarrow E^*$ is injective.*

Proof. (i) follows from the fact that $\text{FBL}^p[E]$ is the closed sublattice of $H^p[E]$ generated by the image of δ^E .

(ii) Suppose that the sublattice generated by $(T \circ \delta^E)(E)$ is dense in $\text{FBL}^p[F]$. Since Φ_T is positively homogeneous, it suffices to verify that its restriction to B_{F^*} is injective. Take $y^*, z^* \in B_{F^*}$ such that $\Phi_T(y^*) = \Phi_T(z^*)$. By (6.2), we have

$$(T\delta_x)(y^*) = \delta_x(\Phi_T(y^*)) = \delta_x(\Phi_T(z^*)) = (T\delta_x)(z^*) \quad (x \in E).$$

This implies that $f(y^*) = f(z^*)$ for every f belonging to the sublattice generated by $(T \circ \delta^E)(E)$. By hypothesis, this sublattice is dense in $\text{FBL}^p[F]$ with respect to the FBL^p -norm, which according to lemma 2.1(iii) dominates the uniform norm on B_{F^*} , so we conclude that $f(y^*) = f(z^*)$ for every $f \in \text{FBL}^p[F]$. In particular, taking $f = \delta_y$ for $y \in F$, we see that $y^*(y) = z^*(y)$, which proves that $y^* = z^*$. $\square$

Example 6.14. Given a bounded linear operator $S: E \rightarrow F$ between Banach spaces E and F , there is a standard way to associate a lattice homomorphism between the corresponding free Banach lattices with it. Indeed, for $1 \leq p < \infty$, $\delta^F \circ S: E \rightarrow \text{FBL}^p[F]$ is a bounded linear operator of norm $\|S\|$ into a p -convex Banach lattice, so the universal property of $\text{FBL}^p[E]$ implies that there is a unique lattice homomorphism $\overline{S} = \widehat{\delta^F \circ S}: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ such that

$$\overline{S} \circ \delta^E = \delta^F \circ S, \tag{6.6}$$

and $\|\bar{S}\| = \|S\|$. The map $\Phi_{\bar{S}}: F^* \rightarrow E^*$ induced by $\bar{S}$ is simply the adjoint S^* of S , as the following calculation, valid for every $y^* \in F^*$ and $x \in E$, shows:

$$\Phi_{\bar{S}}(y^*)(x) = (\delta_x \circ \Phi_{\bar{S}})(y^*) = (\bar{S}\delta_x)(y^*) = \delta_{Sx}(y^*) = y^*(Sx) = S^*(y^*)(x).$$

Consequently, we have

$$\bar{S}f = f \circ S^* = C_{S^*}(f) \quad (f \in \text{FBL}^p[E]), \quad (6.7)$$

and lemma 6.7(i) implies that

$$C_{S^*}(\overline{I_{w^*}[E]}) \subseteq \overline{I_{w^*}[F]}. \quad (6.8)$$

We can use this inclusion to obtain stability results for the properties that questions 4.1 and 4.2 are concerned with.

Proposition 6.15. *Let $1 \leq p < \infty$, and suppose that F is a complemented subspace of a Banach space E for which $\text{FBL}^p[E] = \overline{I_{w^*}[E]}$. Then $\text{FBL}^p[F] = \overline{I_{w^*}[F]}$.*

Proof. The hypothesis that F is complemented in E means that the inclusion map $J: F \hookrightarrow E$ has a bounded linear left inverse $L: E \rightarrow F$; that is, $LJ = I_F$. This implies that $(C_{L^*} \circ C_{J^*})f = f \circ J^* \circ L^* = f$ for every $f \in H[F]$, and consequently

$$\begin{aligned} \overline{I_{w^*}[F]} &= C_{L^*}(C_{J^*}(\overline{I_{w^*}[F]})) \subseteq C_{L^*}(\overline{I_{w^*}[E]}) = C_{L^*}(\text{FBL}^p[E]) \\ &= \overline{L}(\text{FBL}^p[E]) \subseteq \text{FBL}^p[F], \end{aligned}$$

where we have used (6.8), the hypothesis and (6.7). This completes the proof because the opposite inclusion $\text{FBL}^p[F] \subseteq \overline{I_{w^*}[F]}$ is always true. $\square$

Remark 6.16. In view of theorem 4.3, it is possible that proposition 6.15 has no genuine content in the sense that the hypothesis that $\text{FBL}^p[E] = \overline{I_{w^*}[E]}$ for some $1 \leq p < \infty$ may only be satisfied when E is finite-dimensional.

Proposition 6.17. *Let E be a reflexive Banach space for which $\overline{I_{w^*}[E]} = H_{w^*}^p[E]$ for some $1 \leq p < \infty$. Then $\overline{I_{w^*}[F]} = H_{w^*}^p[F]$ for every quotient space F of E .*

Proof. Let $Q: E \rightarrow F$ denote the quotient map, and write $q: B_E \rightarrow B_F$ for its restriction to the unit balls. Then q is continuous with respect to the relative weak topologies and surjective (because E and F are reflexive), so we can define an isometric lattice homomorphism $C_q: C(B_F) \rightarrow C(B_E)$ by composition: $C_q f = f \circ q$. Hence, by [27, Theorem 1.4.19], its adjoint $C_q^*: C(B_E)^* \rightarrow C(B_F)^*$ is a surjective, interval-preserving operator.

As previously mentioned, for every $f \in H_{w^*}^p[F]$, we can find a probability measure $\mu \in C(B_F)_+^*$ such that (4.14) is satisfied; that is,

$$|f(x^*)| \leq \|f\|_{\text{FBL}^p[F]} f_\mu^p(x^*) \quad (x^* \in F^*), \quad (6.9)$$

where we recall that the function $f_\mu^p \in H^p[F]$ is defined by (4.13). In view of the previous paragraph, we can take $\nu \in C(B_E)_+^*$ such that $C_q^*(\nu) = \mu$. Since E is reflexive, $f_\nu^p \in H_{w^*}^p[E] = \overline{I_{w^*}[E]}$ by theorem 5.5 and the hypothesis.

For $y^* \in B_{F^*}$, the function $\psi_p(y^*) \in C(B_F)$ defined in lemma 5.4 satisfies

$$C_q(\psi_p(y^*))(x) = \psi_p(y^*)(q(x)) = |y^*(q(x))|^p = |(Q^*y^*)(x)|^p = \psi_p(Q^*y^*)(x)$$

for each $x \in B_E$, so

$$\begin{aligned} \left(f_\mu^p(y^*)\right)^p &= \langle \psi_p(y^*), \mu \rangle = \langle C_q(\psi_p(y^*)), \nu \rangle \\ &= \langle \psi_p(Q^*y^*), \nu \rangle = \left(f_\nu^p(Q^*y^*)\right)^p = \left(C_{Q^*}(f_\nu^p)(y^*)\right)^p. \end{aligned}$$

This proves that $f_\mu^p = C_{Q^*}(f_\nu^p)$, which belongs to $\overline{I_{w^*}[F]}$ by (6.8), and therefore $f \in \overline{I_{w^*}[F]}$ by (6.9) because $\overline{I_{w^*}[F]}$ is an ideal of $H_{w^*}^p[F]$. $\square$

We shall next address two very natural questions concerning a lattice homomorphism T between free Banach lattices: When is the induced map Φ_T linear? And how can we tell whether T arises from a bounded linear operator between the underlying Banach spaces via the construction described in example 6.14?

Proposition 6.18. *Let $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ be a lattice homomorphism for some $1 \leq p < \infty$ and some Banach spaces E and F . The following conditions are equivalent:*

- (a) *The induced map $\Phi_T: F^* \rightarrow E^*$ is linear.*
- (b) *The induced map $\Phi_T: F^* \rightarrow E^*$ is an adjoint operator; that is, $\Phi_T = S^*$ for some bounded linear operator $S: E \rightarrow F$.*
- (c) *There is a bounded linear operator $S: E \rightarrow F$ for which $T \circ \delta^E = \delta^F \circ S$.*
- (d) *$T = \overline{S}$ for some bounded linear operator $S: E \rightarrow F$.*

Proof. We begin by showing that conditions (a) and (b) are equivalent. It is clear that (b) implies (a). Conversely, suppose that Φ_T is linear. By theorem 6.4, its restriction $\Phi_T|_{B_{F^*}}$ is weak*-to-weak* continuous. This implies that $\Phi_T = S^*$ for some bounded linear operator $S: E \rightarrow F$, as one can prove by adding an application of the Krein–Smulian theorem to the standard proof that a weak*-to-weak* continuous linear operator between dual Banach spaces is the adjoint of a bounded linear operator.

To see that conditions (b) and (c) are also equivalent, we observe that for $x \in E$, $y^* \in F^*$ and a bounded linear operator $S: E \rightarrow F$, we have

$$(\delta^F \circ S)(x)(y^*) = \delta_{Sx}(y^*) = y^*(Sx) = (S^*y^*)(x),$$

while (6.2) implies that

$$(T \circ \delta^E)(x)(y^*) = (T\delta_x)(y^*) = (\delta_x \circ \Phi_T)(y^*) = \Phi_T(y^*)(x).$$

Comparing these two equations, we conclude that $T \circ \delta^E = \delta^F \circ S$ if and only if $\Phi_T = S^*$.

Finally, we verify that (c) and (d) are equivalent. Equation (6.6) shows that (d) implies (c), while the converse implication follows from the fact that $\bar{S}$ is the unique lattice homomorphism satisfying (6.6). $\square$

The following example shows that not all lattice homomorphisms between free Banach lattices arise from bounded linear operators, or equivalently that the maps they induce need not be linear.

Example 6.19. Take $1 \leq p < \infty$, let $(e_n)_{n \in \mathbb{N}}$ denote the unit vector basis for ℓ_1 , and identify ℓ_1^* with ℓ_∞ , as usual. Since

$$\|\delta_{e_n}\|_{\text{FBL}^p[\ell_1]} = \|\delta_{e_n}\|_{\text{FBL}^p[\ell_1]} = \|e_n\|_{\ell_1} = 1 \quad (n \in \mathbb{N}),$$

we can define a bounded linear operator $R: \ell_1 \rightarrow \text{FBL}^p[\ell_1]$ of norm 1 by $Re_n = |\delta_{e_n}|$ for every $n \in \mathbb{N}$. The universal property of $\text{FBL}^p[\ell_1]$ implies that R induces a lattice homomorphism $\hat{R}: \text{FBL}^p[\ell_1] \rightarrow \text{FBL}^p[\ell_1]$ such that $\hat{R} \circ \delta^{\ell_1} = R$, which in turn induces a map $\Phi_{\hat{R}}: \ell_\infty \rightarrow \ell_\infty$ such that $\hat{R}f = f \circ \Phi_{\hat{R}}$ for every $f \in \text{FBL}^p[\ell_1]$ by theorem 6.4. Combining these identities with the definition of R , we obtain

$$\begin{aligned} \Phi_{\hat{R}}(x^*)(e_n) &= (\delta_{e_n} \circ \Phi_{\hat{R}})(x^*) = (\hat{R}\delta_{e_n})(x^*) = (Re_n)(x^*) \\ &= |\delta_{e_n}|(x^*) = |x^*(e_n)| = |x^*|(e_n) \quad (n \in \mathbb{N}, x^* \in \ell_\infty). \end{aligned}$$

This shows that $\Phi_{\hat{R}}(x^*) = |x^*|$ for every $x^* \in \ell_\infty$, so $\Phi_{\hat{R}}$ is non-linear.

Although the map induced by a lattice homomorphism between free Banach lattices need not be linear, it is sometimes possible to find a bounded linear operator which agrees with it on certain vectors.

Proposition 6.20. Let $\Phi \in \text{PH}^1[F^*, E^*]$ for some Banach spaces E and F , and suppose that F^* has an unconditional basis $(f_n^*)_{n \in \mathbb{N}}$. Then the map $R: f_n^* \mapsto \Phi(f_n^*)$ for $n \in \mathbb{N}$ extends to a bounded linear operator $R: F^* \rightarrow E^*$.

Proof. We extend R by linearity to the dense subspace of F^* spanned by $(f_n^*)_{n \in \mathbb{N}}$ and claim that this map is bounded by $K \|\Phi\|_{\text{PH}^1[F^*, E^*]}$, where K denotes the unconditional basis constant of (f_n^*) . Take $(a_j)_{j=1}^m \subset \mathbb{R}$ for some $m \in \mathbb{N}$, and choose $\sigma_j = \pm 1$ such that $\sigma_j a_j \geq 0$ for each $j \in \{1, \dots, m\}$. Then, by (1.4), we have

$$\|(\sigma_j a_j f_j^*)_{j=1}^m\|_{1,\text{weak}} \leq K \left\| \sum_{j=1}^m a_j f_j^* \right\|,$$

which implies that

$$\begin{aligned} \|\Phi\|_{\text{PH}^1[F^*, E^*]} K \left\| \sum_{j=1}^m a_j f_j^* \right\| &\geq \|(\Phi(\sigma_j a_j f_j^*))_{j=1}^m\|_{1,\text{weak}} \\ &= \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^m \varepsilon_j \Phi(\sigma_j a_j f_j^*) \right\| = \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^m \varepsilon_j \sigma_j a_j \Phi(f_j^*) \right\| \\ &\geq \left\| \sum_{j=1}^m a_j \Phi(f_j^*) \right\| = \left\| R \left(\sum_{j=1}^m a_j f_j^* \right) \right\|. \end{aligned}$$

This proves the claim, and the result follows. $\square$

A possible strategy for showing that the Banach spaces E and F are linearly isomorphic whenever a lattice isomorphism $T: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ exists for some $1 \leq p < \infty$ would be to find a suitable linearization of the induced map $\Phi_T: F^* \rightarrow E^*$ that one could then use to construct an isomorphism between E and F .

Remark 6.21. Let $\Phi \in \text{PH}^p[F^*, E^*]$ for some $1 \leq p < \infty$ and Banach spaces E and F . Then, for every $m \in \mathbb{N}$ and $(y_j^*)_{j=1}^m \subset F^*$, we have

$$\begin{aligned} \left\| \Phi \left(\sum_{j=1}^m y_j^* \right) - \sum_{j=1}^m \Phi(y_j^*) \right\| &\leq \left\| \Phi \left(\sum_{j=1}^m y_j^* \right) \right\| + \sum_{j=1}^m \|\Phi(y_j^*)\| \\ &\leq \|\Phi\|_{\text{PH}^p[F^*, E^*]} \left\| \sum_{j=1}^m y_j^* \right\| + \sum_{j=1}^m \|\Phi\|_{\text{PH}^p[F^*, E^*]} \|y_j^*\| \leq 2\|\Phi\|_{\text{PH}^p[F^*, E^*]} \sum_{j=1}^m \|y_j^*\|. \end{aligned}$$

This implies that Φ is quasi-linear if it is homogeneous (not just positively homogeneous). Since there is more than one definition of “quasi-linear” in circulation, let us explicitly state the definition we use, following [9, Chapter 16] and [32, Definition 3.2.1] (not [26, Definition 2.a.9]): a map $\Phi: E \rightarrow F$ between Banach spaces E and F is *quasi-linear* if it is homogeneous and there is a constant $K > 0$ such that

$$\|\Phi(x + y) - \Phi(x) - \Phi(y)\| \leq K(\|x\| + \|y\|) \quad (x, y \in E).$$

As we saw in example 6.19, the issue that the map Φ may not be homogeneous is genuine, even when it is induced by a lattice homomorphism between free Banach lattices. However, our next result shows that there are instances where homogeneity is automatic, so the induced map is quasi-linear.

Lemma 6.22. Let E and F be Banach spaces, and let $\Phi: F^* \rightarrow E^*$ be a positively homogeneous bijection for which $\|\Phi\|_{\text{PH}^1[F^*, E^*]} = 1 = \|\Phi^{-1}\|_{\text{PH}^1[E^*, F^*]}$. Suppose that the norm on E^* is strictly convex. Then Φ is homogeneous.

Proof. It suffices to show that $\Phi(y^*) = -\Phi(-y^*)$ for every unit vector $y^* \in F^*$ because Φ is positively homogeneous. Combining the triangle inequality with the fact that $\|\Phi\|_{\text{PH}^1[F^*, E^*]} = 1$ and (1.4), we obtain

$$2 \geq \max \|\Phi(y^*) \pm \Phi(-y^*)\| = \|(\Phi(y^*), \Phi(-y^*))\|_{1, \text{weak}} \geq \|(y^*, -y^*)\|_{1, \text{weak}}$$

because $\|\Phi^{-1}\|_{\text{PH}^1[E^*, F^*]} = 1$. Another application of (1.4) shows that

$$\|(y^*, -y^*)\|_{1, \text{weak}} = \max \|y^* \pm (-y^*)\| = 2,$$

so either $\|\Phi(y^*) + \Phi(-y^*)\| = 2$ or $\|\Phi(y^*) - \Phi(-y^*)\| = 2$. Since $\|\pm \Phi(\pm y^*)\| \leq \|y^*\| = 1$, the strict convexity of the norm on E^* implies that $\Phi(y^*) = \Phi(-y^*)$ in the first case and $\Phi(y^*) = -\Phi(-y^*)$ in the second. However, $\Phi(y^*) = \Phi(-y^*)$ is impossible because Φ is injective and $y^* \neq 0$, so we must be in the second case, which gives the desired conclusion. $\square$

Corollary 6.23. *Let $T: \text{FBL}^1[E] \rightarrow \text{FBL}^1[F]$ be an isometric lattice isomorphism for some Banach spaces E and F , and suppose that the norm on E^* is strictly convex. Then the induced map $\Phi_T: F^* \rightarrow E^*$ is homogeneous.*

Proof. This follows immediately from lemma 6.22 because the hypothesis that T is an isometric lattice isomorphism means that the induced map $\Phi_T: F^* \rightarrow E^*$ is a positively homogeneous bijection with inverse $\Phi_{T^{-1}}$, and theorem 6.4(i) ensures that $\|\Phi_T\|_{\text{PH}^1[F^*, E^*]} = \|T\| = 1$ and $\|\Phi_T^{-1}\|_{\text{PH}^1[E^*, F^*]} = \|T^{-1}\| = 1$. $\square$

We refer to [29, section 10.3] for a detailed analysis of lattice isometries between free Banach lattices.

We conclude with an extension of another result from [29]. By an isomorphic embedding of a Banach space E into a Banach space F , we understand a bounded linear operator $J: E \rightarrow F$ which is injective and has closed range, or in other words J is bounded below by some constant $\eta > 0$ in the sense that $\|Jx\| \geq \eta\|x\|$ for every $x \in E$. In this case $J^*: F^* \rightarrow E^*$ is surjective, so (6.7) implies that the associated lattice homomorphism $\bar{J}: \text{FBL}^p[E] \rightarrow \text{FBL}^p[F]$ is injective for every $1 \leq p < \infty$. The question of whether $\bar{J}$ is an isomorphic embedding (that is, has closed range) was analyzed in depth in [29, section 3.3]. We shall next provide an extension of the main result therein. In the proof, we require the following standard identity between the operator norm of a bounded linear operator S from a Banach space E into ℓ_p^n for some $1 \leq p < \infty$ and $n \in \mathbb{N}$ and the weak p -norm:

$$\|S\| = \|(S^* e_j^*)_{j=1}^n\|_{p, \text{weak}}, \quad (6.10)$$

where $(e_j^*)_{j=1}^n$ denotes the unit vector basis of $(\ell_p^n)^*$.

Proposition 6.24. *Let $J: E \hookrightarrow F$ be an isomorphic embedding of a Banach space E into a Banach space F , and take $1 \leq p < \infty$. The following conditions are equivalent:*

- (a) There is a constant $C \geq 1$ such that, for every $n \in \mathbb{N}$, every bounded linear operator $S: E \rightarrow \ell_p^n$ admits a bounded linear extension $\tilde{S}: F \rightarrow \ell_p^n$ (in the sense that $S = \tilde{S}J$) with $\|\tilde{S}\| \leq C\|S\|$.
- (b) The lattice homomorphism $\bar{J}$ is an isomorphic embedding of $\text{FBL}^p[E]$ into $\text{FBL}^p[F]$.
- (c) The lattice homomorphism C_{J^*} is an isomorphic embedding of $H^p[E]$ into $H^p[F]$.

Proof. The equivalence of conditions (a) and (b) is essentially a restatement of [29, Theorem 3.7, (1) $\Leftrightarrow$ (3)]. (“Essentially” refers to the fact that [29, Theorem 3.7] is stated only for isometric embeddings. However, it is explained in the text above it how to reduce the isomorphic case to the isometric.) Furthermore, the implication (c) $\Rightarrow$ (b) is immediate from (6.7), so it only remains to verify that (a) implies (c).

Suppose that (a) is satisfied for some constant $C \geq 1$, and take $\eta \in (0, 1)$. Then, for every $f \in H^p[E]$, we can find $n \in \mathbb{N}$ and $(x_j^*)_{j=1}^n \subset E^*$ with $\|(x_j^*)_{j=1}^n\|_{p, \text{weak}} = 1$ such that

$$\eta\|f\|_{\text{FBL}^p[E]} \leq \left(\sum_{j=1}^n |f(x_j^*)|^p \right)^{\frac{1}{p}}.$$

Define an operator $S: E \rightarrow \ell_p^n$ by $Sx = (\langle x, x_j^* \rangle)_{j=1}^n$. Clearly S is bounded and linear, and $\|S\| = 1$ by (6.10) because $S^*e_j^* = x_j^*$ for each $j = 1, \dots, n$, so the hypothesis implies that S admits a bounded linear extension $\tilde{S}: F \rightarrow \ell_p^n$ whose norm is at most C . Another application of (6.10) shows that $\|(\tilde{S}^*e_j^*)_{j=1}^n\|_{p, \text{weak}} \leq C$, and consequently we have

$$\begin{aligned} C^p\|C_{J^*}f\|_{\text{FBL}^p[F]}^p &\geq \sum_{j=1}^n |(C_{J^*}f)(\tilde{S}^*e_j^*)|^p = \sum_{j=1}^n |f(J^*\tilde{S}^*e_j^*)|^p \\ &= \sum_{j=1}^n |f(S^*e_j^*)|^p = \sum_{j=1}^n |f(x_j^*)|^p \geq \eta^p\|f\|_{\text{FBL}^p[E]}^p. \end{aligned}$$

This proves that C_{J^*} is bounded below by $\eta/C > 0$, and it is therefore an isomorphic embedding. $\square$

Acknowledgements. This research was initiated during a visit of the second author to Lancaster University and continued during visits of the first author to ICMAT, supported by the London Mathematical Society (grant numbers 21805 and 42109). Furthermore, Tradacete’s research is partially supported by grants PID2020-116398GB-I00, CEX2023-001347-S and CEX2019-000904-S funded by MCIN/AEI/10.13039/501100011033, as well as a 2022 Leonardo Grant for Researchers and Cultural Creators from the BBVA Foundation.

We wish to thank Antonio Avilés, Timur Oikhberg, Mitchell Taylor and Vladimir Troitsky for many helpful discussions and for sharing ideas relevant to the content of this paper.

References

1. F. Albiac and N. J. Kalton, *Topics in Banach Space theory Graduate Texts in Mathematics*, Volume 233 (Springer-Verlag, New York, 2006).
2. C. D. Aliprantis and O. Burkinshaw, *Positive operators* (Springer-Verlag, Dordrecht, 2006) (reprint of the 1985 original).
3. A. Avilés, G. Martínez-Cervantes and J. D. Rodríguez Abellán, On projective Banach lattices of the form $C(K)$ and $FBL[E]$, *J. Math. Anal. Appl.* **9**(1) (2020), 124129.
4. A. Avilés, G. Martínez-Cervantes and J. D. Rodríguez-Abellán, On the Banach lattice c_0 , *Rev. Mat. Complut.* **34**(1) (2021), 203–213.
5. A. Avilés, G. Plebanek and J. D. Rodríguez-Abellán, Chain conditions in free Banach lattices, *J. Math. Anal. Appl.* **465**(2) (2018), 1223–1229.
6. A. Avilés, J. Rodríguez and P. Tradacete, The free Banach lattice generated by a Banach space, *J. Funct. Anal.* **274**(10) (2018), 2955–2977.
7. A. Avilés and P. Tradacete, Amalgamation and injectivity in Banach lattices, *Int. Math. Res. Not.* **2023**(2) (2023), 956–997.
8. K. A. Baker, Free vector lattices, *Canad. J. Math.* **20** (1968), 58–66.
9. Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional analysis *Amer. Math. Soc. Coll. Publ.* **1**(48) (2000).
10. R. D. Bleier, Free vector lattices, *Trans. Amer. Math. Soc.* **176** (1973), 73–87.
11. C. S. Cardassi, *Strictly p -integral and p -nuclear operators*. In *Analyse harmonique: Groupe de Travail sur les Espaces de Banach Invariants par Translation, Exp. No. 2, 22 pp. Publ. Math. Orsay 89-02* (Université de Paris-Sud, Département de Mathématiques, Orsay, 1989).
12. S. Dantas, G. Martínez-Cervantes, J. Rodríguez Abellán and A. R. Zoca, Norm-attaining lattice homomorphisms, *Rev. Mat. Iberoam.* **38**(3) (2022), 981–1002.
13. D. de Hevia and P. Tradacete, Free complex Banach lattices, *J. Funct. Anal.* **284**(10) (2023), 109888.
14. B. de Pagter and A. W. Wickstead, Free and projective Banach lattices, *Proc. Roy. Soc. Edinb. Sect. A* **145**(1) (2015), 105–143.
15. J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing operators*. Cambridge Studies in Advanced Mathematics, Volume 43 (Cambridge University Press, Cambridge, 1995).
16. J. Diestel and J. Swart, *The Riesz theorem*. In *Handbook of Measure Theory*, Volume I, 401–447 (Editor E. Pap, North-Holland, Amsterdam, 2002).
17. N. Dunford and J. T. Schwartz, *Linear operators. Part I: General theory* (Interscience Publishers, Inc, New York, 1958).
18. M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach space theory: The basis for linear and nonlinear analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC* (Springer, New York, 2011).
19. A. Goulet de Rugy, La structure idéale des M-espaces, *J. Math. Pures Appl.* **51** (1972), 331–373.
20. W. T. Gowers, A solution to Banach’s hyperplane problem, *Bull. London Math. Soc.* **26**(6) (1994), 523–530.
21. W. T. Gowers and B. Maurey, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851–874.
22. W. T. Gowers and B. Maurey, Banach spaces with small spaces of operators, *Math. Ann.* **307** (1997), 543–568.
23. A. Grothendieck, Sur certaines classes de suites dans les espaces de Banach et le théorème de Dvoretzky–Rogers, *Bol. Soc. Mat. São Paulo* **8**(1953), 81–110.

24. P. Hájek, V. M. Santalucía, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces*. *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC* (Springer, New York, 2008).
25. H. Jardón-Sánchez, N. J. Laustsen, M. Taylor, P. Tradacete and V. G. Troitsky, Free Banach lattices under convexity conditions. *RACSAM* **116**(1) (2022), 15.
26. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Volume II (Springer-Verlag, Berlin, 1979).
27. P. Meyer-Nieberg, *Banach lattices* (Springer, 1991).
28. T. Oikhberg, Directed sets in free Banach lattices, forthcoming.
29. T. Oikhberg, M. Taylor, P. Tradacete and V. G. Troitsky, Free Banach lattices, *J. Eur. Math. Soc.* (2024), published online first.
30. M. Rogalski, L'espace des fonctions homogènes sur un cône à chapeau universel, *C. R. Acad. Sci. Paris Sér. A–B* **271** (1970), A401–A403.
31. M. Rogalski, L'espace des fonctions homogènes sur certains cônes bien coiffés, *Bull. Sci. Math.* **95** (1971), 305–325.
32. F. Cabello Sánchez and J. M. F. Castillo, *Homological Methods in Banach Space Theory*. *Cambridge Studies in Advanced Mathematics*, Volume 203 (Cambridge University Press, Cambridge, 2023).
33. V. G. Troitsky, Simple constructions of $\text{FBL}(A)$ and $\text{FBL}[E]$, *Positivity* **23** (2019), 1173–1178.
34. A. W. Wickstead, Banach lattices with trivial centre, *Proc. Roy. Irish Acad. Sect. A* **88**(1) (1988), 71–83.