Canad. Math. Bull. Vol. 62 (3), 2019 pp. 653–661
http://dx.doi.org/10.4153/S0008439518000693

Yamabe Solitons and Ricci Solitons on Almost co-Kähler Manifolds

Young Jin Suh and Uday Chand De

Abstract. The object of this paper is to study Yamabe solitons on almost co-Kähler manifolds as well as on (k,μ)-almost co-Kähler manifolds. We also study Ricci solitons on (k,μ)-almost co-Kähler manifolds.

1 Introduction

It is well known that a Riemannian metric g of an n-dimensional complete Riemannian manifold (M^n, g) is said to be a *Yamabe soliton* [19] if it satisfies

$$\pounds_V g = (\lambda - r)g \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a smooth vector field V on M^n, where r is the scalar curvature of g and $\pounds$ denotes the Lie-derivative operator. A Yamabe soliton is said to be shrinking, steady, or expanding according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively, and λ is said to be the soliton constant.

Given a smooth Riemannian manifold (M^n, g_0), the evolution of the metric g_0 in time t to $g = g(t)$ through the equation

$$\frac{\partial}{\partial t} g_t = -rg, g(0) = g_0$$

is known as the *Yamabe flow* (which was introduced by Hamilton [19]).

The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. Just as a Ricci soliton is a special solution of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by a one parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M, and homotheties, *i.e.*, $g(\cdot, t)) = \sigma(t)\phi_*(t)g_0$ (for more details, see [13, 29]).

Given a Yamabe soliton, if $V = Df$ holds for a smooth function f on M^n, equation (1.1) becomes

$$\text{Hess}\, f = \frac{1}{2}(\lambda - r)g,$$

Received by the editors June 8, 2018.
Published online on Cambridge Core May 7, 2019.
The first author was supported by the National Research Foundation of Korea, Grant Proj. No. NRF-2018-R1D1A1B-05040381.
AMS subject classification: **53C25**, 53C21, 53C44.
Keywords: contact manifold, Yamabe soliton, constant scalar curvature, (k,μ)-almost co-Kähler manifold, k-almost co-Kähler manifold.

where Hess f denotes the Hessian of f and D denotes the gradient operator of g on M^n. In this case f is called the *potential function* of the Yamabe soliton and g is said to be a *gradient Yamabe soliton*. A Yamabe soliton (resp. gradient Yamabe soliton) is said to be trivial when V is Killing (resp. f is constant).

Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [29]. Wang [30] also studied Yamabe solitons on a three-dimensional Kenmotsu manifold. In this paper, our aim is to study Yamabe solitons on almost co-Kähler manifolds and obtain some local classification theorems.

Now, we introduce some basic facts regarding Ricci solitons: A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci solitons according to [19]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where £ is the Lie derivative, and S is the Ricci tensor of type $(0, 2)$. Metrics satisfying (1.2) are interesting and useful in physics and are often referred to as *quasi-Einstein* [8, 9, 17]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equations of Ricci solitons in connection with string theory.

The Ricci soliton is said to be shrinking, steady, or expanding according to whether λ is negative, zero, or positive. If the vector field V is the gradient of a potential function $-f$, then g is called a *gradient Ricci soliton*, and equation (1.2) takes the form

$$\nabla\nabla f = S + \lambda g,$$

where ∇ denotes the Riemannian connection.

We also recall the following significant result of Perelman [28]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [19]), as well as in dimension 3 (Ivey [21]). For details, we refer the reader to Chow and Knopf [12] and Derdzinski [16]. Recently, C. Calin and M. Crasmareanu [4] studied Ricci solitons in f-Kenmotsu manifolds. Also, Bejan et al. [1] 'studied Ricci solitons in manifolds with quasi-constant curvature. In a recent paper, Wang [30] studied Ricci solitons with the potential vector fields pointwise collinear with the Reeb vector fields on K-almost co-Kähler manifolds.

From another point of view, we can state that the co-Kähler manifolds are really an odd-dimensional version of the Kähler manifolds (for more details, see Li [23]).

Co-Kähler manifolds have been studied by Wang [31], Cappelletti-Montano and Pastore [6], and many others.

In addition, a sufficient condition for a compact K-almost co-Kähler manifold with certain η-Einstein condition to be co-Kähler was presented in [6]. Yamabe solitons have been studied by several authors such as [15, 20, 29] and many others. Motivated by the above studies, one of our main aims in this paper is to study Yamabe solitons on almost co-Kähler manifolds as well as on (k, μ)-almost co-Kähler manifolds.

https://doi.org/10.4153/S0008439518000693 Published online by Cambridge University Press

This paper is organized as follows. In Section 2, after a brief introduction, we discuss some preliminaries that will be used in the later sections. In Section 3, we consider Yamabe solitons on almost co-Kähler manifolds and prove that if an almost co-Kähler manifold admits Yamabe soliton (g, ξ), then the manifold is K-almost co-Kähler. Section 4 is devoted to studying Yamabe solitons on (k, μ)-almost co-Kähler manifolds. Finally, we study Ricci solitons on (k, μ)-almost co-Kähler manifolds.

2 Preliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying ([2, 3])

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big)$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$, and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η)- structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y) = \Phi(X, Y),$$

for all X, Y tangent to M. The 1-form η is then a contact form, and ξ is its characteristic vector field. Also, Φ is known as the fundamental 2-form.

If ξ is a Killing vector field, then M^{2n+1} is said to be a K-contact manifold ([2, 3]). A contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three [22].

According to Blair [3], the normality of an almost contact structure is expressed by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Neijenhuis tensor of ϕ defined by

$$[\phi, \phi] = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields X, Y on M.

https://doi.org/10.4153/S0008439518000693 Published online by Cambridge University Press

In this paper, by an almost co-Kähler manifold, we mean an almost contact metric manifold such that both the 1-form η and the 2-form Φ are closed (see [3, 5]). In particular, an almost co-Kähler manifold is said to be a co-Kähler manifold if the associated almost contact structure is normal, which is also equivalent to $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$ (see [3]). It is well known that the Riemannian product of a real line and a (almost) Kähler manifold admits a (almost) co-Kähler structure. However, there exist some examples of (almost) co-Kähler manifolds that are not globally the product of a (almost) Kähler manifold and a real line (see, for example, Chinea et al. [10], Marrero and Padron [24], and Olszak [25, 26]).

On an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we shall set $h = \frac{1}{2}\pounds_\xi\phi$ and $h' = h \circ \phi$ (notice that both h and h' are symmetric operators with respect to the metric g). Then the following formulas can be found in Dacko [14], Endo [18], and Olszak [25, 26]:

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad \operatorname{tr} h = \operatorname{tr} h' \tag{2.3}$$

$$\nabla_\xi\phi = 0, \quad \nabla\xi = h', \quad \operatorname{div}\xi = 0 \tag{2.4}$$

$$S(\xi, \xi) + \|h\|^2 = 0.$$

Here tr and div denote the trace and divergence operators with respect to the metric g, respectively. The Ricci tensor S is defined by $S(X, Y) = \operatorname{tr}\{\cdot \to R(\cdot, X)Y\}$, and Q the Ricci operator defined by $g(QX, Y) = S(X, Y)$.

If, in addition, we put $l = R(\cdot, \xi)\xi$, then we also show

$$\phi l \phi - l = 2h^2, \tag{2.5}$$

where the Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

On an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, using the second term of (2.3), we obtain $(\pounds_\xi g)(X, Y) = 2g(h'X, Y)$. This means that the Reeb vector field ξ is Killing if and only if the $(1,1)$-type tensor field h vanishes.

Definition 2.1 An almost co-Kähler manifold is said to be a *K-almost co-Kähler manifold* if the Reeb vector field ξ is Killing.

We denote by $\mathcal{D}$ the distribution defined by $\mathcal{D} = \ker\eta$ on an almost co-Kähler manifolds. Then using relations (2.1), (2.2), and $d\Phi = 0$, one can define an almost K-Kähler structure $(g_\mathcal{D}, \phi_\mathcal{D})$ on D. According to Olszak [26], the associated almost Kähler structure is integrable if and only if

$$(\nabla_X\phi)Y = g(hX, Y)\xi - \eta(Y)hX$$

for any vector fields $X, Y \in \chi(M)$. This implies that an almost co-Kähler manifold is co-Kähler if and only if it is K-almost co-Kähler and the associated almost Kähler structure is integrable. Obviously, any three-dimensional almost co-Kähler manifold is co-Kähler if and only if it is K-almost co-Kähler.

3 Yamabe Solitons on Almost Co-Kähler Manifolds

Let us consider a Yamabe soliton that is of the type (M^{2n+1}, g, ξ) on an almost co-Kähler manifold, that is, $V = \xi$. From (1.1), we have

$$\pounds_V g = (\lambda - r)g. \tag{3.1}$$

Substituting $V = \xi$ in (3.1), we obtain

$$\pounds_\xi g = (\lambda - r)g. \tag{3.2}$$

If on a Riemannian m-manifold, $\pounds_X g = \rho g$, then $\operatorname{div}(\xi) = \rho \frac{m}{2}$, and so $\operatorname{div} \xi = 0$ forces to $\rho = 0$. Therefore, in our case $\pounds_\xi g = 0$. Thus, ξ is a Killing vector field. Thus, an almost co-Kähler manifold becomes a K-almost co-Kähler manifold. In view of the above, we can state the following theorem.

Theorem 3.1 *If an almost co-Kähler manifold admits Yamabe soliton (g, ξ), then the manifold is K-almost co-Kähler manifold.*

It is known that any three-dimensional almost co-Kähler manifold is co-Kähler if and only if it is K-almost co-Kähler [25]. Thus, we obtain the following corollary.

Corollary 3.2 *If an almost co-Kähler manifold $(M^3, \phi, \xi, \eta, g)$ admits a Yamabe soliton (g, ξ), then it is a co-Kähler manifold.*

Let us assume that V is pointwise collinear with the Reeb vector field, that is, $V = b\xi$, where b is non-zero smooth function. Using (2.4) we have $\nabla_X V = X(b)\xi + bh'X$ for any $X \in \chi(M)$. Thus, it follows from (3.2) that

$$g(\nabla_Y b\xi, Z) + g(\nabla_Z b\xi, Y) = (\lambda - r)g(Y, Z).$$

This implies that

$$X(b)\eta(Y) + Y(b)\eta(X) + 2g(h'X, Y) = (\lambda - r)g(X, Y). \tag{3.3}$$

Next we shall consider a local ϕ-basis $\{e_i : 1 \le i \le 2n+1\}$ on M^{2n+1} on the tangent space T_pM for each point p in M^{2n+1}. Substituting $X = Y = e_i$ in (3.3) and summing over $i, 1 \le i \le 2n+1$, we obtain

$$\xi(b) = \left(n + \frac{1}{2}\right)(\lambda - r). \tag{3.4}$$

Putting $Y = \xi$ in (3.3) we have

$$X(b) = \left(\frac{1}{2} - n\right)(\lambda - r)\eta(X).$$

Thus,

$$\xi(b) = \left(\frac{1}{2} - n\right)(\lambda - r). \tag{3.5}$$

In view of (3.4) and (3.5) , we have

$$r = \lambda.$$

https://doi.org/10.4153/S0008439518000693 Published online by Cambridge University Press

Therefore the scalar curvature r is constant, since λ is constant. Thus, for Yamabe solitons of the type $(M^{2n+1}, g, b\xi)$ on almost co-Kähler manifolds, the scalar curvature r is constant. Again putting $\lambda = r$ in (3.1), we have $\pounds_{b\xi} g = 0$. Thus, $V = b\xi$ is a Killing vector field. Therefore the soliton is trivial. In view of the above we can state the following theorem.

Theorem 3.3 *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost co-Kähler manifold. If the metric g is a Yamabe soliton and the vector field V is a non zero pointwise collinear with the Reeb vector field ξ, then the soliton is trivial.*

4 (k,μ)-almost Co-Kähler Manifolds and Yamabe Solitons

By a (k,μ)-almost co-Kähler manifold we mean an almost co-Kähler manifold such that the Reeb vector filed ξ belongs to the generalized (k,μ)-nullity distribution, that is,

$$R(X,Y)\xi = k\big[\eta(Y)X - \eta(X)Y\big] + \mu\big[\eta(Y)hX - \eta(X)hY\big] \tag{4.1}$$

for any vector fields X, Y in $\chi(M)$ and some smooth functions k and μ. In this paper, a (k,μ)-almost co-Kähler manifold with $k < 0$ will be called a proper (k,μ)-almost coKähler manifold or a non-coKähler (k,μ)-almost co-Kähler manifold. Such manifolds with both k and μ being constants were first introduced by Endo [18] and were generalized to (k,μ,ν)-spaces by Dacko and Olszak in [14] (see also Carriazo and Martin-Molina [7] and [27]). Using (4.1), we have $l = -k\phi^2 + \mu h$, and putting this into (2.5) gives that $h^2 = k\phi^2$.

Clearly, M^{2n+1} is K-almost co-Kähler if and only if $k = 0$. According to [14], under certain D-homothetic deformation, any (k,μ,ν)-almost co-Kähler manifold with $k < 0$ turns out to be a $(-1, \frac{\mu}{\sqrt{-k}})$-space.

Now we state the following lemmas, which will be used in the next theorem.

Lemma 4.1 ([31]) *Let M^{2n+1} be a (k,μ)-almost co-Kähler manifold of dimension greater than 3 with $k < 0$. Then the Ricci operator is given by*

$$Q = \mu h + 2nk\eta \otimes \xi, \tag{4.2}$$

where k is a non-zero constant and μ is a smooth function satisfying $d\mu \wedge \eta$.

Lemma 4.2 (Yano [32]) *On an n-dimensional Riemannian manifold (M^n, g) endowed with a conformal vector field V, we have*

$$(\pounds_V S)(X,Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X,Y),$$
$$\pounds_V r = -2\rho r + 2(n-1)\Delta\rho$$

for any vector fields X and Y, where D denotes the gradient operator and $\Delta := -\operatorname{div} D$ denotes the Laplacian operator of g.

For a Yamabe soliton the vector field V is a conformal vector field, that is,

$$\pounds_V g = 2\rho g,$$

https://doi.org/10.4153/S0008439518000693 Published online by Cambridge University Press

where ρ is called the *conformal coefficient* (in this case by relation (1.1) we have $\rho = \frac{\lambda - r}{2}$). In particular, a conformal vector field with a vanishing conformal coefficient reduces to a Killing vector field.

Notice that the Reeb vector field ξ is a unit vector field, that is, $g(\xi, \xi) = 1$. Taking the Lie-derivative of this relation along the vector field V and using $\eta(\xi) = 1$ and (1.1), we obtain

$$\eta(\pounds_V \xi) = -(\pounds_V \eta)(\xi) = \frac{r - \lambda}{2}.$$

As the Riemannian metric g in (k, μ)-almost co-Kähler manifolds is a Yamabe soliton, applying $\rho = \frac{\lambda - r}{2}$ and Lemma 4.1, we have

$$(\pounds_V S)(X, Y) = -(2n - 1)g(\nabla_X Dr, Y) - \frac{1}{2}(\Delta r)g(X, Y).$$

$$\pounds_V r = r(r - \lambda) - 2n\Delta r \tag{4.3}$$

for any vector fields X, Y. On the other hand, from (4.2), we have

$$S(X, Y) = \mu g(hX, Y) + 2nk\eta(X)\eta(Y). \tag{4.4}$$

Consider a local ϕ-basis $\{e_i : 1 \le i \le 2n + 1\}$ on M^{2n+1} on the tangent space $T_p M$ for each point p in M^{2n+1}. Substituting $X = Y = e_i$ in (4.4) and summing over i, $1 \le i \le 2n + 1$ we obtain $r = 2nk$. Hence the scalar curvature r is constant. Thus, from (4.3), we get either $r = 0$ or $r = \lambda$. Since $k < 0$, $r = \lambda = 2nk < 0$. Therefore, $r = \lambda$, and hence from (1.1), $\pounds_V g = 0$. Thus, the V is Killing. This implies the soliton is trivial. In view of the above result, we can state the following theorem.

Theorem 4.3 *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a (k, μ)-almost co-Kähler manifold. If the metric g is a Yamabe soliton, then the soliton is trivial and expanding.*

5 (k, μ)-almost Co-Kähler Manifolds and Ricci Soliton

This section is devoted to studying Ricci solitons on (k, μ)-almost co-Kähler manifold. Therefore,

$$(\pounds_V g)(X, Y) + +2S(X, Y) + 2\lambda g(X, Y) = 0,$$

for all smooth vector fields X and Y. This implies

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$

Substituting $V = \xi$ in the above equation we have

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$

In virtue of (2.4), (4.2), we obtain from the above equation

$$2g(h'X, Y) + 2\mu g(hX, Y) + 4nk\eta(X)\eta(Y) + 2\lambda g(X, Y) = 0. \tag{5.1}$$

Putting $Y = \xi$ in (5.1) we have

$$\lambda = -2nk.$$

Since $k < 0$, $\lambda = -2nk > 0$. Thus, in view of the above result, we can state the following theorem.

Theorem 5.1 *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a (k, μ)-almost co-Kähler manifold. If the metric g is a Ricci soliton, then the soliton is expanding.*

Remark In a recent paper, Cho [11] proved that an almost cosymplectic manifold M admits Ricci soliton (g, ξ) if and only if ξ is Killing and M is Ricci flat. Since an almost cosymplectic manifold and an almost co-Kähler manifold are the same, an almost co-Kähler manifold becomes K-almost co-Kähler manifold if the Reeb vector field ξ is Killing. Hence, if an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits Ricci soliton (g, ξ), then it is a K-almost co-Kähler manifold.

References

[1] C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi-constant curvature*. Publ. Math. Debrecen 78(2011), 235–243. https://doi.org/10.5486/PMD.2011.4797.

[2] D. E. Blair, *Contact manifold in Riemannian geometry*. Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.

[3] D. E. Blair, *Riemannian geometry on contact and symplectic manifolds*. Progress in Mathematics, 203, Birkhäuser Boston, Inc, Boston, MA, 2002. https://doi.org/10.1007/978-1-4757-3604-5.

[4] C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*. Bull. Malays. Math. Sci. Soc. 33(2010), 361–368.

[5] B. Cappelletti-Montano, A. D. Nicola, and I. Yudin, *A survey on cosymplectic geometry*. Rev. Math. Phys. 25(2013), 1343002 (2013). https://doi.org/10.1142/S0129055X13430022.

[6] B. Cappelletti-Montano and A. M. Pastore, *Einstein-like conditions and cosymplectic geometry*. J. Adv. Math. Stud. 3(2010), 27–40.

[7] A. Carriazo and V. Martin-Molina, *Almost cosymplectic and almost Kenmotsu (k, μ, ν)-spaces*. Mediterr. J. Math. 10(2013), 1551–1571. https://doi.org/10.1007/s00009-013-0246-4.

[8] T. Chave and G. Valent, *Quasi-Einstein metrics and their renoirmalizability properties*. Helv. Phys. Acta. 69(1996), 344–347.

[9] T. Chave and G. Valent, *On a class of compact and non-compact quasi-Einstein metrics and their renoirmalizability properties*. Nuclear Phys. B. 478(1996), 758–778. https://doi.org/10.1016/0550-3213(96)00341-0.

[10] D. Chinea, M. de Leon, and J. C. Marrero, *Topology of cosymplectic manifolds*. J. Math. Pures Appl. 72(1993), 567–591.

[11] J. T. Cho, *Ricci solitons on almost contact geometry*. Proceedings of 17th International workshop on Differential Geometry and the 7th KNUGRG-OCAMI Differential Geometry Workshop [Vol. 17], Natl. Inst. Math. Sci. (NIMS), Taejon, 2013, pp. 85–95.

[12] B. Chow and D. Knopf, *The Ricci flow: An introduction*. Mathematical Surveys and Monographs, 110, American Mathematical Society, Providence, RI, 2004.

[13] B. Chow, P. Lu, and L. Ni, *Hamilton Ricci flow*. Graduate Studies in Mathematics, 77, American Mathematical Society, Providence, RI; Science Press, Beijing, New York, 2006. https://doi.org/10.1090/gsm/077.

[14] P. Dacko and Z. Olszak, *On almost cosymplectic (k, μ)-space*. Banach Center Publ. 69, Polish Acad. Sci. Inst. Math., Warsaw, 2005, pp. 211–220. https://doi.org/10.4064/bc69-0-17.

[15] P. Daskalopoulos and N. Sesum, *The classification of locally conformally flat Yamabe solitons*. Adv. Math. 240(2013), 346–369. https://doi.org/10.1016/j.aim.2013.03.011.

[16] A. Derdzinski, A., *Compact Ricci solitons*. (Polish) Wiad Mat. 48(2012), no. 1, 1–32.

[17] S. Deshmukh, *Jacobi-type vector fields on Ricci solitons*. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 55(2012), 41–50.

[18] H. Endo, *Non-existence of almost cosymplectic manifolds satisfying a certain condition*. Tensor (N. S.) 63(2002), 272–284.

[19] R. S. Hamilton, *The Ricci flow on surfaces*. In: *Mathematics and general relativity*, Contemp. Math., 71, American Mathematica Society, Providence, RI, 1988, pp. 237–262. https://doi.org/10.1090/conm/071/954419.

[20] S.-Y. Hsu, *A note on compact gradient Yamabe solitons*. J. Math. Anal. Appl. 388(2012), 725–726. https://doi.org/10.1016/j.jmaa.2011.09.062.
[21] T. Ivey, *Ricci solitons on compact 3-manifolds*. Differential Geom. Appl. 3(1993), 301–307. https://doi.org/10.1016/0926-2245(93)90008-O.
[22] J. B. Jun and U. K. Kim, *On* 3*-dimentional almost contact metric manifold*. Kyungpook Math. J. 34(1994), 293–301.
[23] H. Li, H., *Topology of co-symplectic/co-Kähler manifolds*. Asian J. Math. 12(2008), 527–543. https://doi.org/10.4310/AJM.2008.v12.n4.a7.
[24] J. C. Marrero and E. Padron, *New examples of compact cosymplectic solvmanifolds*. Arch. Math. (Brno) 34(1998), 337–345.
[25] Z. Olszak, *On almost cosymplectic manifolds*. Kodai Math. J. 4(1981), 239–250.
[26] Z. Olszak, *On almost cosymplectic manifolds with Kählerian leaves*. Tensor (N. S.) 46(1987), 117–124.
[27] H. Oztürk, H., N. Aktan, C. and Murathan, *Almost α-cosymplectic (k,μ,ν)-spaces*. 2010. arxiv:1007.0527v1.
[28] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. arxiv:Math.DG/0211159.
[29] R. Sharma, *A 3-dimensional Sasakian metric as a Yamabe Soliton*. Int. J. Geom. Methods Mod. Phys. 9(2012), 1220003. https://doi.org/10.1142/S0219887812200034.
[30] Y. Wang, *Yamabe solitons in three dimensional Kenmotsu manifolds*. Bull. Belg. Math. Soc. Stenvin 23(2016), 345–355.
[31] Y. Wang, *A generalization of Goldberg conjecture for co-Kähler manifolds*. Mediterr. J. Math. 13(2016), 2679–2690.
[32] K. Yano, *Integral Formulas in Riemannian geometry*. Pure and Applied Mathematics, 1, Marcel Dekker, New York, 1970.

Department of Mathematics, Kyungpook National University, Taegu 702-701, South Korea
e-mail: yjsuh@knu.ac.kr

Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata - 700019, West Bengal, India
e-mail: uc_de@yahoo.com