SOME ISOMETRIC CHARACTERIZATIONS OF l_∞^n

by JIANG ZHU

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0. Introduction. The previous results on isometrically characterizing l_∞^n in terms of operator ideal norms can be summarized as follows.

Let E be an n-dimensional Banach space.

(1) If $\lambda(E)=1$, then $E\simeq l_\infty^n$ (see [**3**], [**5**], [**6**]), where $\lambda(E)$ is the projection constant of E (see [**4**]).

(2) If $\pi(E)=n$, then $E\simeq l_\infty^n$ (see [**1**], [**2**]).

(3) If $\Delta_2(E)=\sqrt{n}$, then $E\simeq l_\infty^n$ (see [**8**]), where $\Delta_2(E)$ is the 2-dominated constant of E (see [**4**]).

(4) If for any linear operator $T\colon l_1^{n+1}\to E$, $\nu_1(T)=\nu_1^{(n)}(T)$, then $E\simeq l_\infty^n$ (see [**7**]), where ν_1 is the 1-nuclear norm (see [**4**]).

In this paper we have the following theorems.

THEOREM 1. *Let $E=(K^n,\|\cdot\|_E)$ be an n-dimensional Banach space and let M be a positive integer. If $p_0\geq 2$ is such that $\pi_p(E)\leq M^{1/p}$ for all $p\geq p_0$, then E embeds isometrically into l_∞^M.*

COROLLARY. *If $\pi_p(E)=n^{1/p}$ for all $p\geq p_0$, then $E\simeq l_\infty^n$.*

THEOREM 2. *If $\pi_p(i_{2E})=n^{1/p}$ for all $p\geq p_0$, then $E\simeq l_\infty^n$, where $i_{2E}\colon(K^n,|\cdot|_2)\to(K^n,\|\cdot\|_E)$ is the John operator.*

Since $n^{1/p}\leq\pi_p(i_{2E})\leq\pi_p(E)$ for all E with $\dim(E)=n$, the Corollary also follows immediately from Theorem 2.

THEOREM 3. *If for any linear operator $T\colon l_\infty^{n+1}\to E$ we have $\pi_1(T)=\pi_1^{(n)}(T)$, then $E\simeq l_\infty^n$.*

This can be regarded as a dual result to (4).

1. Preliminaries. Let $E=(K^n,\|\cdot\|_E)$ be an n-dimensional Banach space, where $K=\mathbb{R}$ or $\mathbb{C}$. Let $l_p^n=(K^n,\|\cdot\|_p)$, where $\|x\|_p=(\sum_{i\leq n}|x(i)|^p)^{1/p}$ (for $1\leq p<\infty$), $\|x\|_\infty=\max_{i\leq n}|x(i)|$ We say that $E=(K^n,\|\cdot\|_E)$ and $F=(K^n,\|\cdot\|_F)$ are isometric and write $E\simeq F$ if there exists a linear operator T from E to F such that $\|T\|\,\|T^{-1}\|=1$. The John operator is the identity map $i_{2E}\colon(K^N,|\cdot|_2)\to(K^n,\|\cdot\|_E)$, where $(K^n,|\cdot|_2)$ is the Euclidean space whose unit ball has maximum volume among all ellipsoids contained in the unit ball of E.

For a linear operator $T\colon E\to F$, the p-summing norms ($p\geq 1$) are defined by

$$\pi_p^{(k)}(T)=\sup\left\{\left(\sum_{i\leq k}\|Tx_i\|^p\right)^{1/p}:x_1,\ldots,x_k\in E,\mu_p(x_1,\ldots,x_k)=1\right\}$$

and

$$\pi_p(T)=\sup_k\pi_p^{(k)}(T),$$

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where

$$\mu_p(x_1, \ldots, x_k) = \sup\left\{\left(\sum_{i\le k} |f(x_i)|^p\right)^{1/p} : f \in E^*, \|f\|_* = 1\right\}.$$

If E is finite dimensional, $\pi_p(E) := \pi_p(\mathrm{id}_E)$.

2. Proofs of Theorem 1 and Theorem 3. It is well known that any separable Banach space is isometric to a subspace of l_∞. Let $m \in N \cup \{\infty\}$ be minimal with the property that there exists an isometric embedding $i: E \to l_\infty^m$. Our aim is to show that $m \le M$ (to prove Theorem 1) and $m = n$ (to prove Theorem 3).

Let $e_1^*, \ldots, e_m^*$ be the standard dual basis of l_∞^m, and for $1 \le j \le m$ let $f_j = i^*(e_j^*)$. Then we have $\|x\|_E = \max\{|f_j(x)| : 1 \le j \le m\}$ for every $x \in E$. By the minimality of m, it is clear that none of the functionals f_j is a multiple of any other (in the case $m = \infty$, we choose a subsequence of $(f_j)_{j=1}^\infty$ such that it is a minimal norming set), and that, for each j, there exists a unit vector x_j such that f_j is its unique supporting functional.

Proof of Theorem 1. Now suppose that $\infty > m > M$. By a simple compactness argument, there exists a such that $0 < a < 1$ and $f(x_i) > a \Rightarrow f(x_j) \le a$ for $f \in E^*$ such that $\|f\|_{E^*} = 1$ and for $1 \le i \le m$, $1 \le j \le m$, $i \ne j$.

Then certainly

$$\mu_p(x_1, \ldots, x_m) \le (1 + (m-1)a^p)^{1/p}$$

and $\left(\sum_{i\le m} \|x_i\|^p\right)^{1/p} = m^{1/p}$. So we have

$$\pi_p^p(E) \ge \frac{m}{1 + (m-1)a^p},$$

which is strictly larger than M if p is large enough. If $m = \infty$, we choose $\infty > m' > M$, and use $x_1, \ldots, x_{m'}$ in the same way as before to show that $\pi_p^p(E) > M$ for p large enough.

This contradicts our assumption that $\pi_p(E) \le M^{1/p}$ for p sufficiently large. ■

Proof of Theorem 3. If $m > n$, define $T: l_\infty^{n+1} \to E$ by $Te_j = x_j$ for $j = 1, \ldots, n+1$. Then $\pi_1(T) = \sum_{j\le n+1} \|Te_j\| = \sum \|x_j\| = n+1$. Since $\pi_1(T) = \pi_1^{(n)}(T)$, there exist $y_1, \ldots, y_n \in l_\infty^{n+1}$ such that

$$\mu_1(y_1, \ldots, y_n) = \max\left\{\sum_{i\le n} |y_i(j)| : 1 \le j \le n+1\right\} = 1 \tag{1}$$

and

$$\sum_{i\le n} \|Ty_i\| = n+1. \tag{2}$$

Hence

$$\sum_{i\le n} \|Ty_i\| \le \sum_{i\le n} \left\| \sum_{j\le n+1} y_i(j) Te_j \right\| \le \sum_{i,j} |y_i(j)| \le n+1. \tag{3}$$

So (2) holds only when

$$\sum_{i\le n} |y_i(j)| = 1 \quad \text{for} \quad j = 1, \ldots, n+1 \tag{4}$$

and

$$\|Ty_i\| = \sum_{j\le n+1} |y_i(j)| \quad \text{for} \quad i = 1, \ldots, n. \tag{5}$$

By (4) there exist i_0, j_1, j_2 such that $1 \le i_0 \le n$, $1 \le j_1 \le j_2 \le n+1$ and $y_{i_0}(j_k) \ne 0$ for $k = 1$, 2. On the other hand, there exists $f \in E^*$ such that $\|f\|_{E^*} = 1$ and $f(Ty_{i_0}) = \|Ty_{i_0}\|$. So (5) implies that $|f(x_j)| = 1$ or $y_{i_0}(j) = 0$. But by the definition of x_j, $|f(x_{j_1})| < 1$ or $|f(x_{j_2})| < 1$. This contradicts (5). ■

3. Proof of Theorem 2. Without losing generality, we assume that $|\cdot|_2 = \|\cdot\|_2$. Then $\|\cdot\|_E \le \|\cdot\|_2 \le \|\cdot\|_{E^*}$. By John's theorem (see **[8]**), there exist $x_1, \ldots, x_N \in K^n$ and positive numbers $\lambda_1, \ldots, \lambda_N$, where $N \le n(n+1)/2$ in the real case, $N \leqslant n^2$ in the complex case, such that $\|x_i\|_E = \|x_i\|_2 = \|x_i\|_{E^*} = 1$, x_i is not a multiple of any other x_j, and

$$\sum_{i\le N} \lambda_i = n,$$

$$\sum_{i<N} \lambda_i x_i \otimes x_i = \mathrm{id}_{K^n}.$$

It is clear that there exist b_i such that $0 < b_i < 1$ for $i = 1, \ldots, N$ and,

$$\text{if } \|f\|_2 \le 1 \text{ and } |(x_i, f)| > b_i, \text{ then } |(x_i, f)| \le b_j \text{ for all } j \ne i.$$

The main step of the proof is the following claim.

Claim. *$N = n$ and $x_1, \ldots, x_n$ form an orthogonal basis in l_2^n.*

We want to show that $\lambda_i \ge 1$ for all i such that $1 \le i \le N$. Suppose that one can find an $i_0 \le N$ such that $0 < \lambda_{i_0} < 1$. Choose p large enough such that

$$b_{i_0}^p < \frac{(1-\lambda_{i_0})}{2n}.$$

Fix such p and choose a positive number c satisfying

$$\frac{1-\lambda_{i_0}}{2} < c^p < \min\left\{1, \frac{n(1-\lambda_{i_0})}{2(n-1)}\right\}.$$

Now we are in the position to estimate a lower bound of $\pi_p(i_{2E})$ by using the $N+1$ elements cx_{i_0}, $\lambda_1^{1/p}x_1, \ldots, \lambda_N^{1/p}x_N$. Denote them by $y_1, \ldots, y_{N+1}$.

Some simple computations show

$$\mu_2(y_1, \ldots, y_{N+1}) < \max\left\{1 + \frac{1-\lambda_{i_0}}{2n}, c^p + \frac{1+\lambda_{i_0}}{2}\right\}$$

in l_2^n. Meanwhile

$$\sum_{i\le N+1} \|y_i\|_E^p = c^p + n.$$

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Hence (note our choice of c)

$$\pi_p(i_{2E})^p \geq \min\left\{\frac{c^p+n}{(1-\lambda_{i_0})/2n+1}, \frac{c^p+n}{c^p+(1+\lambda_{i_0})/2}\right\} > n.$$

The contradiction implies the claim.

Without losing generality we can assume that $\{x_1, \ldots, x_n\}$ is the unit vector basis $\{e_1, \ldots, e_n\}$.

By Pietsch's theorem (see [4]), there exists a sequence $\{f_j\}_{j=1}^{\infty} \subseteq l_2^n$ such that for any $x \in E$

$$\|x\|_E^p \leq \sum_{j=1}^{\infty} |(x, f_j)|^p \tag{6}$$

and

$$\sum_{j=1}^{\infty} \|f_j\|_2^p \leq \pi_p(i_{2E})^p = n. \tag{7}$$

In (6) put $x = e_i$ for $i = 1, \ldots, n$. Then it is easy to see that each f_j must have the form $\beta_j e_k$. Put $A_i = \{f_j : f_j \text{ is of the form } \beta_j e_i\}$ for $i = 1, \ldots, n$. From (6) (for $x = e_i$) and (7), one has

$$\sum_{f_j \in A_i} |(e_i, f_j)|^p = 1$$

for $p \geq p_0$. Hence

$$\|x\|_E^p \leq \sum_{j=1}^{\infty} |(x, f_j)|^p \leq \sum_{i \leq n} \sum_{f_j \in A_i} |(x, f_j)|^p$$

$$= \sum_{i \leq n} |(x, e_i)|^p \sum_{f_j \in A_i} |(f_j, e_i)|^p = \|x\|_p^p$$

for all $p \geq p_0$. Letting $p \to \infty$, we see that $\|x\|_E \leq \|x\|_\infty$. But since $\|e_i\|_{E^*} = 1$, it follows that $\|x\|_\infty = \max_{i \leq n} |(x, e_i)| \leq \max_{i \leq n} \|x\|_E \|e_i\|_{E^*} = \|x\|_E$. So $\|\cdot\|_E = \|\cdot\|_\infty$. ■

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DEPARTMENT OF MATHEMATICS
LANCASTER UNIVERSITY
LANCASTER