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Stratified Subcartesian Spaces

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Abstract. We show that if the family $\mathcal{O}$ of orbits of all vector fields on a subcartesian space P is locally finite and each orbit in $\mathcal{O}$ is locally closed, then $\mathcal{O}$ defines a smooth Whitney A stratification of P. We also show that the stratification by orbit type of the space of orbits M/G of a proper action of a Lie group G on a smooth manifold M is given by orbits of the family of all vector fields on M/G.

1 Introduction

Stratification theory is based on the natural idea of dividing a singular space into manifolds. It deals with study of topological spaces endowed with a partition by smooth manifolds satisfying specific conditions. Many of the singular spaces appearing in analysis have the structure of stratified spaces satisfying Whitney's condition B [11], and the theory of stratified spaces is an important tool with a broad range of applications; see [7] and references quoted there.

Sikorski's theory of differential spaces is a tool in the study of the differential geometry of a large class of singular spaces [9]. A differential space P is said to be subcartesian if every point $p \in P$ has a neighbourhood diffeomorphic to a subset of a Euclidean space [1]. In particular, an arbitrary subset P of $\mathbb{R}^n$, with the ring of smooth functions generated by restrictions to P of smooth functions on $\mathbb{R}^n$, is subcartesian.

Every subcartesian space has a canonical partition by smooth manifolds given by orbits of the family of all vector fields on the space [10]. The aim of this paper is to discuss stratifications of subcartesian spaces and compare them with partitions by orbits of the family of all vector fields. We show that the partition of a subcartesian space P by the family $\mathcal{O}$ of orbits of all vector fields satisfies the frontier condition and Whitney's condition A. From this we conclude that if the family $\mathcal{O}$ is locally finite and each orbit in $\mathcal{O}$ is locally closed, then $\mathcal{O}$ defines a smooth Whitney A stratification of P. A locally finite family $\mathcal{O}$ of locally closed orbits of all vector fields need not satisfy Whitney's condition B. However, some smooth Whitney B stratifications are given by orbits of all vector fields. We show that the stratification by orbit type of the space of orbits M/G of a proper action of a Lie group G on a smooth manifold M is given by orbits of the family of all vector fields on M/G.

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2 Decomposed Spaces

A decomposition of a topological space P is a partition of P by a locally finite family $\mathcal{D}$ of smooth manifolds $M \subset P$ such that each manifold $M \in \mathcal{D}$ with its manifold topology is a locally closed topological subspace of P, satisfying the following condition.

Condition 2.1 (Frontier Condition) *For $M, M' \in \mathcal{D}$, if $M' \cap \bar{M} \neq \varnothing$, then either $M' = M$ or $M' \subset \bar{M} \backslash M$.*

The pair $(P, \mathcal{D})$ is called a decomposed space. Local finiteness of $\mathcal{D}$ means that, for each point $p \in P$, there exists a neighbourhood U of p in P intersecting only a finite number of manifolds $M \in \mathcal{D}$. A subset M of a topological space P is locally closed if for each $x \in M$ there exists a neighbourhood U of x in P such that $M \cap U$ is closed in U. If P is a manifold, an injectively immersed submanifold M of P is embedded if and only if M is locally closed in P.

Decomposed spaces form a category with morphisms $\varphi\colon (P_1, \mathcal{D}_1) \to (P_2, \mathcal{D}_2)$ given by continuous map $\varphi\colon P_1 \to P_2$ such that, for each $M_1 \in \mathcal{D}_1$, there exists $M_2 \in \mathcal{D}_2$ such that $\varphi(M_1) \subset M_2$, and the restriction of φ to M_1 is a smooth map from M_1 to M_2.

For a decomposed space $(P, \mathcal{D})$, let Q be a topological subspace of P, and $\mathcal{D}_Q = \{M \cap Q \mid M \in \mathcal{D}\}$. Suppose that, for each $M \in \mathcal{D}$, $M \cap Q$ is a submanifold of M locally closed in Q, and the family $\mathcal{D}_Q$ is locally finite. Then $\mathcal{D}_Q$ satisfies the Frontier Condition because, if M and M' in $\mathcal{D}$ are such that $(M' \cap Q) \cap \overline{(M \cap Q)} \neq \varnothing$, then $M' \cap \bar{M} \neq \varnothing$, and either $M' = M$ or $M' \subset \bar{M} \backslash M$, so that, either $(M' \cap Q) = (M \cap Q)$ or $(M' \cap Q) \subset (M \cap Q)$. Therefore, $(Q, \mathcal{D}_Q)$ is a decomposed space. In particular, if Q is an open subset of P, then $(U, \mathcal{D}_U)$ is a decomposed space.

Suppose $(P, \mathcal{D})$ is a decomposed space, Q is a smooth manifold, and $\mathcal{D}_{P\times Q} = \{M \times Q \mid M \in \mathcal{D}\}$. Then $(P \times Q, \mathcal{D}_{P\times Q})$ is also a decomposed space, and the projection map $P \times Q \to P$ gives a morphism from $(P \times Q, \mathcal{D}_{P\times Q})$ to $(P, \mathcal{D})$. A decomposed space $(P, \mathcal{D})$ is locally trivial if, for every point $M \in \mathcal{D}$ and each $x \in M$, there exists an open neighbourhood U of x in P, a decomposed space $(P', \mathcal{D}')$ with a distinguished point $y \in P'$ such that the singleton $\{y\} \in \mathcal{D}'$, and an isomorphism $\varphi\colon (U, \mathcal{D}_U) \to (P' \times (U \cap M), \mathcal{D}'_{P' \times (U\cap M)})$, such that $\varphi(x) = y$.

Decompositions of a topological space P can be partially ordered by inclusion. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two decompositions of P, we say that $\mathcal{D}_1$ is a *refinement* of $\mathcal{D}_2$ and write $\mathcal{D}_1 \geq \mathcal{D}_2$, if, for every $M_1 \in \mathcal{D}_1$, there exists $M_2 \in \mathcal{D}_2$ such that $M_1 \subseteq M_2$. We say that $\mathcal{D}$ is a minimal (coarsest) decomposition of P if it is not a refinement of a different decomposition of P. Note that if P is a manifold, then the minimal decomposition of M consists of a single manifold $M = P$. Similarly, we say that $\mathcal{D}$ is a maximal (finest) decomposition of P if $\mathcal{D}' \geq \mathcal{D}$ implies $\mathcal{D}' = \mathcal{D}$.

3 Stratified Spaces

Let A and B be subsets of a topological space P. If $x \in A \cap B$, we say that A and B are equivalent at x if there exists a neighbourhood U of x in P such that $A \cap U = B \cap U$.

The equivalence class at x of a subset A of P containing x is called the germ of A at x and denoted $[A]_x$.

A stratification of a topological space P is a map $\mathcal{S}$ that associates with each $x \in P$ a germ $\mathcal{S}_x$ of a manifold embedded in P such that the following condition is satisfied.

Condition 3.1 (Stratification Condition) *For every $z \in P$ there exists a neighbourhood U of z and a decomposition $\mathcal{D}$ of U such that for all $y \in U$ the germ $\mathcal{S}_y$ coincides with the germ of the manifold $M \in \mathcal{D}$ that contains y.*

Every decomposition $\mathcal{D}$ of P defines a stratification $\mathcal{S}$ of P that associates with every $x \in P$ the germ $\mathcal{S}_x$ at x of the manifold $M \in \mathcal{D}$ that contains x.

Definition 3.2 Two decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ of P are equivalent if they define the same stratification $\mathcal{S}$ of P.

Let $\mathcal{S}$ be a stratification of P. There is a unique decomposition $\mathcal{D}_0$ of P by connected manifolds that defines $\mathcal{S}$. It is the finest element of the class of decomposition of P corresponding to $\mathcal{S}$. From the point of view of this paper it is convenient to identify $\mathcal{S}$ with $\mathcal{D}_0$.

4 Differential and Subcartesian Spaces

A differential structure on a topological space P is a family $C^\infty(P)$ of functions on P satisfying the following conditions.

Condition 4.1 (Differential Structure) (i) *The family of sets $\{f^{-1}((a,b)) \mid f \in C^\infty(P)$, and $a, b \in \mathbb{R}\}$ is a sub-basis for the topology of P.*

(ii) *For every $k \in \mathbb{N}$, every $f_1, \dots, f_k \in C^\infty(P)$, and $F \in C^\infty(\mathbb{R}^k)$, the composition $F(f_1, \dots, f_k)$ is in $C^\infty(P)$.*

(iii) *If a function f on P is such that, for every $x \in P$, there exists an open neighbourhood U_x of x in Q and a function $f_x \in C^\infty(P)$ satisfying $f_{|U_x} = f_{x|U_x}$, then f is in $C^\infty(P)$.*

A topological space endowed with a subring of continuous functions satisfying the above conditions is called a differential space.

A homeomorphism $\varphi\colon P \to Q$ of differential spaces is smooth if its pull-back φ^* maps $C^\infty(Q)$ to $C^\infty(P)$. It is a diffeomorphism if it is invertible and $\varphi^{-1}\colon Q \to P$ is smooth. A subcartesian space is a Hausdorff differential space P such that each point $x \in P$ has a neigbourhood that is diffeomorphic to a subset of a Cartesian space $\mathbb{R}^N$.

We can adapt notions of decomposition and stratification of a topological space to a differential space by requiring smoothness of all maps involved. Thus, a *smooth decomposition* of a differential space P is a decomposition $\mathcal{D}$ of P as a topological space such that, for each $M \in \mathcal{D}$, the inclusion map $M \hookrightarrow P$ is smooth. Similarly, a *smooth stratification* of a differential space P is a smooth decomposition of P by connected manifolds.

For each point p of a differential space P, a derivation of $C^\infty(P)$ at p is a linear map $u\colon C^\infty(P) \to \mathbb{R}$ satisfying Leibniz's rule

$$u(f_1 f_2) = f_1(p)u(f_2) + u(f_1)f_2(p) \quad \text{for all} \quad f_1, f_2 \in C^\infty(P).$$

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The set of all derivations of $C^\infty(P)$ at p is called the *tangent space* to P at p. It is denoted T_pP. The *tangent cone* at $p \in P$ is the subset T^C_pP of T_pP consisting of derivations at p that are given by differentiation along smooth curves in P passing through p. In other words, $u \in T_pP$ is in T^C_pP if there exists a smooth curve $c\colon [0,1] \to P$ such that $u(f) = \frac{d}{dt} f(c(t))_{|t=0}$. Reparametrization of curves gives rise to the cone structure in $T^C_p(P)$.

A (global) *derivation* of $C^\infty(P)$ is a linear map $X\colon C^\infty(P) \to C^\infty(P)$ satisfying Leibniz's rule

$$X(f_1 f_2) = f_1 X(f_2) + X(f_1) f_2 \quad \text{for all} \quad f_1, f_2 \in C^\infty(P).$$

Let I be an interval in $\mathbb{R}$ with a non-empty interior. A smooth map $c\colon I \to P$ is an integral curve of a derivation X if, for every $t \in I$ and $f \in C^\infty(P)$,

$$\tfrac{d}{dt} f(c(t)) = X(f)(c(t)).$$

We want to extend the notion of an integral curve to the case when I consists of a single point, *i.e.*, $I = [a,a]$ for $a \in \mathbb{R}$. In this case the left-hand side of the above equation is not defined. We consider a map $c\colon I = [a,a] \to P\colon a \mapsto c(a)$ to be an integral curve of every derivation of $C^\infty(P)$. With this definition, for every derivation X of the differential structure $C^\infty(P)$ of a subcartesian space P and every $x \in P$, there exists a unique maximal integral curve of X passing through x [10].

Definition 4.2 A derivation X of $C^\infty(P)$ is a vector field on a subcartesian space P if translations along integral curves of X give rise to a local one-parameter group of local diffeomorphisms of P.

Let $\mathcal{X}$ be the family of all vector fields on P. For each $X \in \mathcal{X}$, we denote by $\exp tX$ the local one-parameter group of local diffeomorphisms generated by translations along integral curves of X. The orbit of $\mathcal{X}$ through a point $x \in P$ is

$$O_x = \{(\exp t_1 X_1 \circ \cdots \circ \exp t_n X_n)(x) \mid n \in \mathbb{N},\ (t_1, \dots, t_n) \in \mathbb{R}^n,\ X_1, \dots, X_n \in \mathcal{X}\}.$$

For each $x \in P$, the orbit O_x of the family X of all vector fields on P is a manifold, and the inclusion map $O_x \hookrightarrow P$ is smooth [10]. The collection $\mathcal{O}$ of all orbits of $\mathcal{X}$ is a partition of P by smoothly included manifolds.

Theorem 4.3 *The partition $\mathcal{O}$ of a subcartesian space P by orbits of the family of all vector fields on P is a smooth stratification of P if $\mathcal{O}$ is a locally finite, and each orbit $O \in \mathcal{O}$ is locally closed.*

Proof By definition, orbits of the family of all vector fields are connected. Moreover, for each orbit $O \in \mathcal{O}$, the inclusion map $O \hookrightarrow P$ is smooth. Hence, it suffices to show that the family $\mathcal{O}$ satisfies the Frontier Condition. Suppose $x \in O' \cap \bar{O}$ with $O' \neq O$. We first show that $O' \subset \bar{O}$. Note that the orbit O is invariant under the family of one-parameter local groups of local diffeomorphisms of P generated by vector fields. Since $x \in \bar{O}$, it follows that, for every vector field X on S, $\exp(tX)(x)$ is in $\bar{O}$ if it is defined. But, O' is the orbit of $\mathcal{X}$ through x. Hence, $O' \subset \bar{O}$. ■

A smoothly decomposed diffferential space $(P, \mathcal{D})$ is smoothly locally trivial if, for every point $M \in \mathcal{D}$ and each $x \in M$, there exists an open neighbourhood U of x in P, a smoothly decomposed differential space $(P', \mathcal{D}')$ with a distinguished point $y \in P'$ such that the singleton $\{y\} \in \mathcal{D}'$, and an isomorphism $\varphi\colon (U, \mathcal{D}_U) \to (P' \times (U \cap M), \mathcal{D}'_{P' \times (U \cap M)})$ such that $\varphi(x) = y$. It should be noted that a smoothly decomposed differential space $(P, \mathcal{D})$ may be locally trivial as a (topological) decomposed space but not smoothly locally trivial. The following example, taken from Mather [8], illustrates this situation.

Example 4.4 Consider $F(x, y, z) = xy(x + y)(x + \alpha(x)y)$ for a smooth one-to-one function $\alpha(x)$ with values different from 0 and 1. The zero level S of F, given by

$$S = \{(x, y.z) \in \mathbb{R}^3 \mid xy(x + y)(y + \alpha(z)x) = 0\},$$

is the union of four surfaces intersecting along the z-axis. It has eight 2-dimensional strata: $\pm x > 0$, $\pm y > 0$, $\pm(x + y) > 0$, and $\pm(y + \alpha(z)) > 0$, and a 1-dimensional stratum consisting of the z-axis.

For each z_0, the tangent cone to S at $(0, 0, z_0)$ is the union of four planes $x = 0$, $y = 0$, $x+y = 0$, and $x+\alpha(z_0)y = 0$ intersecting along the z-axis. Projections of these planes to the (x, y)-plane are four lines intersecting at the origin. If values of α are different from 0 and 1, then all four lines are distinct and their cross-ratio is $\gamma(z_0) = 1 + \alpha(z_0)$. By assumption, the function $\alpha(z)$ is one-to-one. Since the cross-ratio is an invariant of linear transformations preserving the origin, all diffeomorphisms of S preserve points on the z-axis. This implies that the stratification of S described above is not locally trivial.

The argument above also implies that the z-axis is not an orbit of the family of all vector fields on S. The partition of S by the family of orbits of all vector fields on S consists of 2-dimensional orbits, which coincide with two dimensional strata, and 0-dimensional orbits $\{(0, 0, z_0)\}$ for each $z_0 \in \mathbb{R}$.

Let $\mathcal{D}$ be a smooth decomposition of a differential space P. We say that $\mathcal{D}$ admits *local extension of vector fields* if, for each $M \in \mathcal{D}$ each vector field X_M on M and each point $x \in M$, there exists a neighbourhood V of x in M, and a vector field X on P such that $X_{|V} = X_{M|V}$. In other words, the vector field X is an extension to P of the restriction of X_M to V.

Theorem 4.5 *Every smooth locally trivial decomposition of a subcartesian space P admits local extensions of vector fields.*

Proof Let X_M be a vector field on $M \in D$. Given $x_0 \in M$, let U be a neighbourhood of x_0 in M admitting an isomorphism $\varphi\colon U \to (M \cap U) \times P'$ for some smoothly decomposed differential space $(P', \mathcal{D}')$ such that $\{\varphi(x_0)\} \in \mathcal{D}'$. Let $\exp(tX_M)$ be the local one-parameter group of local diffeomorphisms of M generated by X_M and $X_{(M\cap U)\times P'}$ be a derivation of $C^\infty((M \cap U) \times P')$ defined by

$$(X_{(M\cap U)\times P'}h)(x, y) = \tfrac{d}{dt}h(\exp(tX_M)(x), y)_{|t=0}.$$

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for every $h \in C^\infty((M \cap U) \times P')$ and each $(x, y) \in (M \cap U) \times P'$. Since $X_{(M\cap U)\times P'}$ is defined in terms of a local one-parameter group $(x, y) \mapsto (\exp(tX_M)(x), y)$ of diffeomorphisms, it is a vector field on $(M \cap U) \times P'$.

We can use the inverse of the diffeomorphism $\varphi\colon U \to (M \cap U) \times P'$ to push forward $X_{(M\cap U)\times P'}$ to a vector field $X_U = \varphi_*^{-1} X_{(M\cap U)\times P'}$ on U. Choose a function $f_0 \in C^\infty(P)$ with support in U and such that $f_0(x) = 1$ for x in some neighbourhood U_0 of x_0 contained in U. Let X be a derivation X of $C^\infty(P)$ extending $f_0 X_U$ by zero outside U. In other words, for every $f \in C^\infty(P)$, if $x \in U$, then $(Xf)(x) = f_0(x)(X_U f)(x)$, and if $x \notin U_0$, then $(Xf)(x) = 0$. Clearly, X is a vector field on P extending the restriction of X_M to $M \cap U_0$. ∎

Theorem 4.6 *Let $\mathcal{D}$ be a decomposition of a subcartesian space P admitting local extensions of vector fields, then the partition $\mathcal{O}$ of P by orbits of the family of all vector fields on P is a stratification of P. If all manifolds in $\mathcal{D}$ are connected, then $\mathcal{D}$ is a refinement of $\mathcal{O}$. Moreover, if $\mathcal{D}$ is minimal in the class of decompositions by connected manifolds, then $\mathcal{D} = \mathcal{O}$.*

Proof Let $\mathcal{D}$ be a decomposition of P admitting local extensions of vector fields. Since every vector field X_M on a manifold $M \in \mathcal{D}$ extends locally to a vector field on P, it follows that M is contained in an orbit $O \in \mathcal{O}$.

Every orbit $O \in \mathcal{O}$ is a union of manifolds in the decomposition $\mathcal{D}$. Since $\mathcal{D}$ is locally finite, it follows that, for each $x \in P$, there exists a neigbourhood U of x in P that intersects only a finite number of manifolds in $\mathcal{D}$. Hence, U intersects only a finite number of orbits in $\mathcal{O}$.

Since manifolds in $\mathcal{D}$ are locally closed, for each $M \in \mathcal{D}$ and each $x \in M$, there exists a neighbourhood U of x in P such that $M \cap U$ is closed in U. Without loss of generality, we may assume that there is only a finite number of manifolds $M_1 = M$, $M_2, \dots, M_k$ in $\mathcal{D}$ such that $M_i \cap U \neq \varnothing$ for $i = 1, \dots, k$. Since manifolds in $\mathcal{D}$ form a partition of P, it follows that $U = \bigcup_{i=1}^k M_i \cap U$. We may also assume that each $M_i \cap U$ is closed in U.

Let O be the orbit in $\mathcal{O}$ that contains $M = M_1$. We can relabel the manifolds $M_1, \dots, M_k$ so that

$$O \cap U = O \cap \bigcup_{i=1}^{k} M_i \cap U = \bigcup_{i=1}^{k} O \cap M_i \cap U = \bigcup_{i=1}^{l} M_i \cap U$$

for some $l \leq k$. Since $M_i \cap U$ is closed in U for each $i = 1, \dots, l$, it follows that $O \cap U$ is also closed in U. Hence, orbits $O \in \mathcal{O}$ are locally closed.

Taking into account Theorem 4.3, we see that $\mathcal{O}$ is a stratification of P. If all manifolds in $\mathcal{D}$ are connected, then each $M \in \mathcal{D}$ is contained in an orbit in $\mathcal{O}$ and $\mathcal{D}$ is a refinement of $\mathcal{O}$. If $\mathcal{D}$ is minimal in the class of decompositions by connected manifolds, then it cannot be a refinement of a different decomposition. Hence, $\mathcal{D} = \mathcal{O}$. ∎

5 Whitney Conditions

In his analysis of stratifications, Whitney introduced two conditions on a triple (M, M', x) of C^1-submanifolds M and M' of a manifold W, and $x \in M'$ [11]. Stratifications satisfying Whitney's condition A are called Whitney A stratifications. Similarly, stratifications satisfying Whitney's condition B are called Whitney B stratifications.

Since we are dealing here with subcartesian spaces, we assume that M and M' are C^∞-submanifolds of $\mathbb{R}^N$, and M' is in the closure $\bar{M}$ of M. Let (x_n) be sequence of points in $M \in \mathcal{D}$ converging to $x \in M' \in \mathcal{D}$ such that the sequence of tangent spaces $T_{x_n}M$ converges to a space D in the Grassmannian of m-planes in R^N, where $m = (\dim M)$.

Condition 5.1 (Whitney Condition A) $T_xM' \subseteq D$.

Condition 5.2 (Whitney Condition B) *If y_n is a sequence of points in M' converging to x and the sequence of lines $\langle x_n.y_n \rangle$ converges to a line L through x, then $L \subset D$.*

Let $e = (e_1, \dots, e_N)$ be the canonical basis of $\mathbb{R}^N$. Each orthonormal basis $b = (b_1, \dots, b_N)$ in R^N is of the form $b = eA$ for a unique $A \in O(N)$. An orthonormal basis $b = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ is said to be adapted to an m-dimensional subspace D if the first m vectors $(b_1, \dots, b_m)$ in b form a basis of D. The class of all bases of $\mathbb{R}^N$ adapted to D is given by an element $\gamma \in O(N)/(O(N) \times O(N-m))$. Using the bijection $A \mapsto b = eA$ between $O(N)$ and the space of orthonormal bases on $\mathbb{R}^N$, one can identify the set of all m-dimensional subspaces of R^N with the Grassmannian $O(N)/O(N) \times O(N-m)$.

Let D_n be a sequence of m-dimensional subspaces of $\mathbb{R}^N$. For each n, we denote by $\gamma_n \in O(N)/(O(N) \times O(N-m))$ the class of bases in R^N adapted to D_n. The sequence of subspaces D_n is said to be convergent to an m-dimensional subspace D if the sequence γ_n converges in $O(N)/(O(N) \times O(N-m))$ to γ representing the class of all bases in D.

Assume that D_n converges to D. For each n, we can choose a matrix $A_n \in O(N)$ such that $b_n = eA_n$ is adapted to D_n. Since $O(N)$ is compact, there exists a convergent subsequence A_{n_k}. Let $A = \lim_{k\to\infty} A_{n_k}$. Then $b = eA = \lim_{k\to\infty} b_{n_k}$. If $b = (b_1, \dots, b_N)$ then, for each $i = 1, \dots, N$, the sequence $b_{n_k i}$ of i-th vectors in $b_{n_k} = (b_{n_k 1}, \dots, b_{n_k N})$ converges to b_i.

Let $u_n \in D_n$ be a convergent sequence of vectors in $\mathbb{R}^N$ and $u = \lim_{n\to\infty} u_n$. For each n, we can express u_n in terms of the basis $b_n = (b_{n1}, \dots, b_{nN})$ obtaining $u_n = a_n^1 b_{n1} + \cdots + a_n^N b_{nN}$. Since the basis b_n is orthonormal, for each $i = 1, \dots, N$, we have $a_n^i = u_n \cdot b_{ni}$, where $\cdot$ denotes the canonical scalar product in $\mathbb{R}^N$. Hence,

$$\lim_{k\to\infty} a_{n_k}^i = \lim_{k\to\infty} (u_{n_k} \cdot b_{n_k i}) = (\lim_{k\to\infty} u_{n_k}) \cdot (\lim_{k\to\infty} b_{n_k i}) = u \cdot b_i.$$

Therefore, $u = (u \cdot b_1)b_1 + \cdots + (u \cdot b_N)b_N \in D$ because $b = (b_1, \dots, b_N)$ is a basis in D.

Conversely, if $u = a^1 b_1 + \cdots + a^N b_N$ is a vector in D, then $u_n = a^1 b_{n_k 1} + \cdots + a^N b_{n_k N}$ is in D_{n_k} because $b_{n_k} = (b_{n_k 1}, \dots, b_{n_k N})$ is a basis in in D_{n_k}. Moreover, the sequence $u_{n_k} \in D_{n_k}$ converges to u. Hence we have justified the following observation.

Remark 5.3 Suppose that a sequence D_n of m-dimensional subspaces of $\mathbb{R}^N$ converges to an m-dimensional subspace D. Then every convergent sequence of vectors

$u_n \in D_n$ has a limit in D and vectors in D are limits of convergent sequences of vectors in D_n.

Proposition 5.4 *The partition of a subcartesian space P by the family $\mathcal{O}$ of orbits of all vector fields on P satisfies Whitney's condition A.*

Proof Let O' and O be orbits in $\mathcal{O}$, $x \in O' \cap \bar{O}$, and let (x_n) be sequence of points in M converging to x such that the sequence of tangent spaces $T_{x_n}O$ converges to D in the Grassmannian of m-planes in $\mathbb{R}^N$, where $m = \dim M$. First, we need to show that if a sequence (x_n) of points in O converges to $x \in O'$ such that the spaces $T_{x_n}O$ converge to $D_x \subseteq T_xP$, then $T_xO' \subseteq D_x$. Since P is subcartesian, we may assume without loss of generality that x has a neighbourhood U in P that can be identified with a subset of $\mathbb{R}^N$. Each $T_{x_n}O$ can be identified with the corresponding m-dimensional subspace D_n of $\mathbb{R}^N$, where $m = \dim O$. Similarly, we identify $D_x \subseteq T_xP$ with an m-dimensional subspace D of $\mathbb{R}^N$. By assumption, the sequence D_n converges to D.

Let $k = \dim O'$, $m = \dim O$, and let $X_1, \dots, X_k, X_{k+1,} \dots, X_m$ be vector fields on P such that $X_1(x), \dots, X_k(x)$ is a basis for T_xO' and $X_1, \dots, X_k, X_{k+1,} \dots, X_m$ give rise to a frame in $T(O \cap U)$ for some neighbourhood U of x in P. Without loss of generality, we may assume that U is the neigbourhood of x introduced in the preceding paragraph and that all points of the sequence x_n in O converging to x are contained in $O \cap U$.

For each $i = 1, \dots, m$, the vector field X_i is continuous so that $X_i(x) = \lim_{n\to\infty} X_i(x_n)$. Since $(X_1(x), \dots, X_k(x))$ is a frame for T_xO', it follows that every vector $u \in T_xO'$ is of the form $u = a^1X_1(x) + \cdots + a^kX_k(x)$. Let $u_n = a^1X_1(x_n) + \cdots + a^kX_k(x_n) \in T_{x_n}O$. Then $u = \lim_{n\to\infty} u_n$, and Remark 5.3 implies that $u \in D_x$. Hence, $T_xO' \subseteq D_x$, which implies Whitney's condition A. ■

In general, the family $\mathcal{O}$ of orbits of all vector fields on a subcartesian space P need not satisfy Whitney's condition B.

Example 5.5 (Spiral) Let S be the closure of the spiral defined by $r = e^{-\theta}$ in $\mathbb{R}^2$. That is $S = S_0 \cup S_1$,where $S_0 = \{(0,0)\}$ and $S_1 = \{(e^{-\theta}\cos\theta, e^{-\theta}\sin\theta) \mid \theta \in \mathbb{R}\}$. The slope of S_1 at θ is

$$m_\theta = \frac{(e^{-\theta}\sin\theta)'}{(e^{-\theta}\cos\theta)'} = \frac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta}.$$

The sequence of points $\mathbf{x}_n = (e^{-2\pi n}\cos(2\pi n), e^{-2\pi n}(\sin 2\pi n)) = (e^{-2\pi n}, 0)$ converges to the origin. Moreover, the slope of S_1 at $\mathbf{x}_n$ is $m_{2\pi n} = -1$, which implies that the sequence $T_{\mathbf{x}_n}S_1$ converges to a line $y = -x$.

For each n, the line L_n joining x_n to the origin $0 = (0,0)$ is the y-axis. Hence, the sequence L_n converges to the y-axis that is not contained in $\lim_{n\to\infty} T_{\mathbf{x}_n}S_1$. Thus, our spiral does not satisfies Whitney's condition B.

Nevertheless, there are several Whitney B stratifications which are given by the family $\mathcal{O}$ of orbits of vector fields on a subcartesian space.

Example 5.6 (Whitney's cusp) Whitney's cusp $S \subseteq \mathbb{R}^3$ is the zero level set of $F(x, y, z) = y^2 + x^3 - z^2x^2$. In other words,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + x^3 - z^2x^2 = 0\}.$$

Since $F \in C^\infty(\mathbb{R}^3)$, the implicit function theorem implies that S is a smooth manifold in a neighbourhood of every point (x, y, z) in S such that $DF(x, y, z) \neq 0$. But,

$$DF(x, y, z) = (3x^2 - 2xz^2)dx + 2y\,dy - 2zx^2dz.$$

Hence, $DF(x, y, z) = 0$ on the z-axis

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\},$$

and $S_2 = S \backslash S_1$ is a smooth manifold. The Hessian of F is

$$D^2F(x, y, z) = (6x - 2z^2)dx^2 + 2dy^2 - 2x^2dz - 8xzdxdz.$$

It has rank 2 on

$$S_1^{\pm} = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0, \ \pm z > 0\}$$

and rank 1 at the origin $S_0 = \{(0, 0, 0)\}$.

The function F is invariant under the action of $\mathbb{R}$ on $\mathbb{R}^3$ given by

$$\Phi \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \colon (t, (x, y, z)) \mapsto (e^{2t}x, e^{3t}y, e^tz)$$

generated by a vector field

$$X = 2x\tfrac{\partial}{\partial x} + 3y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z}.$$

This action is transitive on S_1^+ and S_1^-. Moreover, since the Hessian of a smooth function on a manifold is well defined on the set of critical points of the function, it follows that every diffeomorphism of $\mathbb{R}^3$ to itself which leaves F invariant preserves the origin. This implies that the decomposition $S = S_2 \cup S_1^+ \cup S_2^- \cup S_0$ is the partition given by the family of orbits of all smooth vector fields on S. It is of interest to note that this partition is a stratification of S satisfying Whitney's conditions A and B.

6 Orbits of a Proper Group Action on a Manifold

In this section we prove that the stratification by orbit type of the space of orbits of a proper action of a Lie group G on a manifold P is given by the family $\mathcal{O}$ of orbits of all vector fields on the orbit space $R = P/G$ with the differential structure $C^\infty(R)$ given by the ring $C^\infty(P)^G$ of G-invariant smooth functions on P. This results shows that the ring $C^\infty(P)^G$ encodes information about the stratification structure of the orbit space P/G.

We consider here a proper action

$$\Phi\colon G \times P \to P\colon (g, p) \mapsto \Phi(g, p) \equiv \Phi_g(p) \equiv gp$$

of a connected Lie group on a manifold P. Properness of Φ means that, for every convergent sequence (p_n) in P and a sequence (g_n) in G such that the sequence $(g_n p_n)$ is convergent, the sequence (g_n) has a convergent subsequence (g_{n_k}) and

$$\lim_{k\to\infty}(g_{n_k} p_{n_k}) = \Big(\lim_{k\to\infty} g_{n_k}\Big)\Big(\lim_{k\to\infty} p_{n_k}\Big).$$

For $p \in P$, the orbit of G through p is the set $Gp = \{gp \mid g \in G\}$. Let $R = P/G$ denote the space of G-orbits in P with the quotient topology, and let $\rho\colon P \to R\colon p \mapsto Gp$ be the canonical projection. Since the action Φ is proper, the orbit space R is a subcartesian space with the ring $C^\infty(R)$ of smooth functions on R given by

$$C^\infty(R) = \{f \in C^0(R) \mid \rho^* f \in C^\infty(P)\},$$

and the projection map $\rho\colon P \to R$ is smooth [4].

The orbit space $R = P/G$ of a proper action of a Lie group is stratified by orbit type . Since Φ is proper, for each $p \in P$, the isotropy group $G_p = \{g \in G \mid gp = p\}$ of p is compact. For each compact subgroup $H \subseteq G$, $P_H = \{p \in P \mid G_p = H\}$ is the set of all points in P of isotropy type H. Similarly,

$$P_{(H)} = \{p \in P \mid G_p \text{ is congruent to } H\}$$

is the set of all points in P of orbit type H. Both P_H and $P_{(H)}$ are local submanifolds of P. This means that connected components of P_H and $P_{(H)}$ are submanifolds of P. Connected components of the projection of $P_{(H)}$ to R are smooth manifolds. They are strata of the stratification of R by orbit type. For more details, see [6].

Theorem 6.1 *The stratification of $R = P/G$ by orbit type coincides with the partition of R by the family $\mathcal{O}$ of orbits of all vector fields on R.*

Proof Theorem 4.6 implies that it suffices to prove that the stratification of R by orbit types is minimal, and it admits local extensions of vector fields, Minimality of R has been proved by Bierstone [2, 3]. See also Duistermaat [5]. Hence, it remains to prove that the orbit type stratification of R admits local extensions of vector fields.

Let M be a stratum of the stratification of R by orbit type and X_M a smooth vector field on M. We want to show that, for each $x \in M$, there exists a neighbourhood $V \subseteq M$ and a vector field X on R such that the restrictions to V of X and X_M coincide.

Since the action of G on P is proper, for each $p \in \rho^{-1}(x)$, there exists a slice Σ for the action of G at p. That is, Σ is a submanifold of P containing p, invariant under the action of G_p, and satisfying the following conditions

(6.1) $$T_pP = T_p\Sigma \oplus T_p(Gp),$$

(6.2) $$T_{p'}P = T_{p'}\Sigma + T_{p'}(Gp') \text{ for all } p' \in \Sigma,$$

(6.3) For $p' \in \Sigma$ and $g \in G$, if $gp' \in \Sigma$, then $g \in G_p$.

Given a slice Σ, the set

$$\widetilde{W} = \bigcup_{p' \in \Sigma} Gp' = \{gp' \mid p' \in \Sigma, g \in G\}$$

is a G-invariant neighbourhood $\widetilde{W}$ of Gp in P. Its projection $W = \rho(\widetilde{W})$ to R is an open neighbourhood of x in R.

Let $\widetilde{M}$ be the connected component of P_{G_p} that contains p. As we have already stated, $\widetilde{M}$ is a submanifold of P. Moreover, $M = \rho(\widetilde{M})$. The intersection $\widetilde{M} \cap W$ is an open submanifold of $\widetilde{M}$.

Claim $\widetilde{M} \cap \Sigma$ is a submanifold of Σ diffeomorphic to $M \cap W$.

Proof Condition (6.3) states that if p' and gp' are in Σ, then $g \in G_p$. Moreover, p' and gp' in $\widetilde{M}$ implies that $G_{p'} = G_{gp'} = G_p$. Hence, $g \in G_p$ implies that $g \in G_{p'}$ and $gp' = p'$. Thus, $\widetilde{M} \cap \Sigma$ intersects fibres of the projection map $\rho\colon P \to R$ in at most single points. Therefore, the restriction μ of the projection map ρ to $\widetilde{M} \cap \Sigma$ is a bijection of $\widetilde{M} \cap \Sigma$ onto $M \cap W$. To show that $\mu\colon \widetilde{M} \cap \Sigma \to M \cap W$ is a diffeomorphism it suffices to show that μ and μ^{-1} are smooth.

The space $C^\infty(\widetilde{M} \cap \Sigma)$ is generated by restrictions to $\widetilde{M} \cap \Sigma$ of smooth functions on P. On the other hand, $C^\infty(M \cap W)$ is generated by restrictions to $M \cap W$ of functions in $C^\infty(R) = \{\rho_* f \mid f \in C^\infty(P)^G\}$.

First, we show that $\mu\colon \widetilde{M} \cap \Sigma \to M \cap W$ is smooth. Consider a function $f_{M \cap W} \in C^\infty(M \cap W)$. We need to show that $\mu^* f_{M \cap W} \in C^\infty(\widetilde{M} \cap \Sigma)$. For each point $x' \in M \cap W$, there exists a neighbourhood U of x' in $M \cap W$ and a function $\rho_* f \in C^\infty(R)$ such that the restriction of $\rho_* f$ to U coincides with the restriction to U of $f_{M \cap W}$. For each $p'' \in \mu^{-1}(U)$, we have

$$\mu^* f_{M \cap W}(p'') = f_{M \cap W}(\mu(p'')) = \rho_* f(\mu(p'')) = f(p'') = f_{|\widetilde{M} \cap \Sigma}(p''),$$

since f is G-invariant. Hence, $\mu^* f_{M \cap W}$ restricted to $\mu^{-1}(U)$ coincides with the restriction to $\mu^{-1}(U)$ of $f \in C^\infty(P)^G$. Since this result is valid for each $x' \in M \cap W$, it follows that $\mu^* f_{M \cap W} \in C^\infty(\widetilde{M} \cap \Sigma)$. However, $f_{M \cap W}$ is an arbitrary smooth function on $M \cap W$. Hence, $\mu\colon \widetilde{M} \cap \Sigma \to M \cap W$ is smooth.

Next, we want to show that $\mu^{-1}\colon M \cap \Sigma \to \widetilde{M} \cap W$ is smooth. Consider a function $f_{\widetilde{M} \cap \Sigma}$ in $C^\infty(\widetilde{M} \cap \Sigma)$. We need to show that $f_{\widetilde{M} \cap \Sigma} \circ \mu^{-1}$ is in $C^\infty(M \cap W)$. Given $q \in \widetilde{M} \cap \Sigma$, there exists a compactly supported function f_Σ on Σ that coincides with $f_{\widetilde{M} \cap \Sigma}$ on a neighbourhood $\widetilde{U}$ of q in $\widetilde{M} \cap \Sigma$. Let $\bar{f}_\Sigma$ be the G_p invariant function on Σ obtained by averaging f_Σ over G_p. Then, $f_{\Sigma|\widetilde{U}} = \bar{f}_{\Sigma|\widetilde{U}}$ because $\widetilde{U} \subseteq \widetilde{M} \cap \Sigma$, and all points in $\widetilde{M} \cap \Sigma$ have isotropy group G_p. Let $f_{\widetilde{W}}$ be a function on $\widetilde{W}$ defined by

$$f_{\widetilde{W}}(gp') = \bar{f}_\Sigma(p') \quad \text{for all} \quad p' \in \Sigma \text{ and } g \in G. \tag{6.4}$$

The function $f_{\widetilde{W}}$ is well defined by equation (6.4) and is G-invariant because $\bar{f}_\Sigma$ is G_p-invariant. If $g'p' = g''p''$, with p' and p'' in Σ, then $p' = (g')^{-1}g''p''$ implies that $(g')^{-1}g'' \in G_p$ and

$$f_{\widetilde{W}}(g'p') = \bar{f}_\Sigma(p') = \bar{f}_\Sigma((g')^{-1}g''p'') = \bar{f}_\Sigma(p'') = f_{\widetilde{W}}(p'').$$

Moreover, for every $g, g' \in G$, and $p' \in \Sigma$,

$$f_{\widetilde{W}}(g(g'p')) = f_{\widetilde{W}}(gg'p') = \bar{f}_\Sigma(p') = f_{\widetilde{W}}(g'p').$$

Since $\bar{f}_\Sigma$ is compactly supported, it follows that there is a G-invariant open set $\widetilde{V}$ in P containing the support of $f_{\widetilde{W}}$ and such that the closure of $\widetilde{V}$ is in $\widetilde{W}$. Therefore, there exists an extension of $f_{\widetilde{W}}$ to a smooth function f on P that f vanishes on the complement of $\widetilde{V}$. Moreover, $f \in C^\infty(P)^G$ because $f_{\widetilde{W}}$ is G-invariant. For each $p' \in \widetilde{U}$, we have

$$f_{\widetilde{M}\cap\Sigma} \circ \mu^{-1}(\rho(p')) = f_{\widetilde{M}\cap\Sigma}(p') = f(p') = \rho_* f(\rho(p')).$$

Hence, $f_{\widetilde{M}\cap\Sigma} \circ \mu^{-1}$ restricted to an open neighbourhood $U = \rho(\widetilde{U}) = \mu(\widetilde{U})$ of $\rho(q)$ in $M \cap W$ coincides with the restriction to U of $\rho_* f \in C^\infty(R)$. Since it holds for every point $\rho(q) \in M \cap \Sigma$, it follows that $(\mu^{-1})^* f_{\widetilde{M}\cap\Sigma} = f_{\widetilde{M}\cap\Sigma} \circ \mu^{-1} \in C^\infty(M \cap \Sigma)$. Hence, $\mu^{-1} \colon M \cap \Sigma \to \widetilde{M} \cap \Sigma$ is smooth.

This completes the proof of the claim. ∎

Continuing with the proof of Theorem 6.1, consider a smooth vector field X_M on M. For each $x \in M$, consider $p \in \rho^{-1}(x)$ and let $\widetilde{M}$ be the connected component of $\rho^{-1}(M)$ containing p. Let Σ be a slice at p for the action of G on P, and $W = \rho(\Sigma)$. We have shown that $\mu \colon \widetilde{M} \cap \Sigma \to M \cap W$ is a diffeomorphism Let $f_{M\cap W}$ be a compactly supported smooth function on $M \cap W$ such that $f_{M\cap W}(x') = 1$ for all x' in a neighbourhood of x in $M \cap W$. The product $f_{M\cap W} X_M$ is a vector field on $M \cap W$ which can be pushed forward by $\mu^{-1} \colon M \cap W \to \widetilde{M} \cap \Sigma$ to a vector field $\mu_*^{-1}(f_{M\cap W} X_M)$ on $\widetilde{M} \cap \Sigma$. Let $c \colon t \mapsto c(t)$ be an integral curve of $\mu_*^{-1}(f_{M\cap W} X_M)$. Since $c(t)$ is contained in M, for each $g \in G_p$ we have $gc(t) = c(t)$. Hence, c is invariant under the action of G_p. This implies that $\mu_*^{-1}(f_{M\cap W} X_M)$ is G_p-invariant. That is, for each $g \in G_p$,

$$T\Phi_g \circ \mu_*^{-1}(f_{M\cap W} X_M) \circ \Phi_{g^{-1}} = \mu_*^{-1}(f_{M\cap W} X_M).$$

Since $\mu_*^{-1}(f_{M\cap W} X_M)$ is compactly supported in a neighbourhood of $p = \mu^{-1}(x)$ in $\widetilde{M} \cap \Sigma$, it can be extended by zero to a vector field X_Σ on Σ. Note that X_Σ is G_p-invariant, since, for each $g \in G_p$ and $p' \in \Sigma$, either (6.1) $p' \in M \cap \Sigma$ or (6.2) $p' \notin M \cap \Sigma$. If (6.1) $p' \in M \cap \Sigma$, then

$$\begin{aligned} T\Phi_g \circ X_\Sigma \circ \Phi_{g-1}(p') &= T\Phi_g \circ \mu_*^{-1}(f_{M\cap W} X_M) \circ \Phi_{g-1}(p') \\ &= \mu_*^{-1}(f_{M\cap W} X_M)(p') = X_\Sigma(p'). \end{aligned}$$

If (6.2) $p' \notin M \cap \Sigma$ and $g^{-1}p' \in M \cap \Sigma$, then $G_{g^{-1}p'} = G_p$ and $g \in G_p$ implies that $g \in G_p = G_{g^{-1}p'}$ so that $g^{-1}p' = g(g^{-1}p') = p'$ and we have contradiction with the assumption that $p' \notin M \cap \Sigma$. Hence, $g^{-1}p' \notin M \cap \Sigma$. In this case $X_\Sigma(p') = X_\Sigma(g^{-1}p') = 0$, and

$$T\Phi_g \circ X_\Sigma \circ \Phi_{g-1}(p') = T\Phi_g \circ X_\Sigma(g^{-1}p') = T\Phi_g(0) = 0 = X_\Sigma(p').$$

In any case,

$$T\Phi_g \circ X_\Sigma(p') = X_\Sigma(gp')$$

for all $g \in G_p$ and $p' \in \Sigma$

We can extend X_Σ to a G-invariant vector field $X_{\widetilde{W}}$ on $\widetilde{W}$ by setting

$$X_{\widetilde{W}}(gp') = T\Phi_g(X_\Sigma(p'))$$

for every $g \in G$ and $p' \in \Sigma$. It is well defined since, if $gp' = g'p''$ for $p', p'' \in \Sigma$, then $g^{-1}g' \in G_p$ and

$$T\Phi_{g'}(X_\Sigma(p'')) = T\Phi_g(T\Phi_{g^{-1}g'}(X_\Sigma(p'')) = T\Phi_g(X_\Sigma(\Phi_{g^{-1}g'}(p'')).$$

If $p'' \in \Sigma \cap \widetilde{M}$, then

$$G_{p''} = G_p \quad \text{and} \quad \Phi_{g^{-1}g'}(p'') = g^{-1}g'p'' = p'' \quad \text{and} \quad X_{\widetilde{W}}(gp') = X_{\widetilde{W}}(g'p'').$$

If $p'' \notin \Sigma \cap \widetilde{M}$ and $g^{-1}g' \in G_p$, then

$$p' \notin \Sigma \cap \widetilde{M} \quad \text{and} \quad X_{\widetilde{W}}(gp') = 0 = X_{\widetilde{W}}(g'p'').$$

Finally, we can extend $X_{\widetilde{W}}$ to a G-invariant vector field X on P, by setting $X(p') = X_{\widetilde{W}}(p')$ for $p' \in \widetilde{W}$ and $X(p'') = 0$ for $p'' \notin \widetilde{W}$. Since X is G-invariant, it restricts to a derivation of $C^\infty(P)^G$ which is equivalent to a derivation of $C^\infty(R)$. This derivation is a vector field on R because it comes from a vector field on P.

Thus, the stratification of R by orbit type admits local extensions of vector fields. Since it is also minimal, Theorem 4.6 implies that it coincides with the partition of R by the family $\mathcal{O}$ of orbits of all vector fields on R. ■

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