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MUTUAL INJECTIVE HULLS

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ABSTRACT. The mutual injective hull of an arbitrary family of modules is constructed. Applications to the calculations of quasi-injective and π-injective hulls of direct sums are given.

1. **Introduction.** Consider the direct sum $M = \oplus \sum_{i \in I} M_i$ of a family of right modules $\{M_i \mid i \in I\}$. Assume, for simplicity, that I is finite or R is right noetherian. If each M_i is injective, then M is injective. However, if each M_i is merely quasi-injective (or π-injective) it does not necessarily follow that M is quasi-injective (resp. π-injective). In either case, M is quasi-injective (π-injective) if and only if the M_i's are "mutually injective" in the sense that whenever $j \neq k$, M_j is M_k-injective. If the M_i's are arbitrary, the question then arises to determine formulae for the quasi-injective hull $q(M)$ and the π-injective hull $\pi(M)$ of M as a direct sum of essential extensions of the M_i's. These essential extensions should then be quasi-injective (π-injective) and mutually injective. We start by focusing on the problem of producing, for an arbitrary family of modules $\{N_j \mid j \in J\}$, over an arbitrary ring R a family of "smallest" essential extensions $\{\tilde{N}_j \mid j \in J\}$ which are mutually injective, namely a *mutual injective hull* of $\{N_j \mid j \in J\}$.

Theorem 3.1 proves the existence and Theorem 3.2 provides formulae for the mutual injective hulls of arbitrary families of modules. The construction of mutual injective hulls allows us then to show that if I is finite or R is right noetherian

$$q(M) = \oplus \sum_{i \in I} q(\tilde{M}_i) = \oplus \sum_{i \in I} \widetilde{q(M_i)}$$

and

$$\pi(M) = \oplus \sum_{i \in I} \pi(\tilde{M}_i) = \oplus \sum_{i \in I} \widetilde{\pi(M_i)}.$$

2. **Definitions and preliminaries.** A family of right modules $\{N_j \mid j \in J\}$ over an arbitrary ring R is said to be *mutually injective* if, for all $j \neq k$ in J, N_j is N_k-injective. Given a family $\{N_j \mid j \in J\}$ with corresponding injective hulls $\{E(N_j) \mid j \in J\}$, a family $\{\tilde{N}_j \mid j \in J\}$ is said to be a *mutual injective hull* for $\{N_j \mid j \in J\}$ (in $\{E(N_j) \mid j \in J\}$) if

(i) for all $j \in J$, $N_j \subseteq \tilde{N}_j \subseteq E(N_j)$,

(ii) $\{\tilde{N}_k \mid k \in J\}$ is mutually injective, and

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(iii) for any mutually injective family $\{N'_k \mid k \in J\}$ if for all $j \in J$, $N_j \subseteq N'_j \subseteq E(N_j)$, then for all $j \in J$, $\tilde{N}_j \subseteq N'_j$.

Throughout this paper, all modules are right and unital unless otherwise stated. For an R-module M, the symbols $E(M)$ (or $\hat{M}$), $q(M)$ and $\pi(M)$ will denote, respectively, the injective hull, quasi-injective hull and π-injective hull of M. An R-module M is injective relative to an R-module N (or N-injective) if

$$\hom_R\big(N, E(M)\big)N \subseteq M.$$

M is quasi-injective if M is injective relative to itself, *i.e.*, M is invariant under $\mathrm{End}\big(E(M)\big)$. M is π-injective if for any two submodules A, $B \subset M$ with $A \cap B = 0$, each of the projection maps $A \oplus B \longrightarrow A$ and $A \oplus B \longrightarrow B$ can be extended to an endomorphism of M. Equivalently, M is π-injective iff for each idempotent $e \in \mathrm{End}\big(E(M)\big)$, $eM \subset M$. It is well known that a module is π-injective if and only if it is quasi-continuous. For any two modules M_1 and M_2, $M_1 \subset' M_2$ will denote that M_1 is essential in M_2.

We record below some well-known lemmas which are used throughout the paper.

LEMMA 2.1. *The quasi-injective hull of M is given by $q(M) = \mathrm{End}\big(E(M)\big)M$ and the π-injective hull of M is given by $\pi(M) = SM$, where S is the subring generated by the idempotent elements of* $\mathrm{End}\big(E(M)\big)$.

LEMMA 2.2. *M is π-injective if and only if $E(M) = \oplus\sum_{i\in I} M_i$ implies $M = \oplus\sum(M_i \cap M)$.*

PROOF. See ([3], Theorem 1.1).

LEMMA 2.3. *(a) If M is N-injective, then M is both K- and (N/K)-injective for each submodule K of N.*

(b) If M is N_i-injective, $i \in I$, then M is $\oplus\sum_{i\in I} N_i$-injective.

(c) If M is N-injective and $M \subseteq' M'$ is N-injective, then M' is N-injective.

(d) If M is N-injective, then M is SN-injective for any subring S of $\mathrm{End}\big(E(N)\big)$. *In particular, M is $\pi(N)$- (and $q(N)$-) injective.*

PROOF. It is straightforward. (See, for example, ([1] Proposition 16.13, p. 188).

LEMMA 2.4. *Let M and N be R-modules. Let $M_1 = \hom_R\big(E(N), E(M)\big)N + M$. Then:*

(i) M_1 is N-injective;

(ii) if M' is any other R-module such that $M \subset M' \subset E(M)$ and M' is N-injective, then $M_1 \subset M'$.

PROOF. Follows immediately from the definition of N-injectivity.

We call M_1 (in Lemma 2.4) the *N-injective hull of M* (in $E(M)$), and denote it by $E_N(M)$.

LEMMA 2.5. *If $M = \oplus\sum_{i\in I} M_i$ is π-injective, then M_i is π-injective and $\{M_i \mid i \in J\}$ is mutually injective. In case I is finite or R is noetherian, the converse also holds.*

PROOF. See [4].

3. Main results.

THEOREM 3.1. *(Existence). Every family of modules $\{N_j \mid j \in J\}$ over an arbitrary ring R with corresponding injective hulls $\{E(N_j) \mid j \in J\}$ has a mutual injective hull.*

PROOF. Using the relative injective hulls introduced in Lemma 2.4, we define the sequence

$$N_i^{(0)} = N_i,$$
$$N_i^{(k)} = E_{\sum_{j\neq i} N_j^{(k-1)}}(N_i^{(k-1)}), \qquad k = 1, 2, 3, \ldots.$$

Let $\tilde{N}_i = \cup N_i^{(k)}$. We show $\tilde{N}_i$ is $\tilde{N}_j$-injective for $i \neq j$. So, let $f: \tilde{N}_j \longrightarrow E(\tilde{N}_i)$ be an R-homomorphism. By our construction, $N_i^{(k)}$ is $\sum_{j\neq i} N_j^{(k-1)}$-injective hence $N_j^{(k-1)}$-injective for $j \neq i$. Also, $E(\tilde{N}_i) = E(N_i)$. So $f(\tilde{N}_j) \subset \cup_k N_i^{(k)} = \tilde{N}_i, j \neq i$. Thus $\tilde{N}_i$ is $\tilde{N}_j$-injective.

Let $\{N_i'\}$ be a mutually injective family such that $N_i \subset N_i' \subset \hat{N}_i$, for all i. We show that $\tilde{N}_i \subset N_i'$ for all i. Now $N_i^{(1)}$ is relative $\sum_{j\neq i} N_j$-injective hull of N_i in $E(N_i)$, and since N_i' is also $\sum_{j\neq i} N_j$-injective, it follows that

$$N_i^{(1)} \subset N_i', \text{ for all } i.$$

So N_i' is $\sum_{j\neq i} N_j^{(1)}$-injective, yielding

$$N_i^{(2)} \subset N_i'.$$

Proceeding like this, we obtain $N_i^{(k)} \subset N_i'$ $\forall k$; hence $\tilde{N}_i \subset N_i'$ as desired.

The family $(\tilde{N}_i)_{i\in I}$ of R-modules obtained in the above theorem will be called the *mutual injective hull of the family* $(N_i)_{i\in I}$ in $\big(E(N_i)\big)_{i\in I}$.

The next theorem shows that the sequence of R-modules $\{N_i^{(k)}\}$, $k = 1, 2, \ldots$ constructed above to obtain $\tilde{N}_i$ indeed becomes stationary after the second step, *i.e.*, $N_i^{(2)} = N_i^{(k)}$ for all $k \geq 2$.

THEOREM 3.2. *With the notation as in the proof of Theorem 3.1, $\tilde{N}_i = N_i^{(2)}$. Indeed,*

$$\tilde{N}_i = N_i + \sum_{j\neq i} \hom(\hat{N}_j, \hat{N}_i) N_j + \sum_{j\neq i} \hom(\hat{N}_j, \hat{N}_i) \hom(\hat{N}_i, \hat{N}_j) N_i,$$

where $\hat{N} = E(N)$.

PROOF. For simplicity we will consider a family with only two modules N_1 and N_2. The general case works similarly.

$$N_1^{(1)} = \hom(\hat{N}_2, \hat{N}_1) N_2 + N_1$$
$$N_1^{(2)} = \hom(\hat{N}_2, \hat{N}_1) N_2^{(1)} + N_1^{(1)}$$

as exhibited in the towers:

$$\begin{array}{ll} N_1^{(2)} = \hom(\hat{N}_2, \hat{N}_1) N_2^{(1)} + N_1^{(1)} & N_2^{(2)} = \hom(\hat{N}_1, \hat{N}_2) N_1^{(1)} + N_2^{(1)} \\ N_1^{(1)} = \hom(\hat{N}_2, \hat{N}_1) N_2^{(0)} + N_1^{(0)} & N_2^{(1)} = \hom(\hat{N}_1, \hat{N}_2) N_1^{(0)} + N_2^{(0)} \\ N_1^{(0)} = N_1 & N_2^{(0)} = N_2 \end{array}$$

Thus

$$N_1^{(2)} = \hom(\hat{N}_2,\hat{N}_1)[\hom(\hat{N}_1,\hat{N}_2)N_1^{(0)} + N_2^{(0)}] + \hom(\hat{N}_2,\hat{N}_1)N_2^{(0)} + N_1^{(0)}$$
$$= \hom(\hat{N}_2,\hat{N}_1)\hom(\hat{N}_1,\hat{N}_2)N_1 + \hom(\hat{N}_2,\hat{N}_1)N_2 + N_1.$$

Next,

$$N_1^{(3)} = \hom(\hat{N}_2,\hat{N}_1)N_2^{(2)} + N_1^{(2)}$$

where

$$N_2^{(2)} = \hom(\hat{N}_1,\hat{N}_2)N_1^{(1)} + N_2^{(1)},$$

hence

$$N_2^{(2)} = \hom(\hat{N}_1,\hat{N}_2)\hom(\hat{N}_2,\hat{N}_1)N_2 + \hom(\hat{N}_1,\hat{N}_2)N_1 + N_2.$$

Now

$$\hom(\hat{N}_2,\hat{N}_1)N_2^{(2)} = \hom(\hat{N}_2,\hat{N}_1)N_2 + \hom(\hat{N}_2,\hat{N}_1)\hom(\hat{N}_1,\hat{N}_2)N_1.$$

Adding $N_1^{(2)}$ to both sides, we obtain

$$N_1^{(3)} = N_1^{(2)} + \hom(\hat{N}_2,\hat{N}_1)N_2 + \cdots,$$
$$= N_1 + \hom(\hat{N}_2,\hat{N}_1)N_2 + \hom(\hat{N}_2,\hat{N}_1)\hom(\hat{N}_1,\hat{N}_2)N_1 = N_1^{(2)}.$$

This yields by induction that $N_i^{(k)} = N_i^{(2)}\ \forall k \geq 2$, and hence $\tilde{N}_i = N_i^{(2)}$.

EXAMPLE 3.3. *It is possible to have a family $\{N_j \mid j \in I\}$ where $\tilde{N}_j \neq N_j^{(1)}$.*

PROOF. Let V be an infinite dimensional left vector space over a field F. Let $R = \mathrm{End}_F(V)$, and $S = \mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$. It is well known that ${}_RR$ is injective but R_R is not injective, and S_R is essential in R_R. Set $M_1 = R$, $M_2 = S$. Then $M_1^{(1)} = R$, $M_2^{(1)} = E(R)$, and $\tilde{M}_1 = E(R) = \tilde{M}_2$.

4. **Applications to quasi-injective and π-injective hulls.** Next we return to our consideration of a family of right R-modules $\{M_i \mid i \in I\}$ where either I is finite or R is right noetherian and their direct sum $M = \oplus\sum_{i\in I} M_i$. The mutual injective hulls discussed above in Theorems 3.1 and 3.2 allow us to develop formulae for $q(M)$ and $\pi(M)$. But first,

LEMMA 4.1. *Let $\{N_1 \mid i \in I\}$ be a family of R-modules and f an endomorphism of $E(N_i)$. Then*

$$f(\tilde{N}_i) \subseteq \tilde{N}_i + f(N_i).$$

PROOF. Notice that $N_i^{(k)} = \hom(\sum_{j\neq i} N_j^{(k-1)},\hat{N}_i)\sum_{j\neq i} N_j^{(k-1)} + N_i$. Let $g \in \hom(\sum_{j\neq i} N_j^{(k-1)},\hat{N}_i)$. Then $fg \in \hom(\sum_{j\neq i} N_j^{(k-1)},\hat{N}_i)$. Hence $f(N_i^{(k)}) \subseteq N_i^{(k)} + f(N_i)$. Now the proof follows by the definition of $\tilde{N}_i$.

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COROLLARY 4.2. *If N_i is quasi or π-injective, then so is $\tilde{N}_i$.*

THEOREM 4.3. *Let $\{M_i \mid i \in I\}$ be a family of R-modules where either I is finite or R is right noetherian. Then*

$$\pi\Big(\oplus\sum_{i\in I} M_i\Big) = \oplus\sum_{i\in I}\pi(\tilde{M}_i) = \oplus\sum_{i\in I}\widetilde{\pi(M_i)}.$$

PROOF. Set $\pi = \pi(\oplus\sum_{i\in I} M_i)$. Then $E(\pi) = \oplus\sum_{i\in I} E(M_i)$, and so

$$\pi = \oplus\sum_{i\in I} E(M_i)\cap\pi,$$

which yields $\{E(M_i)\cap\pi \mid i\in I\}$ is mutually injective. Let $\{\tilde{M}_i \mid i \in I\}$ be the mutual injective hull of $\{M_i\}$ in $\{E(M_i)\}$. Then

$$\tilde{M}_i \subset \pi\cap E(M_i) \text{ for all } i.$$

Hence, $\pi(\tilde{M}_i) \subset \pi$, $\forall i$. Thus $\oplus\sum\pi(\tilde{M}_i) \subset \pi$. But $\{\pi(\tilde{M}_i) \mid i\in I\}$ is mutually injective (Lemma 2.3). Hence, $\sum\pi(\tilde{M}_i)$ is π-injective and equals π.

We next show the equality $\pi = \oplus\sum_{i\in I}\widetilde{\pi(M_i)}$. Since $\{\pi(\tilde{M}_i) \mid i\in I\}$ is a mutually injective family and for all $i \in I$, $\pi M_i \subseteq \pi\tilde{M}_i$, we have by definition of mutual injective hull of this family $\widetilde{\pi(M_i)} \subseteq \pi(\tilde{M}_i)$, and so

$$\oplus\sum\widetilde{\pi(M_i)} \subseteq \oplus\sum\pi(\tilde{M}_i). \tag{1}$$

Next,

$$\begin{aligned}\oplus\sum_{i\in I}\pi\tilde{M}_i = \pi\Big(\oplus\sum_{i\in I} M_i\Big) &\subseteq \pi\Big(\oplus\sum_{i\in I}\pi M_i\Big)\\ &= \oplus\sum\pi(\widetilde{\pi M_i}) = \oplus\sum\widetilde{\pi(M_i)},\end{aligned} \tag{2}$$

by Corollary 4.2. Then by (1) and (2), $\pi = \oplus\sum\widetilde{\pi M_i}$, as claimed.

A similar result for quasi-injective hulls is contained in

THEOREM 4.4. *If I is finite or R is right noetherian, then*

$$q\Big(\oplus\sum_{i\in I} M_i\Big) = \oplus\sum_{i\in I} q(\tilde{M}_i) = \oplus\sum_{i\in I}\widetilde{q(M_i)}.$$

The first equality can actually be strengthened to obtain

$$q\Big(\oplus\sum_{i\in I} M_i\Big) = \oplus\sum_{i\in I} q(M_i^{(1)}).$$

PROOF. The proof of the first two equalities is identical to the proof of Theorem 4.3. For the latter equality, since

$$M_i^{(1)} = M_i + \sum_{j\neq i}\hom(\hat{M}_j,\hat{M}_i)\hom(\hat{M}_i,\hat{M}_j)M_i + \sum_{j\neq i}\hom(\hat{M}_j,\hat{M}_i)M_j,$$

we get

$$qM_i^{(1)} = qM_i + \sum_{j\neq i}\hom(\hat{M}_j,\hat{M}_i)\hom(\hat{M}_i,\hat{M}_j)M_i + \sum_{j\neq i}\hom(\hat{M}_j,\hat{M}_i)M_j,$$

which is clearly M_j-injective for all $j \neq i$, proving our claim.

EXAMPLE 4.5. *There exist modules M_1, M_2 such that $\pi(M_1 \oplus M_2) \neq \pi(M_1^{(1)}) \oplus \pi(M_2^{(1)})$.*

PROOF. Let $M_1 = R$ be a non-left artinian ring with right composition length 2 and essential simple right socle $S = M_2$ ([2], p. 337). Then $M_1^{(1)} = R$, $M_2^{(1)} = E(R)$. So, $\pi(M_1^{(1)}) \oplus \pi(M_2^{(1)}) = R \oplus E(R) \neq \pi(M_1 \oplus M_2) = E(R) \oplus E(R)$.

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