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ANALYTIC CONTINUATION OF POWER SERIES BY REGULAR GENERALIZED WEIGHTED MEANS

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ABSTRACT. The behaviour of summability transforms of power series outside their circles of convergence has been studied by many authors. In the case of the geometric series Luh [6] and Tomm [10] showed that there exist regular methods A which provide an analytic continuation into any given simply connected region G that contains the unit disc but not the point 1. Moreover, the A-transforms of the geometric series may be required to converge to any chosen analytic function on prescribed regions outside the unit circle. In this paper, these results are extended to power series representing other meromorphic functions. It is also shown that the summability methods involved may be chosen to be generalized weighted means previously introduced by Faulstich [1].

1. **Introduction.** Many authors have studied the behaviour of summability transforms of power series outside their circles of convergence. See, for example, [2], [3], [4], [5], [7], [8], [9], [11], [13] and [14]. Naturally, the idea of possible analytic continuation by summability methods has been a strong motivation for these studies, and so one might ask the following questions:

Into what regions can a given power series have an analytic continuation through regular matrix transforms? And is it possible that, on sets outside the circle of convergence, the same transforms converge to functions other than the one which is represented by the given power series?

A surprisingly general and positive answer to this question was given by Luh (Theorem 5.1 in [6]) in the case of the geometric series. He showed that for every simply connected region G which contains the open unit disc but not the point 1, there is a matrix A such that the A-transforms of the geometric series converge to $1/(1-z)$ on G. In fact, the A-transforms may additionally be required to converge to a large class of functions on a collection of sets outside G. Although the methods A which are constructed in [6] are not in general regular, Luh's result suggested that it might also hold with regular A. (This was proved in [10].) It was shown by Faulstich [1] that A may be chosen to be a

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"generalized weighted means method":

DEFINITION 1. Suppose that $(d_k)_{k=0}^{\infty}$ is a sequence of complex numbers and that $D_n=\sum_{k=0}^{n} d_k$ for $n=0,1,\ldots$. If $(l_n)_{n=0}^{\infty}$ is a strictly increasing sequence of non-negative integers satisfying $D_{l_n}\neq 0$ for $n=0,1,\ldots$ then the matrix $(d_{n,k})$ defined by

$$d_{n,k}=\begin{cases} d_k/D_{l_n} & \text{for} \quad 0\le k\le l_n \\ 0 & \text{for} \quad k>l_n \end{cases}$$

is called a *generalized weighted means method.*

It is easy to show (cf. [1]) that $(d_{n,k})$ is regular if and only if the following two conditions hold:

$$\lim_{n\to\infty} D_{l_n}=\infty, \tag{1}$$

$$\sup_{n\ge 0} |D_{l_n}|^{-1}\cdot \sum_{k=0}^{l_n} |d_k|<\infty. \tag{2}$$

Like the methods A obtained in [6] the matrices $(d_{n,k})$ constructed in [1] are not in general regular. In this paper, the results of Luh and Faulstich will be improved (see Theorem 1) in two ways. We construct regular generalized weighted means methods. Also, our results hold not only for the geometric series but for a whole class of power series.

NOTATIONS. For every set $S\subset\mathbb{C}$, let $\mathring{S}$ denote the interior and $\bar{S}$ the closure of S. A sequence of functions (f_n) will be called compactly convergent to a function f on S if it converges to f on every compact subset of S. Also, we shall use the following abbreviations throughout:

$$\Delta_r=\{z\in\mathbb{C}:|z|\le r\} \quad \text{for} \quad r\ge 0, \qquad \Delta=\Delta_1.$$

If K is a compact set then $A(K)$ denotes the Banach space of all functions which are continuous on K and holomorphic on $\mathring{K}$.

2. **Statement of the results.** First, we describe the class of functions appropriate to our results. Throughout this paper, let f denote a function which is meromorphic in $\mathbb{C}$ and has a Taylor series expansion

$$f(z)=\sum_{m=0}^{\infty} c_m z^m \tag{3}$$

whose radius of convergence, r, is positive. We also require that the coefficients c_m satisfy the condition

$$\lim_{m\to\infty} R^m c_m=\infty \quad \text{for every} \quad R>r. \tag{C}$$

It was proved in [12] that (C) holds if
(C′) f has exactly one pole on the circle $|z|=r$, or if

$$(\tilde{C})\qquad \lim_{m\to\infty} R^m \tilde{c}_m = \infty \quad \text{for every} \quad R > r$$

where $\tilde{f}(z)=\sum_{m=0}^{\infty}\tilde{c}_m z^m$ is any rational function which has the same poles and the same singular parts as f has on Δ_r.

If $D=(d_{n,k})$ is a row-finite summability method (every generalized weighted means method is row-finite) then (σ_n^D) will denote the sequence of D-transforms of the series in (3), that is

$$(4)\qquad \sigma_n^D(z)=\sum_{k=0}^{\infty} d_{n,k}\cdot\sum_{m=0}^{k} c_m z^m \quad \text{for all} \quad z\in\mathbb{C} \quad \text{and} \quad n=0,1,\ldots.$$

Our generalization of Theorem 5.1 in [6] is

THEOREM 1. *Suppose that G_0 is a simply connected region that contains $\mathring{\Delta}_r$ but no pole of f. Also suppose that at most countably many*

simply connected regions $G_1, G_2, \ldots,$

Jordan arcs $J_1, J_2, \ldots,$

point sets $\{z_1\}, \{z_2\}, \ldots$

are given, and that these sets are disjoint and contain no pole of f nor any point of $\Delta_r \cup G_0$. Furthermore, let $f_0(z)=f(z)$ and, for $\nu\ge 1$, let

$f_\nu(z)$ be holomorphic on G_ν,

$g_\nu(z)$ be continuous on J_ν,

w_ν be any complex number.

Then there exists a regular generalized weighted means method D for which the following statements hold:

$$(5)\qquad \lim_{n\to\infty}\sigma_n^D(z)=f_\nu(z) \text{ compactly on } G_\nu, \text{ for } \nu=0,1,\ldots,$$

$$(6)\qquad \lim_{n\to\infty}\sigma_n^D(z)=g_\nu(z) \text{ uniformly on } J_\nu, \text{ for } \nu=1,2,\ldots,$$

$$(7)\qquad \lim_{n\to\infty}\sigma_n^D(z)=w_\nu \text{ for } \nu=1,2,\ldots.$$

The theorem can be extended to include the condition that $(\sigma_n^D(z))$ diverges for $z\notin \overline{\Delta_r\cup\bigcup_{\nu\ge 0}G_\nu\cup\bigcup_{\nu\ge 1}(J_\nu\cup\{z_\nu\})}$, as is done in [6]. However, we have decided to omit this statement for the sake of a simpler exposition. The proof of Theorem 1 follows in Section 3.

Some examples of functions f, appropriate to Theorem 1 (and the following Theorem 2), are:

(a) $f(z)=(a+bz)/(1-z^2)$ with $a\neq 0$ and $b\neq 0$. (Note that $f(z)=1/(1-z)$ if $a=b=1$. Hence Theorem 1 applies to the geometric series.)

(b) $f(z)=1/\zeta(z)$ where ζ is the Riemann zeta function.

(c) $f(z)=\Gamma(z+1)$

(d) $f(z)=\sec\sqrt{z}$ ($\sqrt{z}$ is the principal branch of $\sqrt{z}$).

In example (a) condition ($\tilde{C}$) holds with $\tilde{f}=f$. And in each of the examples (b), (c), (d) the function f has exactly one pole on the circle of convergence. It is interesting to note that, if $a=0$ (resp. $b=0$) in example (a), then Theorem 1 is not true in general, because in that case f is an odd (resp. even) function, implying that every transform σ_n^D is odd (resp. even), and so we cannot choose, e.g., $f_1(z)=1$ on $G_1=\{z\in\mathbb{C}\colon \operatorname{Re} z>1\}$ and $f_2(z)=0$ on $G_2=\{z\in\mathbb{C}:\operatorname{Re} z<-1\}$.

THEOREM 2. *Let P be the set of all poles of f. Then there exists a regular generalized weighted means method D such that $\lim_{n\to\infty}\sigma_n^D(z)=f(z)$ for all $z\in\mathbb{C}\setminus P$. Moreover, if G is a simply connected region, disjoint to P, and if the set $S=\mathbb{C}\setminus(G\cup P)$ has an empty interior and contains no point of Δ_r, then D can be chosen such that*

$$\lim_{n\to\infty}\sigma_n^D(z)=f(z) \text{ compactly on } G, \tag{8}$$

$$\lim_{n\to\infty}\sigma_n^D(z)=f(z) \text{ compactly on } S. \tag{9}$$

3. **Proof of the Theorems 1 and 2.** The entries d_k/D_{l_n} of the matrices D in our theorems will be constructed from the coefficients of a sequence of polynomials (p_n) whose existence is guaranteed by the following result in [12].

THEOREM A. *Let G be a simply connected region which contains $\mathring{\Delta}_r$ but contains no pole of f (where r and f are as described at the beginning of Section 2). Suppose that K is a compact subset of G and that L is a compact set which does not separate the plane and contains no pole of f nor any point of $G\cup\Delta_r$. Then, for every map $g\in A(L)$ and for every $\varepsilon>0$, there exists a polynomial $p(z)=\sum_{k=0}^N a_kz^k$ with the following five properties:*

$$|a_k|<\varepsilon \quad \text{for} \quad k=0,\ldots,N, \tag{α}$$

$$p(1)=1, \tag{β}$$

$$\sum_{k=0}^N |a_k|<1+\varepsilon, \tag{γ}$$

$$\left|\sum_{k=0}^N a_k \sum_{m=0}^k c_mz^m-f(z)\right|<\varepsilon \quad \text{for all} \quad z\in K, \tag{δ}$$

$$\left|\sum_{k=0}^N a_k \sum_{m=0}^k c_mz^m-g(z)\right|<\varepsilon \quad \text{for all} \quad z\in L. \tag{ε}$$

Proof of Theorem 1. For notational ease, set $G_\nu=\varnothing$, $J_\nu=\varnothing$ and $\{z_\nu\}=\varnothing$

wherever the index ν exceeds the number of sets given. For every pair of indices (n, ν) $(n \geq 0, \nu \geq 0)$, let $K_{n,\nu}$ be the set of all points in G_ν whose moduli do not exceed $n+1$ and whose distance to the complement of G_ν is at least $1/(n+1)$. Also define

$$L_n = \bigcup_{\nu=1}^{n+1} (K_{n,\nu} \cup J_\nu \cup \{z_\nu\}) \quad \text{for} \quad n = 0, 1, \ldots.$$

It is easy to verify that, if $n \in \mathbb{N}_0$ is fixed, then $K_{n,\nu}, J_\nu, \{z_\nu\}$ $(\nu \leq n+1)$ are disjoint compact sets with connected complements. Therefore the L_n are compact sets which do not separate the plane. And it is also easy to verify that no L_n contains a pole of f nor any point of $\Delta_r \cup G_0$. Hence the topological hypotheses of Theorem A are satisfied by every pair $K_{n,0}$, L_n.

Consider the map g defined by

$$g(z) = \begin{cases} f_\nu(z) & \text{for} \quad z \in G_\nu \ (\nu \geq 0) \\ g_\nu(z) & \text{for} \quad z \in J_\nu \ (\nu \geq 1) \\ w_\nu & \text{for} \quad z = z_\nu \ (\nu \geq 1). \end{cases}$$

For every $n \geq 0$, the restriction of g to L_n belongs to $A(L_n)$. Applying Theorem A with $K = K_{n,0}$, $L = L_n$ and $\varepsilon = \varepsilon_n > 0$, we obtain polynomials $p_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$ (say $a_{n,k} = 0$ for $k > l_n$) such that the following conditions hold for every $n \geq 0$:

(11) $$|a_{n,k}| < \varepsilon_n \quad \text{for} \quad k = 0, 1, \ldots,$$

(12) $$\sum_{k=0}^{l_n} a_{n,k} = 1,$$

(13) $$\sum_{k=0}^{l_n} |a_{n,k}| < 1 + \varepsilon_n,$$

(14) $$\left| \sum_{k=0}^{l_n} a_{n,k} \sum_{m=0}^{k} c_m z^m - g(z) \right| < \varepsilon_n \quad \text{for all} \quad z \in L_n \cup K_{n,0}.$$

(For the last condition note that $g = f_0 = f$ on G_0.)

Let us define the positive numbers ε_n inductively by setting $\varepsilon_0 = 1$, and

(15) $$\varepsilon_n = 1/(2^n M_{n-1}(l_{n-1}+1)) \quad \text{for} \quad n \geq 1,$$

where

$$M_n = 1 + \max\left\{ \sum_{m=0}^{l_n} |c_m z^m| : z \in \{0\} \cup L_n \cup K_{n,0} \right\}.$$

Since (11) and (15) imply that $|a_{n,k}| < 2^{-n}$, we obtain a well-defined sequence

(d_k) by setting

(16) $$d_k = \sum_{\mu=0}^{\infty} a_{\mu,k} \quad \text{for} \quad k = 0, 1, \ldots.$$

The sequence $(l_n)_{n=0}^{\infty}$ is strictly increasing since, by (11) and (15), we have for $n \geq 1$:

$$\sum_{k=0}^{l_{n-1}} |a_{n,k}| < \sum_{k=0}^{l_{n-1}} \frac{1}{l_{n-1}+1} = 1.$$

Thus l_n must be greater than l_{n-1} since (12) cannot hold otherwise.

Let us now show that a regular generalized weighted means matrix $D = (d_{n,k})$ is defined by setting

$$d_{n,k} = \begin{cases} d_k/D_{l_n} & \text{for} \quad 0 \leq k \leq l_n \\ 0 & \text{for} \quad k > l_n, \end{cases}$$

where $D_n = d_0 + \cdots + d_n$.

It follows from (16) that $D_{l_n} = \sum_{\mu=0}^{\infty} \sum_{k=0}^{l_n} a_{\mu,k}$, and (15) and (11) imply that

$$\sum_{\mu>n} \sum_{k=0}^{l_n} |a_{\mu,k}| < \sum_{\mu>n} \frac{l_n+1}{2^{\mu}(l_{\mu-1}+1)} \leq \sum_{\mu>n} 2^{-\mu} \leq 1.$$

Thus we have that

$$\left| D_{l_n} - \sum_{\mu=0}^{n} \sum_{k=0}^{l_n} a_{\mu,k} \right| < 1 \quad \text{for} \quad n = 0, 1, \ldots$$

and also that

(17) $$\sum_{\mu>n} \sum_{k=0}^{l_n} |a_{\mu,k}| < 1 \quad \text{for} \quad n = 0, 1, \ldots.$$

In the first one of these two inequalities the sums $\sum_{k=0}^{l_n} a_{\mu,k}$ are equal to 1 (because of (12) and since $l_n \geq l_\mu$ for $\mu \leq n$), and so we obtain

(18) $$|D_{l_n} - (n+1)| < 1 \quad \text{for} \quad n = 0, 1, \ldots.$$

This estimate shows that $D_{l_n} \neq 0$ for $n = 0, 1, \ldots$. Hence the $d_{n,k}$ are well-defined. Also, (1) is an immediate consequence of (18). Moreover, the definition (16) shows that

$$\sum_{k=0}^{l_n} |d_k| = \sum_{k=0}^{l_n} \left| \sum_{\mu=0}^{\infty} a_{\mu,k} \right| \leq \sum_{\mu=0}^{\infty} \sum_{k=0}^{l_n} |a_{\mu,k}|.$$

The tail, $\sum_{\mu>n}$, of this latter series is less than 1 (by (17)), and for $0 \leq \mu \leq n$ the estimate (13) gives that

$$\sum_{k=0}^{l_n} |a_{\mu,k}| = \sum_{k=0}^{l_\mu} |a_{\mu,k}| < 1 + \varepsilon_\mu \leq 1 + 2^{-\mu}.$$

Adding up all the terms gives

$$|D_{l_n}| \le \sum_{k=0}^{l_n} |d_k| < \sum_{\mu=0}^{n} (1+2^{-\mu})+1 < n+4 \quad \text{for} \quad n=0, 1, \ldots.$$

Dividing by $|D_{l_n}|$ we find, with (18), that $\lim_{n\to\infty} |D_{l_n}|^{-1} \cdot \sum_{k=0}^{l_n} |d_k| = 1$, which is much more than we need to show (2).

For the proof of the statements (5), (6), (7) suppose that, for some index ν, K is a compact subset of G_ν or that $K = J_\nu$ or that $K = \{z_\nu\}$. In the last two cases K is a subset of L_n for $n \ge \nu$. And in the first case, K is a positive distance from G_ν^c. Thus it follows from the definitions of the $K_{n,\nu}$ that K is a subset of $K_{n,\nu}$ provided that n is sufficiently large. Hence in all three cases there exists an index $n_0 \ge 0$ such that (14) holds for all $z \in K$ and all $n \ge n_0$:

$$\left| \sum_{k=0}^{l_n} a_{n,k} \sum_{m=0}^{k} c_m z^m - g(z) \right| < \varepsilon_n \quad \text{for all } z \in K \text{ and all } n \ge n_0. \tag{19}$$

Since $g(z) = f_\nu(z)$, resp. $g(z) = g_\nu(z)$, resp. $g(z) = w_\nu$ on K, it suffices to show that $\lim_{n\to\infty} \sigma_n^D(z) = g(z)$ uniformly on K.

According to (4), the definition of the $d_{n,k}$ and (16), we may write

$$D_{l_n}\sigma_n^D(z) = \sum_{k=0}^{l_n} d_k \sum_{m=0}^{k} c_m z^m = \sum_{\mu=0}^{\infty} \sum_{k=0}^{l_n} a_{\mu,k} \sum_{m=0}^{k} c_m z^m$$

for $n \ge 0$, $z \in \mathbb{C}$. If $n > n_0$ we have the identity

$$\begin{aligned} D_{l_n}(\sigma_n^D(z) - g(z)) &= (n - n_0 - D_{l_n})g(z) + \sum_{0 \le \mu \le n_0} \sum_{k=0}^{l_n} a_{\mu,k} \sum_{m=0}^{k} c_m z^m \\ &\quad + \sum_{n_0 < \mu \le n} \left(\sum_{k=0}^{l_n} a_{\mu,k} \sum_{m=0}^{k} c_m z^m - g(z) \right) \\ &\quad + \sum_{\mu > n} \sum_{k=0}^{l_n} a_{\mu,k} \sum_{m=0}^{k} c_m z^m \\ &= H + \Sigma_1 + \Sigma_2 + \Sigma_3, \quad \text{say.} \end{aligned} \tag{20}$$

From (18) and the fact that g is continuous on K, it follows that H is uniformly bounded for $z \in K$. Σ_1 is a polynomial which does not depend on n, since $a_{\mu,k} = 0$ for $k > l_\mu$ (and $l_\mu \le l_{n_0} < l_n$). Therefore Σ_1 is also uniformly bounded on K. Thus $|H + \Sigma_1| < M$ for all $z \in K$ where M is a constant.

In Σ_2, we may replace the upper limits of summation, l_n, by l_μ (since $a_{\mu,k} = 0$ for $k > l_\mu$) and use (19) and (15) to show that

$$|\Sigma_2| < \sum_{n_0 < \mu \le n} \varepsilon_\mu < \sum_{n_0 < \mu \le n} 2^{-\mu} < 1.$$

In Σ_3, we have $|\sum_{m=0}^{k} c_m z^m| < M_n$ for all $z \in K$ (remember that K is a subset of $L_n \cup K_{n,0}$ for $n \ge n_0$), and by (11) we also have $|a_{\mu,k}| < \varepsilon_\mu$ in Σ_3. Also using (15) and the monotonicity of the sequences (M_k) and (l_k) we can therefore make the estimate

$$|\Sigma_3| < \sum_{\mu > n} (l_n + 1)\varepsilon_\mu M_n < \sum_{\mu > n} 2^{-\mu} \le 1.$$

Inserting all these inequalities in (20) we obtain that

$$|D_{l_n}| \cdot |\sigma_n^D(z) - g(z)| < M + 2 \quad \text{for all} \quad z \in K.$$

After dividing this inequality by $|D_{l_n}|$, which is greater than n for all $n \ge 0$ (by (18)), we obtain the desired result that

$$\lim_{n \to \infty} \sigma_n^D(z) = g(z) \text{ uniformly on } K.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove parts (8) and (9) of the theorem since for every meromorphic function f, there are always sets G and S that satisfy the topological hypotheses of Theorem 2. (Choose, for instance, G to be the Mittag–Leffler star of f, i.e., the star-shaped region obtained by taking away all points of the form $z = tw$, where w is a pole of f and $t \ge 1$.)

For every $n \ge 0$, define

i) K_n to be the set of all points in G whose moduli do not exceed $n+1$ and whose distance to the complement of G is at least $1/(n+1)$,

ii) L_n to be the set of all points in G^c whose moduli do not exceed $n+1$ and whose distance to P is at least $1/(n+1)$.

With the hypotheses of Theorem 2, it is easy to verify that each K_n is a compact subset of G and that each L_n is a compact set which contains no pole of f nor any point of $G \cup \Delta_r$. It also follows from the hypotheses that $G \subset L_n^c \subset \bar{G}$, whence L_n^c is connected and we may apply Theorem A with $K = K_n$, $L = L_n$, $g = f$, and $\varepsilon = \varepsilon_n > 0$. Thus there is a sequence of polynomials $p_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$ (say $a_{n,k} = 0$ for $k > l_n$) such that the following conditions hold for every $n \ge 0$:

$$\tag{21} \begin{aligned} &|a_{n,k}| < \varepsilon_n \quad \text{for} \quad k = 0, 1, \ldots, \\ &\sum_{k=0}^{l_n} a_{n,k} = 1, \\ &\sum_{k=0}^{l_n} |a_{n,k}| < 1 + \varepsilon_n, \\ &\left| \sum_{k=0}^{l_n} a_{n,k} \sum_{m=0}^{k} c_m z^m - f(z) \right| < \varepsilon_n \quad \text{for all} \quad z \in K_n \cup L_n. \end{aligned}$$

As in the proof of Theorem 1 we define

$$\varepsilon_0 = 1,$$

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and

$$\varepsilon_n = 1/(2^n M_{n-1}(l_{n-1}+1)) \quad \text{for} \quad n \geq 1,$$

where

$$M_n = 1 + \max\left\{\sum_{m=0}^{l_n} |c_m z^m| : z \in \{0\} \in K_n \cup L_n\right\}.$$

Following the proof of Theorem 1, it follows that a well-defined regular generalized weighted means method $D = (d_{n,k})$ is obtained by setting

$$d_k = \sum_{\mu=0}^{\infty} a_{\mu,k} \quad \text{for} \quad k = 0, 1, \ldots,$$

and

$$d_{n,k} = \begin{cases} d_k/D_{l_n} & \text{for} \quad 0 \leq k \leq l_n \\ 0 & \text{for} \quad k > l_n, \end{cases}$$

where $D_n = d_0 + \cdots + d_n$.

Furthermore, it is easy to show that for every compact subset K of G resp. S there is an index $n_0 \geq 0$ such that $K \subset K_n \cup L_n$ for all $n \geq n_0$. Thus, if K is such a compact set, (21) implies that

$$\left|\sum_{k=0}^{l_n} a_{n,k} \sum_{m=0}^{k} c_m z^m - f(z)\right| < \varepsilon_n \quad \text{for all } z \in K \text{ and all } n \geq n_0.$$

The same arguments as those following (19) in the proof of Theorem 1, with g replaced by f, show that

$$\lim_{n\to\infty} \sigma_n^D(z) = f(z) \text{ uniformly on } K.$$

Thus, (8) and (9) are proved.

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