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CHARACTERIZATIONS OF *-MULTIPLICATION DOMAINS

BY

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ABSTRACT. Let * be a finite-type star-operation on an integral domain D. If D is integrally closed, then D is a *-multiplication domain (the *-finite *-ideals form a group) if and only if each upper to 0 in $D[x]$ contains an element f with $c(f)^* = D$. A finite-type star operation on $D[x]$ naturally induces a finite-type star operation on D, and, if each *-prime ideal P of $D[x]$ satisfies $P \cap D = 0$ or $P = (P \cap D)D[x]$, then $D[x]$ is a *-multiplication domain if and only if D is.

Introduction. Throughout this paper D will denote an integral domain, and K will denote its quotient field. We shall be concerned with finite-type star operations. A *star operation* on D is a mapping $I \to I^*$ from the set of non-zero fractional ideals of D to itself which satisfies the following axioms for each element $a \in K$ and each pair of non-zero fractional ideals I, J of D:

(i) $(a)^* = (a)$ and $(aI)^* = aI^*$,

(ii) $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$,

(iii) $I^{**} = I^*$.

The star operation * is said to be of *finite type* if each non-zero ideal I of D satisfies $I^* = \bigcup \{J^* \mid J$ is a non-zero finitely generated ideal contained in $I\}$. For the pertinent facts about star operations, we refer the reader to [2, Sections 32 and 34] or to [6].

If * is a star operation on D, then we call D a **-multiplication domain* if for each non-zero finitely generated ideal I of D there is a finitely generated fractional ideal J of D with $(IJ)^* = D$. If * is the identity star operation, then this is the definition of a Prüfer domain. At the other extreme is the well-known v-operation ($I_v = (I^{-1})^{-1}$ = intersection of the principal fractional ideals containing I), which is the coarsest star operation. The name "Prüfer v-multiplication domain" (PVMD) has been used for "v-multiplication domain" and PVMD's have been studied in [7], [6], [4], and [8]. It is easily shown that every *-multiplication domain is a PVMD.

In this paper we shall characterize PVMD's in terms of their uppers to 0 (see the definition below), an approach which not only yields new results but also offers a unified treatment of several old ones.

As far as possible, we shall state our results for general finite-type star

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operations *. In section 1 we prove our main theorem, a generalization of [3, Theorem 2]: D is a *-multiplication domain $\Leftrightarrow$ each non-zero prime ideal P of $D[x]$ with $P \cap D = 0$ contains an element f such that $c(f)^* = D$. (Here, $c(f)$ denotes the content of f, that is, the ideal generated by the coefficients of f.) This yields a straightforward proof of Griffin's result that D is a *-multiplication domain $\Leftrightarrow D_P$ is a valuation domain for each *-prime ideal P ([4, Theorem 5]). In the second section we point out that star operations on $D[x]$ induce star operations on D, and then we apply the main theorem to the v-operation.

§1. **The main theorem.** An ideal I of $D[x]$ is called an upper to 0 if I is the contraction to $D[x]$ of a nontrivial ideal of $K[x]$. A nontrivial primary ideal I of $D[x]$ is an upper to $0 \Leftrightarrow I \cap D = 0$.

THEOREM 1.1. *Let * be a finite-type star operation on D. The following statements are equivalent.*

(1) *D is a *-multiplication domain.*

(2) *For each nonzero $a, b \in D$ there is a finitely generated fractional ideal J such that $((a, b)J)^* = D$.*

(3) *D is integrally closed and for each $a, b \neq 0$ in D the prime ideal $(ax - b)K[x] \cap D[x]$ of $D[x]$ contains an element f with $c(f)^* = D$.*

(4) *D_M is a valuation domain for each ideal M maximal in the set of *-ideals.*

(5) *D is integrally closed and each prime upper to 0 in $D[x]$ contains f with $c(f)^* = D$.*

(6) *D is integrally closed and each upper to 0 in $D[x]$ contains f with $c(f)^* = D$.*

Proof. (1) $\Rightarrow$ (2) This is trivial.

(2) $\Rightarrow$ (3) Since $D = \bigcap\{D_M \mid M$ is a prime *-ideal$\}$ ([4, Proposition 4]), to prove that D is integrally closed, it is enough to show that D_M is integrally closed. Let $u \in K$ be integral over D_M. It follows from (2) that there is a finitely generated fractional ideal J of D with $((1, u)J)^* = D$. Hence $(1, u)J \not\subseteq M$ and $(1, u)JD_M = D_M$. Thus $(1, u)D_M$ is invertible and must equal $1D_M$ or uD_M. In the former case we certainly have $u \in D_M$. In the latter case, $u^{-1} \in D_M$, which, together with the equation of integrality satisfied by u, implies that $u \in D_M$ as well. Thus D is integrally closed. Now consider $gK[x] \cap D[x]$, where $g = ax - b$. By (2) we have $((a, b)A)^* = D$ with A finitely generated. Choose $k(x) \in K[x]$ with $c(k) = A$. By the content formula ([2, Theorem 28.1]) $c(g)c(k)c(g)^m = c(gk)c(g)^m$ for some integer m. Multiplying both sides by A^m and taking *'s yields $D = (c(g)c(k))^* = c(gk)^*$. Thus $f = gk$ is the desired element.

(3) $\Rightarrow$ (4) Let M be any prime *-ideal. Let a, $b \neq 0$ in D and choose $f \in P = (ax - b)K[x] \cap D[x]$ with $c(f)^* = D$. Then $c(f) \not\subseteq M$ so $P \not\subseteq MD[x]$. Thus,

by a well-known characterization of valuation domains ([2, Theorem 19.15(1) $\Leftrightarrow$ (3) and its proof]) D_M is a valuation domain.

(4) $\Rightarrow$ (5) Let $P = gK[x] \cap D[x]$. By the characterization of valuation domains just mentioned, $P \not\subseteq MD[x]$ for each ideal M maximal in the set of $*$-ideals. Thus for each such M, $c(P) \not\subseteq M$, so that $c(P)^* = D$. Since $*$ is of finite type, $1 \in J^*$ and $J^* = D$, for some finitely generated ideal $J \subseteq c(P)$. Let the generators of J be coefficients of the polynomials $g_1, \ldots, g_n \in P$. It is easy to construct a polynomial $f \in (g_1, \ldots, g_n)$ with $c(f) = c(g_1) + \cdots + c(g_n)$. Clearly, $c(f)^* = D$.

(5) $\Rightarrow$ (6) Let $I = gK[x] \cap D[x]$. Write $g = h_1 \cdots h_m$ with each h_i irreducible in $K[x]$. If $P_i = h_i K[x] \cap D[x]$ then there exists $f_i \in P_i$ such that $c(f_i)^* = D$. If $f_i = h_i k_i$ then $f \equiv f_1 \cdots f_m = g.\ k_1 \cdots k_m \in gK[x] \cap D[x] = I$. That $c(f)^* = D$ follows easily from the content formula and induction.

(6) $\Rightarrow$ (1) Let $A = c(g)$ be a finitely generated ideal of D, and choose $f \in gK[x] \cap D[x]$ with $c(f)^* = D$. If $f = gk$ then $D = c(f)^* \subseteq (c(g)c(k))^*$. We shall complete the proof by showing $c(g)c(k) \subseteq D$. By the content formula $c(g)c(k)c(g)^m = c(gk)c(g)^m \subseteq c(g)^m$ since $gk = f \in D[x]$. That $c(g)c(k) \subseteq D$ now follows from the fact that D is integrally closed.

REMARK 1.2. The integral closure hypothesis cannot be omitted in (3), (5), or (6). This is illustrated by any one-dimensional local (Noetherian) domain which is not integrally closed.

REMARK 1.3. Denote the upper to 0 in each of (3), (5), and (6) by I. Then in each case the existence of "f with $c(f)^* = D$" can be replaced by "$c(I)^* = D$." To verify this it suffices to show that $c(I)^* = D$ implies the existence of $f \in I$ with $c(f)^* = D$. This may be demonstrated as in the proof of (4) $\Rightarrow$ (5) above.

§2. **Applications of the main theorem.** Verification of the following is routine.

PROPOSITION 2.1. *Let $*$ be a star-operation on $D[x]$. Define $*$ on D by $A^* = AD[x]^* \cap D$. Then $*$ is a star operation on D. Moreover, if $*$ is of finite type on $D[x]$, then the induced $*$ is of finite type on D. In fact, $A^*D[x]^* = AD[x]^*$ for each ideal A of D.*

PROPOSITION 2.2. *Let $*$ be induced from the finite-type star operation$*$ on $D[x]$ as above. If D is a $*$-multiplication domain, then each upper to 0 in $D[x]$ is $*$-finite.*

Proof. Let $I = gK[x] \cap D[x]$ be an upper to 0. Since D is integrally closed $I = gc(g)^{-1}D[x]$ ([2, Corollary 34.9]). Pick A finitely generated with $(c(g)A)^* = D$. One easily verifies that $A^* = c(g)^{-1}$. Thus $I = gA^*D[x]$. We next claim that I is a $*$-ideal. To verify this let J be a finitely generated ideal contained in I. Then there is a nonzero element s of D with $sJ \subseteq (g)$. Hence

$sJ^* \subseteq (g)$ and $J^* \subseteq s^{-1}(g) \cap D[x] \subseteq gK[x] \cap D[x] = I$. Therefore, $I = (gA^*D[x])^* = g(A^*D[x])^* = (gAD[x])^*$, which is clearly *-finite.

PROPOSITION 2.3. *Let * be induced from the finite-type star operation* on* $D[x]$ *as above. Assume that for each prime *-ideal* P *of* $D[x]$, *either* $P = (P \cap D)D[x]$ *or* $P \cap D = 0$. *Then* D *is a *-multiplication domain* $\Leftrightarrow$ D *is integrally closed and each prime upper to* 0 *in* $D[x]$ *is a maximal *-ideal (i.e., maximal in the set of *-ideals of* $D[x]$).

Proof. Suppose that D is a *-multiplication domain, and let P be a prime upper to 0. Then P is certainly a *-ideal. If P is not maximal in the set of *-ideals of $D[x]$, then by hypothesis $P \subseteq QD[x]$ for some prime *-ideal Q of D. Thus $c(P) \subseteq Q$. However, by Theorem 1.1 $c(P)^* = D$, a contradiction.

Conversely, assume that each prime upper P to 0 in $D[x]$ is a maximal *-ideal. We shall verify that $c(P)^* = D$, which will complete the proof by Theorem 1.1 and Remark 1.3. If $c(P)^* \neq D$ then $c(P)$ is contained in a prime *-ideal Q of D. This implies that $P \subseteq QD[x]^*$, a contradiction.

PROPOSITION 2.4. *Under the hypotheses of Proposition* 2.3 D *is a *-multiplication domain* $\Leftrightarrow$ $D[x]$ *is.*

Proof. Assume that D is a *-multiplication domain. Let P be a *-prime of $D[x]$; we shall show that $D[x]_P$ is a valuation domain. We distinguish two cases: $P \cap D = 0$ and $P = (P \cap D)D[x]$. In the first case it is well known that $D[x]_P$ is a valuation domain. In the second case $D[x]_P = V[x]_{(P \cap D)V[x]}$, where $V = D_{P \cap D}$ is a valuation domain by Theorem 1.1. Thus $D[x]_P$ is a valuation domain in this case as well.

For the converse let Q be a *-prime of D. Then $QD[x]^* \cap D = Q^* = Q$, and there is a prime ideal I of $D[x]$ with $I \cap D = Q$ and $QD[x]^* \subseteq I$. Now any prime minimal over $QD[x]^*$ is a *-prime ([5, Proposition 1.1 (5)], and by hypothesis $I = (I \cap D)D[x] = QD[x]$. Thus $QD[x]^* = QD[x]$. Since $D[x]_{QD[x]}$ is a valuation domain, D_Q must be a valuation domain also.

We now examine Propositions 2.3 and 2.4 in the context of the v- and t-operations. The t-operation is defined as follows: for a non-zero ideal I of D, $I_t = \cup\{J_v \mid J \subseteq I$ is a finitely generated ideal]. Thus the t-operation is of finite type. By [5, Proposition 4.3], the v- and t-operations on $D[x]$ induce, respectively, the v- and t-operations on D. Hence Proposition 2.2 is valid for these operations.

LEMMA 2.5. *If* D *is integrally closed, then for each prime* t*-ideal* P *of* $D[x]$, *either* $P = (P \cap D)D[x]$ *or* $P \cap D = 0$.

Proof. Assume $P \cap D \neq 0$ and pick a nonzero element $a \in P \cap D$. Let $f \in P$ and let $k \in (a, f)^{-1}$. Since $a \in D$, $k \in K[x]$. Since D is integrally closed $c(fk)_v = (c(f)c(k))_v$ ([2, Proposition 34.8]), whence $c(f)c(k) \subseteq D$. Thus $c(f)(a, f)^{-1} \subseteq$

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$D[x]$ and $c(f) \subseteq (a, f)_v \subseteq P$. It follows that $f \in (P \cap D)D[x]$, and the proof is complete.

REMARK. An alternate proof can be based on [9, Lemme 2].

Theorem 1.1, Propositions 2.2, 2.3, 2.4 and Lemma 2.5 can be combined to give the following.

PROPOSITION 2.6. *The following statements are equivalent for an integrally closed domain* D:

(1) D *is a PVMD.*

(2) *Each upper* I *to* 0 *in* $D[x]$ *is* v*-finite with* $c(I)_v = D$.

(3) *Each upper* $I = gK[x] \cap D[x]$ *satisfies* $I = (g, f)_v$ *for some* $f \in I$, *and* $c(I)_v = D$.

(4) *Each prime upper to* 0 *in* $D[x]$ *is a maximal* t*-ideal of* $D[x]$.

(5) $D[x]$ *is a PVMD.*

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