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# Measurable and Continuous Units of an $E_0$-semigroup

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*Abstract.* Let $P$ be a closed convex cone in $\mathbb{R}^d$ which is spanning, *i.e.*, $P - P = \mathbb{R}^d$ and pointed, *i.e.*, $P \cap -P = \{0\}$. Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup over $P$ and let $E$ be the product system associated to $\alpha$. We show that there exists a bijective correspondence between the units of $\alpha$ and the units of $E$.

## 1 Introduction

Units of an $E_0$-semigroup play an important role in Arveson's classification programme of $E_0$-semigroups [2]. Apparently there are two notions of units, one that is associated with an $E_0$-semigroup and the other that is associated with the product system associated to the given $E_0$-semigroup; however both these notions coincide. We refer the reader to [2, p. 85] (the paragraph preceding Proposition 3.6.5) to convince herself of this fact. The authors in [6] have extended the notion of $E_0$-semigroups, called $E_0$-semigroups over $P$, to the case of closed convex cones. The purpose of this paper is to prove that there exists a bijective correspondence between the units of an $E_0$-semigroup over $P$ and the units of the associated product system.

We fix notation which will be used throughout this paper. Let $P \subset \mathbb{R}^d$ be a closed convex cone which is spanning and pointed, *i.e.*, $P - P = \mathbb{R}^d$ and $P \cap -P = \{0\}$. Denote the interior of $P$ by $\Omega$. Then $\Omega$ is dense in $P$ (see [6, Lemma 3.1]). Also $\Omega - \Omega = \mathbb{R}^d$. Observe that $\Omega$ is an ideal in $P$ in the sense that $\Omega + P \subset \Omega$. Let $\mathcal{H}$ be an infinite dimensional complex separable Hilbert space. Denote the algebra of bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$. The trace class operators on $\mathcal{H}$ will be denoted by $\mathcal{L}^1(\mathcal{H})$. We will use the above notation for the rest of this paper.

By an $E_0$-semigroup over $P$ on $B(\mathcal{H})$, we mean a family $\alpha := \{\alpha_x\}_{x \in P}$ of normal $*$-endomorphisms of $B(\mathcal{H})$ such that

(1) for $x \in P$, $\alpha_x$ is unital, *i.e.*, $\alpha_x(1) = 1$,
(2) for $x, y \in P$, $\alpha_{x+y} = \alpha_x \circ \alpha_y$, and
(3) for $A \in B(\mathcal{H})$ and $T \in \mathcal{L}^1(\mathcal{H})$, the map $P \ni x \mapsto Tr\big(\alpha_x(A)T\big) \in \mathbb{C}$ is continuous.

Using the fact that $*$-homomorphisms are contractive and the fact that finite rank operators are norm dense in $\mathcal{L}^1(\mathcal{H})$, it is easy to see that Condition (3) above can be replaced by the following condition.

(3′) For $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $P \ni x \mapsto \langle \alpha_x(A)\xi \mid \eta \rangle \in \mathbb{C}$ is continuous.

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Since the cone $P$ is fixed for the rest of this paper, we simply call an $E_0$-semigroup over $P$ an $E_0$-semigroup. Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. Fix a continuous multiplier $\omega$ on $P$. By this, we mean a continuous map $\omega : P \times P \to \mathbb{T}$ such that

$$\omega(x,y)\omega(x+y,z) = \omega(x,y+z)\omega(y,z)$$

for all $x, y, z \in P$. The multiplier $\omega$ is fixed for the rest of this paper.

By an $\omega$-unit of the $E_0$-semigroup $\alpha$, we mean a family $v := \{v_x\}_{x\in P}$ of bounded operators on $\mathcal{H}$ such that

(1) for $x \in P$ and $A \in B(\mathcal{H})$, $\alpha_x(A)v_x = v_x A$,
(2) for $x, y \in P$, $v_{x+y} = \omega(x,y)v_x v_y$,
(3) there exists $x \in P$ such that $v_x \neq 0$, and
(4) for $\xi \in \mathcal{H}$, the map $P \ni x \mapsto v_x\xi \in \mathcal{H}$ is continuous.

Since we have fixed the multiplier $\omega$ for the rest of this paper, we will simply call a $\omega$-unit a unit. Let $\{v_x\}_{x\in P}$ be a unit. Since $\Omega$ is dense in $P$ and $\{v_x\}_{x\in P}$ is strongly continuous, we can assume in Condition (3) that there exists $x \in \Omega$ such that $v_x \neq 0$. We denote the collection of units of $\alpha$ by $\mathcal{U}_\alpha$.

For every $E_0$-semigroup, there is an associated product system. Product systems were originally invented by Arveson to classify $E_0$-semigroups when $P = [0,\infty)$ [2]. Let us recall the notion of the product system associated to an $E_0$-semigroup.

Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. We endow $B(\mathcal{H})$ with the measurable structure induced by the $\sigma$-weak topology on $B(\mathcal{H})$. We consider $\Omega \times B(\mathcal{H})$ as a measurable space where the measurable structure is the one induced by the cartesian product of measurable structures on $\Omega$ (*i.e.*, the Borel $\sigma$-algebra on $\Omega$) and on $B(\mathcal{H})$. Let

$$E := \{(x,T) \in \Omega \times B(\mathcal{H}) : \alpha_x(A)T = TA \ \forall A \in B(\mathcal{H})\}.$$

For $x \in \Omega$, set $E(x) := \{T \in B(\mathcal{H}) : (x,T) \in E\}$. Let $p\colon E \to \Omega$ be the first projection. We have the following.

(1) The set $E$ is a measurable subset of $\Omega \times B(\mathcal{H})$.
(2) Let $x \in \Omega$ be given. For $S, T \in E(x)$, $T^*S$ is a scalar which we denote by $\langle S \mid T\rangle$. With respect to the inner product $\langle \cdot|\cdot\rangle$, the vector space $E(x)$ is a Hilbert space.
(3) For $x, y \in \Omega$, the closed linear span of $\{TS : T \in E(x), S \in E(y)\}$ is dense in $E(x+y)$.
(4) There exists a non-zero separable Hilbert space $\mathcal{H}_0$ for which the following holds: for every $x \in \Omega$, there exists a unitary operator $\theta_x\colon E(x) \to \mathcal{H}_0$ such that the map $E \ni (x,T) \mapsto \big(x, \theta_x(T)\big) \in \Omega \times \mathcal{H}_0$ is a Borel isomorphism, where the measurable structure on $\Omega \times \mathcal{H}_0$ is the one induced by the product topology on $\Omega \times \mathcal{H}_0$.

The Borel space $E$ together with the structures given by (1)–(4) is called the product system associated to $\alpha$. We refer the reader to [6] for a proof of the above facts. Note that $E$ has a semigroup structure where the semigroup multiplication is given by

$$(x,T).(y,S) = (x+y, TS).$$

Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup and let $E$ be the product system associated to $\alpha$. By a $\omega$-unit of the product system $E$, we mean a non-zero measurable cross section of $E$ which respects the semigroup operation up to the factor $\omega$. A moment's thought reveals that by a $\omega$-unit, or simply a unit, of $E$, we mean a family of bounded operators $\{v_x\}_{x\in\Omega}$ on $\mathcal{H}$ such that

(1) for $x \in \Omega$ and $A \in B(\mathcal{H})$, $\alpha_x(A)v_x = v_x A$,
(2) for $x, y \in \Omega$, $v_{x+y} = \omega(x, y)v_x v_y$,
(3) there exists $x \in \Omega$ such that $v_x \neq 0$, and
(4) for $T \in \mathcal{L}^1(\mathcal{H})$, the map $\Omega \ni x \mapsto Tr(v_x T)$ is measurable.

Since the measurable structures on $B(\mathcal{H})$ induced by the $\sigma$-weak topology and the weak topology coincide, we can replace Condition (4) by the following condition.

(4′) for $\xi, \eta \in \mathcal{H}$, the map $\Omega \ni x \mapsto \langle v_x \xi \mid \eta\rangle$ is measurable.

Let $\mathcal{U}_E$ denote the collection of units of $E$. Now we can state our main theorem.

***Theorem 1.1*** *Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$ and let $E$ be the product system associated to $\alpha$. Then the restriction map*

$$\mathcal{U}_\alpha \ni \{v_x\}_{x\in P} \mapsto \{v_x\}_{x\in\Omega} \in \mathcal{U}_E$$

*is a bijection.*

## 2 Towards the Proof of Theorem 1.1

We start with a little lemma which is probably well known. But we could not find any reference and thus we have included the proof.

***Remark 2.1*** We need the following two well-known facts.

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(x + y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$. Then there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.
(2) Let $f: \mathbb{R}^d \to \mathbb{R}$ be a measurable function that $f(x + y) = f(x) + f(y)$ for $x, y \in \mathbb{R}^d$. It is easily deducible from (1) that there exists $\lambda \in \mathbb{R}^d$ such that $f(x) = \langle\lambda \mid x\rangle$. Here $\langle \mid \rangle$ denotes the standard inner product on $\mathbb{R}^d$.

***Lemma 2.1*** *We have the following.*

(1) *Let $f: \Omega \to \mathbb{R}$ be measurable and $f(x + y) = f(x) + f(y)$ for all $x, y \in \Omega$. Then there exists $\lambda \in \mathbb{R}^d$ such that $f(x) = \langle\lambda \mid x\rangle$ for $x \in \Omega$.*
(2) *Let $f: P \to \mathbb{R}$ be measurable and $f(x + y) = f(x) + f(y)$ for all $x, y \in P$. Then there exists $\lambda \in \mathbb{R}^d$ such that $f(x) = \langle\lambda \mid x\rangle$ for $x \in P$.*

**Proof** Let $f: \Omega \to \mathbb{R}$ be as in (1). Define $g: \mathbb{R}^d \to \mathbb{R}$ as follows: for $z \in \mathbb{R}^d$, write $z = x - y$ with $x, y \in \Omega$ and set $g(z) = f(x) - f(y)$. We leave it to the reader to verify that $g$ is well-defined. We claim that $g$ is measurable.

Note that $\mathbb{R}^d = \bigcup_{a\in\Omega}(\Omega - a)$ and $\mathbb{R}^d$ is separable. Thus there exists a sequence $(a_k) \in \Omega$ such that $\mathbb{R}^d = \bigcup_{k=1}^{\infty}(\Omega - a_k)$. Fix $k \geq 1$. Note that for $z \in \Omega - a_k$, $g(z) = f(z + a_k) - f(a_k)$. The measurability of $f$ implies that $g$ is measurable on $\Omega - a_k$ for every $k \geq 1$. This implies that $g$ is measurable. It is clear that $g$ is additive. We claim that $g$ agrees with $f$ on $\Omega$. Let $x \in \Omega$ be given. Choose $s \in \Omega$. Then $x = (x + s) - s$. Note that

$$g(x) = f(x + s) - f(s) = f(x) + f(s) - f(s) = f(x).$$

This proves our claim. By Remark 2.1, it follows that there exists $\lambda \in \mathbb{R}^d$ such that $\langle \lambda \mid x \rangle = g(x)$ for every $x \in \mathbb{R}^d$. Hence $f(x) = \langle \lambda \mid x \rangle$ for every $x \in \Omega$. This proves (1).

Let $f\colon P \to \mathbb{R}$ be as in (2). Denote the restriction of $f$ to $\Omega$ by $f_0$. By (1), there exists $\lambda \in \mathbb{R}^d$ such that $\langle \lambda \mid x \rangle = f_0(x) = f(x)$ for every $x \in \Omega$. Now let $x \in P$ be given. Choose $s \in \Omega$. Observe that

$$\begin{aligned} f(x) &= f(x+s) - f(s) \\ &= f_0(x+s) - f_0(s) \\ &= \langle \lambda \mid x+s \rangle - \langle \lambda \mid s \rangle \\ &= \langle \lambda \mid x \rangle. \end{aligned}$$

This completes the proof. □

Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$ which will be fixed throughout this paper. Denote the product system associated to $\alpha$ by $E$. For $x \in P$, let

$$E(x) := \{T \in B(\mathcal{H}) : \alpha_x(A)T = TA \ \forall A \in B(\mathcal{H})\}.$$

Let $x \in P$ be given. Note that for $T, S \in E(x)$, $T^*S$ commutes with every element of $B(\mathcal{H})$. Thus for $T, S \in E(x)$, $T^*S$ is a scalar.

***Lemma 2.2*** *For $A \in B(\mathcal{H})$ and $\xi \in \mathcal{H}$, the map $P \ni x \mapsto \alpha_x(A)\xi$ is continuous.*

**Proof** Let $U \in B(\mathcal{H})$ be a unitary and $\xi \in \mathcal{H}$ be given. Let $(x_n)$ be a sequence in $P$ such that $x_n \to x \in P$. Then $\alpha_{x_n}(U)$, a sequence of unitaries, converges weakly to $\alpha_x(U)$, which is a unitary. Hence $\alpha_{x_n}(U)$ converges strongly to $\alpha_x(U)$. This implies that $\alpha_{x_n}(U)\xi \to \alpha_x(U)\xi$. Now the proof is completed by using the fact that any bounded operator on $\mathcal{H}$ can be written as a finite linear combination of unitaries. □

***Lemma 2.3*** *We have the following.*

(1) *Let $\{v_x\}_{x \in \Omega} \in \mathcal{U}_E$ be given. Then there exists a unique $\lambda \in \mathbb{R}^d$ such that for $x \in \Omega$, $v_x^* v_x = e^{\langle \lambda | x \rangle}$. In particular, for every $x \in \Omega$, $v_x \neq 0$.*

(2) *Let $\{v_x\}_{x \in P} \in \mathcal{U}_\alpha$ be given. Then there exists a unique $\lambda \in \mathbb{R}^d$ such that for $x \in P$, $v_x^* v_x = e^{\langle \lambda | x \rangle}$. In particular, for every $x \in P$, $v_x \neq 0$.*

**Proof** We imitate the proof of Proposition 3.6.2 of [2]. Let $\{v_x\}_{x \in \Omega} \in \mathcal{U}_E$ be given. Choose $a \in \Omega$ such that $v_a \neq 0$. For $x \in \Omega$, let $g(x) = v_x^* v_x$. The $\sigma$-weak measurability of $\{v_x\}_{x \in \Omega}$ implies that $g$ is measurable. Now observe that

$$\begin{aligned} g(x+y) &= v_{x+y}^* v_{x+y} \\ &= \overline{\omega(x,y)} v_y^* v_x^* \omega(x,y) v_x v_y \\ &= g(x) v_y^* v_y \\ &= g(x) g(y). \end{aligned}$$

Let $x \in \Omega$ be given. We claim that $g(x) > 0$. Clearly $g(x) \geq 0$. Since $v_a \neq 0$, $g(a) > 0$. Note that the sequence $a - \frac{x}{n}$ tends to $a \in \Omega$ as $n \to \infty$. Since $\Omega$ is open, $a - \frac{x}{n} \in \Omega$

eventually. Choose $n \geq 1$ such that $a - \frac{x}{n} \in \Omega$. Then $na - x \in \Omega$. Now the equality

$$g(x)g(na-x) = g(na) = g(a)^n \neq 0$$

forces that $g(x) \neq 0$. This proves our claim.

For $x \in \Omega$, let $f(x) = \log g(x)$. Then $f$ is measurable and $f(x+y) = f(x) + f(y)$ for $x, y \in \Omega$. By Lemma 2.1, there exists $\lambda \in \mathbb{R}^d$ such that $f(x) = \langle \lambda \mid x \rangle$ for $x \in \Omega$. Thus there exists $\lambda \in \mathbb{R}^d$ such that for $x \in \Omega$, $v_x^* v_x = g(x) = e^{\langle \lambda | x \rangle}$. The uniqueness of such a $\lambda$ follows from the fact that $\Omega$ is spanning, *i.e.*, $\Omega - \Omega = \mathbb{R}^d$. This proves (1).

Let $\{v_x\}_{x \in P} \in \mathcal{U}_\alpha$ be given. Then $\{v_x\}_{x \in \Omega} \in \mathcal{U}_E$. Hence there exists $\lambda \in \mathbb{R}^d$ such that $v_x^* v_x = e^{\langle \lambda | x \rangle}$ for $x \in \Omega$. The strong continuity of $\{v_x\}_{x \in P}$ implies that the map $P \ni x \mapsto v_x^* v_x \in [0, \infty)$ is continuous. Since $\Omega$ is dense in $P$, it follows that for $x \in P$, $v_x^* v_x = e^{\langle \lambda | x \rangle}$. The uniqueness of such a $\lambda$ follows from the fact that $P$ is spanning. This proves (2). Now the proof is complete. $\square$

***Definition 2.2*** We make the following definition.

(1) Let $\{u_x\}_{x \in \Omega} \in \mathcal{U}_E$. We say that $\{u_x\}_{x \in \Omega}$ is a normalised unit of $E$ if $u_x^* u_x = 1$ for every $x \in \Omega$.
(2) Let $\{u_x\}_{x \in P} \in \mathcal{U}_\alpha$. We say that $\{u_x\}_{x \in P}$ is a normalised unit of $\alpha$ if $u_x^* u_x = 1$ for every $x \in P$.

We need to recall a few facts from [4] before proving the next proposition. We refer the reader to Chapter 1 of [4] for details. The dual cone of $P$, denoted $P^*$, is defined as

$$P^* := \{a \in \mathbb{R}^d : \langle a \mid x \rangle \geq 0 \ \forall x \in P\}.$$

It is known that the dual of $P^*$ is $P$. Moreover $P^*$ is spanning and pointed. The interior of $P^*$, denoted $\Omega^*$, is given by [4, Proposition I.1.4]:

$$\Omega^* = \{a \in \mathbb{R}^d : \langle a \mid x \rangle > 0 \ \forall x \in P \backslash \{0\}\}.$$

By Lemma I.1.5 of [4], it follows that given $a \in \Omega^*$ and $k \geq 1$, there exists $M > 0$ such that for $x \in P$,

$$e^{-\langle a | x \rangle} \leq \frac{M}{(1 + \|x\|)^k}. \tag{2.1}$$

***Remark 2.3*** Let $(x_k)$ be a sequence in $\mathbb{R}^d$ such that $x_k \to x$. Then $1_{\Omega + x_k} \to 1_{\Omega + x}$ a.e. For a proof of this, see Lemma 3.1 of [6].

***Proposition 2.4*** *Let $\{v_x\}_{x \in \Omega} \in \mathcal{U}_E$ be given. Then $\{v_x\}_{x \in \Omega}$ is strongly continuous.*

**Proof** Let $\lambda \in \mathbb{R}^d$ be such that for $x \in \Omega$, $v_x^* v_x = e^{\langle \lambda | x \rangle}$. For $x \in \Omega$, set $u_x := e^{\frac{-\langle \lambda | x \rangle}{2}} v_x$. Then $\{u_x\}_{x \in \Omega} \in \mathcal{U}_E$. Note that $\{u_x\}_{x \in \Omega}$ is a family of isometries. It is clear that $\{v_x\}_{x \in \Omega}$ is strongly continuous if and only if $\{u_x\}_{x \in \Omega}$ is strongly continuous. Since $\{u_x\}_{x \in \Omega}$ is a family of isometries, it is enough to show that $\{u_x\}_{x \in \Omega}$ is weakly continuous. We claim that $\{u_x\}_{x \in \Omega}$ is weakly continuous.

For $\xi \in \mathcal{H}$ and $a \in \Omega^*$, let $\xi(a)$ be the unique vector in $\mathcal{H}$ such that for $\eta \in \mathcal{H}$,

$$\langle \xi(a) \mid \eta \rangle = \int_\Omega e^{-\langle a | x \rangle} \langle u_x \xi \mid \eta \rangle \, dx.$$

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The fact that such a vector $\xi(a)$ exists follows from the measurability of $\{u_x\}_{x\in\Omega}$, the norm boundedness of $\{u_x\}_{x\in\Omega}$, and from equation (2.1).

*Claim:* The linear span of $\{\xi(a) : \xi \in \mathcal{H}, a \in \Omega^*\}$ is dense in $\mathcal{H}$.

Suppose not. Then there exists a non-zero vector $\eta \in \mathcal{H}$ such that $\langle \xi(a) \mid \eta\rangle = 0$ for every $\xi \in \mathcal{H}$ and for every $a \in \Omega^*$. Let $\{\xi_1, \xi_2, \dots\}$ be an orthonormal basis for $\mathcal{H}$. Fix $i \in \{1,2,\dots\}$. For every $a \in \Omega^*$, the integral

$$\int_\Omega e^{-\langle a|x\rangle}\langle u_x\xi_i \mid \eta\rangle = 0.$$

By Proposition 3.4 of [6], it follows that for every $i$, $\langle u_x\xi_i \mid \eta\rangle = 0$ a.e. Thus there exists a subset $E \subset \Omega$ of measure zero such that for $x \notin E$, $\langle \xi_i \mid u_x^*\eta\rangle = 0$ for every $i = 1,2,\dots$. Hence $u_x^*\eta = 0$ for every $x \notin E$.

Now let $x \in \Omega$ be given. Fix $s \in \Omega$. The sequence $x - \frac{s}{n} \to x \in \Omega$. Thus eventually $x - \frac{s}{n} \in \Omega$. This implies that the intersection $(x - \Omega) \cap \Omega$ is a non-empty open subset of $\mathbb{R}^d$. Since $E$ is of measure zero, the interesection $(x - \Omega) \cap \Omega \cap E^c$ is non-empty. Let $z \in (x - \Omega) \cap \Omega \cap E^c$. Choose $y \in \Omega$ such that $z = x - y$. Now observe that

$$\begin{aligned} u_x^*\eta &= u_{z+y}^*\eta \\ &= \overline{\omega(z,y)}u_y^*u_z^*\eta \\ &= 0 \quad (\text{since } z \notin E). \end{aligned}$$

Hence $u_x^*\eta = 0$ for every $x \in \Omega$. Since $\alpha_x(A)u_x = u_xA$ for $x \in \Omega$ and $A \in B(\mathcal{H})$, it follows that for $x \in \Omega$ and $A \in B(\mathcal{H})$, $u_x^*\alpha_x(A)\eta = 0$.

Let $x_n$ be a sequence in $\Omega$ such that $x_n \to 0$. Consider an element $A \in B(\mathcal{H})$. We assert that the sequence $u_{x_n}^*A\eta \to 0$. Now observe that

$$\begin{aligned} \|u_{x_n}^*A\eta\| &= \|u_{x_n}^*A\eta - u_{x_n}^*\alpha_{x_n}(A)\eta\| \\ &\le \|u_{x_n}^*\|\,\|A\eta - \alpha_{x_n}(A)\eta\| \\ &\le \|A\eta - \alpha_{x_n}(A)\eta\| \\ &\to 0 \quad (\text{by Lemma 2.2}). \end{aligned}$$

This proves our assertion.

Now let $s \in \Omega$ be given. Set $s_n = \frac{s}{n+1}$ and $t_n = s - s_n = \frac{n}{n+1}s \in \Omega$. Note that $s_n \to 0$. Now observe that

$$\begin{aligned} u_{t_n}\eta &= u_{s_n}^*u_{s_n}u_{t_n}\eta \\ &= \overline{\omega(s_n,t_n)}u_{s_n}^*u_{s_n+t_n}\eta \\ &= \overline{\omega(s_n,t_n)}u_{s_n}^*u_s\eta \\ &\to 0 \quad (\text{by our previous assertion}). \end{aligned}$$

This is a contradiction because $\{u_{t_n}\}$ is a sequence of isometries and $\|\eta\| = \|u_{t_n}\eta\|$. This contradiction implies that the linear span of $\{\xi(a) : \xi \in \mathcal{H}, a \in \Omega^*\}$ is dense in $\mathcal{H}$. This proves our claim.

Now we show $\{u_x\}_{x\in\Omega}$ is weakly continuous. Since $\{u_x\}_{x\in\Omega}$ is norm bounded, it suffices to show that for $\xi \in \mathcal{H}$, $a \in \Omega^*$, $\eta \in \mathcal{H}$, the map $\Omega \ni x \mapsto \langle u_x\xi(a) \mid \eta\rangle \in \mathbb{C}$ is continuous.

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Thus let $\xi \in \mathcal{H}$, $a \in \Omega^*$ and $\eta \in \mathcal{H}$ be given. Let $(x_n)$ be a sequence in $\Omega$ such that $x_n \to x \in \Omega$. Observe that

$$\begin{aligned}\langle u_{x_n}\xi(a) \mid \eta\rangle &= \langle \xi(a) \mid u_{x_n}^*\eta\rangle \\ &= \int_\Omega e^{-\langle a|y\rangle}\langle u_y\xi \mid u_{x_n}^*\eta\rangle\,dy \\ &= \int_\Omega e^{-\langle a|y\rangle}\langle u_{x_n}u_y\xi \mid \eta\rangle\,dy \\ &= \int_\Omega e^{-\langle a|y\rangle}\overline{\omega(x_n,y)}\langle u_{x_n+y}\xi \mid \eta\rangle\,dy \\ &= \int_{\mathbb{R}^d} e^{-\langle a|z-x_n\rangle}\overline{\omega(x_n,z-x_n)}1_{\Omega+x_n}(z)\langle u_z\xi \mid \eta\rangle\,dz \\ &\to \int_{\mathbb{R}^d} e^{-\langle a|z-x\rangle}\overline{\omega(x,z-x)}1_{\Omega+x}(z)\langle u_z\xi \mid \eta\rangle\,dz.\end{aligned}$$

The last line in the above calculation is justified using the dominated convergence theorem. The application of the dominated convergence theorem is justified by Remark 2.3, the continuity of $\omega$, and equation (2.1). A calculation similar to the one above implies that

$$\int_{\mathbb{R}^d} e^{-\langle a|z-x\rangle}\overline{\omega(x,z-x)}1_{\Omega+x}(z)\langle u_z\xi \mid \eta\rangle\,dz = \langle u_x\xi(a) \mid \eta\rangle.$$

This completes the proof. □

We make a small digression to discuss how Proposition 2.4 can be derived from existing methods in the literature. As observed in the proof of Proposition 2.4, it suffices to consider the case when $\{v_x\}_{x\in\Omega}$ is a family of isometries.

(1) The work of Laca and Raeburn [5] suggests that the multiplier $\omega$ extends to a multiplier on $\mathbb{R}^d$. However, we cannot directly apply the results of [5] as it provides us with an extension which is Borel but not continuous. It is possible to choose a continuous extension as follows. Thanks to [1, Corollary 3.8], it follows that there exists $\psi: P \to \mathbb{T}$ such that $\psi$ is continuous and a matrix $A$ such that

$$\omega(x,y) = \psi(x)\psi(y)\overline{\psi(x+y)}e^{i\langle Ax|y\rangle}$$

for $x, y \in P$. Let $\widetilde{\psi}: \mathbb{R}^d \to \mathbb{T}$ be a continuous extension of $\psi$. Define $\widetilde{\omega}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$ by

$$\widetilde{\omega}(x,y) = \widetilde{\psi}(x)\widetilde{\psi}(y)\overline{\widetilde{\psi}(x+y)}e^{i\langle Ax|y\rangle}.$$

Then $\widetilde{\omega}$ is the desired extension. We denote $\widetilde{\omega}$ by $\omega$ itself.

(2) Note that $\{v_x\}_{x\in\Omega}$ is a $\omega$-projective isometric representation which is measurable. The proof of Theorem 2.2 of [5] allows us to "dilate" the $\omega$-projective isometric representation to a $\omega$-projective unitary representation of $\mathbb{R}^d$ which is measurable.

(3) The previous step could also be achieved as follows. Following Bargmann [3] and Mackey (see [8, Chapter 7] for a complete discussion), let

$$G_\omega := \{(x,\lambda) : x \in \mathbb{R}^d, \lambda \in \mathbb{T}\}.$$

Endow $G_\omega$ with the product topology. The set $G_\omega$ has a group structure where the group multiplication is defined by

$$(x,\lambda)(y,\mu) = \big(x+y, \omega(x,y)\lambda\mu\big).$$

The group $G_\omega$ is a locally compact topological group. The subset $\Omega \times \mathbb{T}$ is an open subsemigroup such that $(\Omega\times\mathbb{T})^{-1}(\Omega\times\mathbb{T}) = G_\omega$. In other words, $\Omega\times\mathbb{T}$ is an Ore semigroup.

For $(x,\lambda) \in \Omega\times\mathbb{T}$, let $\widetilde{v}_{(x,\lambda)} = \lambda v_x$. Then $\widetilde{v}$ is an isometric representation of $\Omega\times\mathbb{T}$ on $\mathcal{H}$ which is measurable. Let $\widetilde{u}$ be the minimal unitary dilation of $\widetilde{v}$, say, on $\mathcal{K}$. Then $\mathcal{K}$ contains $\mathcal{H}$ as a subspace and for $(x,\lambda)\in\Omega\times\mathbb{T}$, $\widetilde{u}_{(x,\lambda)}|_{\mathcal{H}} = \lambda v_x$. Moreover $\widetilde{u}$ is a weakly measurable unitary representation of the locally compact group $G_\omega$. Hence $\widetilde{u}$ is strongly continuous (see [7, Theorem 9.2.20]). Thus for $\xi\in\mathcal{H}$, the map

$$\Omega \ni x \mapsto v_x\xi = \widetilde{u}_{(x,1)}\xi \in \mathcal{H}$$

is continuous. In other words, $\{v_x\}_{x\in\Omega}$ is strongly continuous.

Despite the deductions explained above, we believe our proof is elementary and will be of independent interest.

Next we prove that a normalised unit of $E$ extends uniquely to a normalised unit of $\alpha$.

***Proposition 2.5*** *Let $\{u_x\}_{x\in\Omega} \in \mathcal{U}_E$ be a normalised unit. Then there exists a unique normalised unit $\{\widetilde{u}_x\}_{x\in P} \in \mathcal{U}_\alpha$ such that $\widetilde{u}_x = u_x$ for every $x\in\Omega$.*

**Proof** The uniqueness part follows from the fact that $\Omega$ is dense in $P$.

Let $\{u_x\}_{x\in\Omega} \in \mathcal{U}_E$ be a normalised unit. Since $\{u_x\}_{x\in\Omega}$ is a strongly continuous family of isometries, it follows that $\{u_x\}_{x\in\Omega}$ is $\sigma$-weakly continuous, *i.e.*, for every $T\in\mathcal{L}^1(\mathcal{H})$, the map $\Omega\ni x\mapsto Tr(u_xT)\in\mathbb{C}$ is continuous. In what follows, unless otherwise specified, when we speak of convergence of a sequence of operators, we mean the $\sigma$-weak convergence.

Let $x\in P$ be given and let $(x_n)$, $(y_n)$ be sequences in $\Omega$ such that $x_n\to x$ and $y_n\to x$. Suppose that $u_{x_n}\to u$ and $u_{y_n}\to v$. We claim that $u=v$. Let $s\in\Omega$ be given. Then $u_{x_n+s}\to u_{x+s}$ and $u_{y_n+s}\to u_{x+s}$. Now observe that

$$u_{x_n+s} = \omega(s,x_n)u_su_{x_n} \to \omega(s,x)u_su.$$

Hence $\omega(s,x)u_su = u_{x+s}$. Working with the sequence $(y_n)$, we obtain the equality $\omega(s,x)u_sv = u_{x+s}$. Hence $\omega(s,x)u_su = \omega(s,x)u_sv$. Since $u_s$ is an isometry, it follows that $u=v$.

Let $x\in P$ and let $(x_n)$ be a sequence in $\Omega$ such that $x_n\to x$. We claim that $(u_{x_n})$ converges. The $\sigma$-weak compactness of the unit ball of $B(\mathcal{H})$ implies that $(u_{x_n})$ has a convergent subsequence. By what we have shown in the preceding paragraph, it follows that every convergent subsequence of $(u_{x_n})$ converges to the same limit. This implies that $(u_{x_n})$ converges. Set

$$\widetilde{u}_x := \lim_{n\to\infty} u_{x_n}.$$

By what we have shown in the previous paragraph, it follows that $\widetilde{u}_x$ is well-defined. Note that $\|\widetilde{u}_x\|\le 1$. It is clear that $\widetilde{u}_x = u_x$ for $x\in\Omega$.

Let $x, y \in P$ be given. Choose sequences $(x_n)$ and $(y_n)$ in $\Omega$ such that $x_n \to x$ and $y_n \to y$. Observe that for $n, m \geq 1$,

$$u_{x_n+y_m} = \omega(x_n, y_m) u_{x_n} u_{y_m}. \tag{2.2}$$

Fixing $m$ and letting $n \to \infty$ in equation (2.2), we obtain that for every $m \geq 1$,

$$u_{x+y_m} = \omega(x, y_m) \widetilde{u}_x u_{y_m}.$$

Now letting $m \to \infty$ in the above equality, we obtain that $\widetilde{u}_{x+y} = \omega(x, y)\widetilde{u}_x \widetilde{u}_y$.

We claim that $\{\widetilde{u}_x\}_{x \in P}$ is $\sigma$-weakly continuous. Let $(x_n)$ be a sequence in $P$ such that $x_n \to x \in P$. Choose $s \in \Omega$. Then $(x_n + s)$ is a sequence in $\Omega$ which converges to $x + s \in \Omega$. Hence $u_{x_n+s} \to u_{x+s}$. Now note that

$$\begin{aligned} \widetilde{u}_{x_n} &= u_s^* u_s \widetilde{u}_{x_n} \\ &= \overline{\omega(s, x_n)} u_s^* u_{s+x_n} \to \overline{\omega(s, x)} u_s^* u_{s+x} \end{aligned}$$

Note that $\overline{\omega(s, x)} u_s^* u_{s+x} = \overline{\omega(s, x)} u_s^* \omega(s, x) u_s \widetilde{u}_x = \widetilde{u}_x$. This shows that $\widetilde{u}_{x_n} \to \widetilde{u}_x$ and proves our claim.

Next we prove that for every $x \in P$ and $A \in B(\mathcal{H})$, $\alpha_x(A)\widetilde{u}_x = \widetilde{u}_x A$. Let $x \in P$ and $A \in B(\mathcal{H})$ be given. Choose a sequence $(x_n) \in \Omega$ such that $x_n \to x$. Since $u_{x_n}$ converges $\sigma$-weakly to $\widetilde{u}_x$, it follows that $u_{x_n}$ converges weakly to $\widetilde{u}_x$. Note that

$$u_{x_n}^* \alpha_{x_n}(A^*) = A^* u_{x_n}^*.$$

Clearly $A^* u_{x_n}^*$ converges weakly to $A^* \widetilde{u}_x^*$. The sequence $u_{x_n}^*$ converges weakly to $\widetilde{u}_x$ and the sequence $\alpha_{x_n}(A^*)$ converges strongly to $\alpha_x(A^*)$ (Lemma 2.2). Moreover the sequence $u_{x_n}^*$ is norm bounded by 1. Hence the sequence $u_{x_n}^* \alpha_{x_n}(A^*)$ converges weakly to $\widetilde{u}_x^* \alpha_x(A^*)$. As a consequence, we obtain that $\widetilde{u}_x^* \alpha_x(A^*) = A^* \widetilde{u}_x^*$. Taking adjoints, we obtain $\alpha_x(A)\widetilde{u}_x = \widetilde{u}_x A$.

For $x \in P$ and $s \in \Omega$, $\widetilde{u}_x u_s = \overline{\omega(x, s)} u_{x+s}$. From this, it is immediate that $\widetilde{u}_x \neq 0$ for every $x \in P$.

For $x \in P$, let $g(x) = \widetilde{u}_x^* \widetilde{u}_x$. Then $g(x) > 0$ for every $x \in P$. One proves as in Lemma 2.3 that for $x, y \in P$, $g(x + y) = g(x)g(y)$. The $\sigma$-weak continuity of $\{\widetilde{u}_x\}_{x \in P}$ implies that $g$ is measurable. For $x \in P$, let $f(x) = \log g(x)$. Then $f$ is a measurable function on $P$ which is additive. By Lemma 2.1, it follows that there exists $\lambda \in \mathbb{R}^d$ such that $f(x) = \langle \lambda \mid x \rangle$ for $x \in P$. Hence $g(x) = e^{\langle \lambda | x \rangle}$ for $x \in P$. But observe that $g(x) = 1$ for $x \in \Omega$. Hence $\langle \lambda \mid x \rangle = 0$ for every $x \in \Omega$. Since $\Omega$ spans $\mathbb{R}^d$, this implies that $\lambda = 0$. In other words, $\{\widetilde{u}_x\}_{x \in P}$ is a family of isometries.

Now the fact that $\{\widetilde{u}_x\}_{x \in P}$ is a family of isometries and is $\sigma$-weakly continuous implies that $\{\widetilde{u}_x\}_{x \in P}$ is a strongly continuous family of isometries. This completes the proof. □

**Proof of Theorem 1.1** The injectivity of the map of Theorem 1.1 follows from the fact that $\Omega$ is dense in $P$. Now let $\{v_x\}_{x \in \Omega} \in \mathcal{U}_E$ be given. Let $\lambda \in \mathbb{R}^d$ be such that for $x \in \Omega$, $v_x^* v_x = e^{\langle \lambda | x \rangle}$. For $x \in \Omega$, set $u_x := e^{-\frac{\langle \lambda | x \rangle}{2}} v_x$. Then $\{u_x\}_{x \in \Omega} \in \mathcal{U}_E$ is normalised. Let $\{\widetilde{u}_x\}_{x \in P}$ be the extension of $\{u_x\}_{x \in \Omega}$ given by Proposition 2.5. Now for $x \in P$, set

$$\widetilde{v}_x = e^{\frac{\langle \lambda | x \rangle}{2}} \widetilde{u}_x.$$

Then $\{\widetilde{v}_x\}_{x\in P} \in \mathcal{U}_\alpha$ and $\widetilde{v}_x = v_x$ for $x \in \Omega$. This shows that the map of Theorem 1.1 is surjective and completes the proof. □

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