CERTAIN FOURIER TRANSFORMS OF DISTRIBUTIONS: II

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1. Introduction. In an earlier paper **(1)**, published in this journal, a necessary condition was given which the reciprocal of a polynomial without multiple roots must satisfy in order to be a characteristic function. This condition is, however, valid for a wider class of functions since it can be shown **(2**, theorem 2 and corollary to theorem 3**)** that it holds for all analytic characteristic functions. The proof given in **(1)** is elementary and has some methodological interest since it avoids the use of theorems on singularities of Laplace transforms. Moreover the method used in **(1)** yields some additional necessary conditions which were not given in **(1)** and which do not seem to follow easily from the properties of analytic characteristic functions. The purpose of this note is to supplement **(1)** by establishing these conditions and by deriving the results for rational functions which need not be the reciprocals of polynomials without multiple zeros. It should be remarked that Takano **(3)** extended the conditions of **(1)** to the reciprocals of polynomials having multiple roots.

2. Necessary conditions for rational characteristic functions. In this note we derive the following

THEOREM. *If the rational function $\phi(t)$ is a characteristic function then the following three conditions are satisfied:*

(a) *$\phi(t)$ has no poles on the real axis. All the poles and zeros of $\phi(t)$ are either all located on the imaginary axis or they occur in pairs $\pm b + ia$, symmetric with regard to this axis. If $\phi(t)$ is written as the quotient of two polynomials then the degree of its numerator can not exceed the degree of its denominator.*

(b) *If $b + ia$ (a, b real, $a \neq 0$, $b \neq 0$) is a pole of $\phi(t)$ then $\phi(t)$ has at least one pole $i\alpha$ (α real) such that $\operatorname{sgn} \alpha = \operatorname{sgn} a$ and $|\alpha| \leqslant |a|$.*

(c) *Let a be the imaginary part of the poles of $\phi(t)$ which are closest to the real axis in the upper (lower) half plane. If $\phi(t)$ has not only the pole ia but also poles $\pm b + ia$ (a, b real $a \neq 0$, $b \neq 0$) then*

(i) *no pole of the form $b + ia$ can have a higher multiplicity than the pole ia*

(ii) *if the only poles of $\phi(t)$ with imaginary part a are ia and $\pm b + ia$ and if these poles have equal multiplicity s we have*

$$A_{1,s}\, a^s - 2|C_{1,s}|\, (a^2 + b^2)^{\frac{1}{2}s} \geqslant 0.$$

Here $A_{1,s}$ is the coefficient of $(1 - it/a)^{-s}$ and $C_{1,s}$ is the coefficient of $(1 - it/(a + ib))^{-s}$, in the expansion of $\phi(t)$ into partial fractions.

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Conditions (a) and (b) correspond to conditions (a) and (b) of **(1)**; the proof is practically identical with the proof given in the earlier paper so that we have now to establish only (c). We shall therefore assume from now on that $\phi(t)$ satisfies (a) and (b). We can then write

$$\phi(t) = \frac{P(it)}{Q(it)} \tag{1}$$

where t is a real variable while $P(z)$ and $Q(z)$ are polynomials in z with real coefficients.

3. Proof of condition (c). We divide the zeros of $Q(it)$ into four groups:
(i) zeros $i\beta_h$ $(h = 1, \ldots , \mu)$ on the positive imaginary axis;
(ii) zeros $-i\alpha_j$ $(j = 1, \ldots , \nu)$ on the negative imaginary axis;
(iii) p pairs of complex zeros in the upper half plane iw_k and $i\bar{w}_k$ where $w_k = c_k + id_k$ $(k = 1, \ldots , p)$;
(iv) n pairs of complex roots in the lower half plane $-iv_m$ and $-i\bar{v}_m$ where $v_m = a_m + ib_m$ $(m = 1, \ldots , n)$.
The quantities $a_m, b_m, c_k, d_k, \alpha_j, \beta_h$ are positive numbers. We denote by q_m, ρ_j, r_k, σ_h the multiplicities of the zeros $v_m, \alpha_j, w_k, \beta_h$ respectively. It is sufficient to prove (c) only for one of the half planes. (See the argument used in **(1)**.) Therefore it is no restriction to suppose that $Q(it)$ has zeros in the lower half plane. The assumptions of condition (c) imply then that $\nu > 0$ and $n > 0$. According to condition (a) of our theorem, the degree of $P(z)$ cannot exceed the degree of $Q(z)$; we obtain therefore from (1) the decomposition of $\phi(t)$ into partial fractions

$$\begin{aligned} \phi(t) = {} & \sum_{j=1}^{\nu} \sum_{\lambda=1}^{\rho_j} \frac{A_{j,\lambda}}{(1 - it/\alpha_j)^{\lambda}} + \sum_{m=1}^{n} \sum_{\lambda=1}^{q_m} \left[\frac{C_{m,\lambda}}{(1 - it/v_m)^{\lambda}} + \frac{\bar{C}_{m,\lambda}}{(1 - it/\bar{v}_m)^{\lambda}} \right] \\ & + \sum_{h=1}^{\mu} \sum_{\lambda=1}^{\sigma_h} \frac{B_{h,\lambda}}{(1 + it/\beta_h)^{\lambda}} + \sum_{k=1}^{p} \sum_{\lambda=1}^{r_k} \left[\frac{D_{k,\lambda}}{(1 + it/w_k)^{\lambda}} + \frac{\bar{D}_{k,\lambda}}{(1 + it/\bar{w}_k)^{\lambda}} \right]. \end{aligned} \tag{2}$$

If either μ or p should be equal to zero, then the corresponding sum is omitted. From formulae (3.11), (3.12), (3.21), (3.22) of our earlier paper we can compute the frequency function

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \phi(t)\, dt.$$

We see that, for $x > 0$,

$$\begin{aligned} f(x) = {} & \sum_{j=1}^{\nu} \sum_{\lambda=1}^{\rho_j} \frac{A_{j,\lambda}\, \alpha_j^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha_j x} \\ & + \sum_{m=1}^{n} e^{-xa_m} \sum_{\lambda=1}^{q_m} \frac{x^{\lambda-1}}{\Gamma(\lambda)} [C_{m,\lambda}\, v_m^{\lambda}\, e^{-ixb_m} + \bar{C}_{m,\lambda}\, \bar{v}_m^{\lambda}\, e^{ixb_m}]. \end{aligned} \tag{3}$$

A similar expression can be found for $f(x)$ if $x < 0$ and $\mu \neq 0$ or $p \neq 0$ but is not needed in the sequel. If $\mu = p = 0$ then $f(x) = 0$ for $x < 0$.

We choose our notation so that

$$
\begin{aligned}
&a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n, \\
&\alpha_1 < \alpha_2 < \ldots < \alpha_\nu.
\end{aligned} \tag{4.1}
$$

We denote by N the greatest integer such that $a_j = a_1$ for $j = 1, 2, \ldots, N$. Clearly $1 \leqslant N \leqslant n$ and we have $a_N < a_{N+1}$ in case $N < n$. It is convenient to put

$$t = \max\,[q_1, q_2, \ldots q_N]. \tag{4.2}$$

We can then write (3) for large positive values of x as

$$f(x) = e^{-x\alpha_1} x^{\rho_1 - 1}\left[\frac{A_{1,\rho_1}\alpha_1^{\rho_1}}{\Gamma(\rho_1)} + o(1)\right] + e^{-xa_1} x^{t-1}[h(x) + o(1)] \tag{5}$$

where

$$h(x) = \sum_{m=1}^{N} \frac{1}{\Gamma(t)}\,[C_{m,t}\, v_m^t\, e^{-ixb_m} + \bar{C}_{m,t}\,\bar{v}_m^t\, e^{ixb_m}]. \tag{5.1}$$

The summation is here to be extended over all integers m $(1 \leqslant m \leqslant N)$ for which $q_m = t$.

Since we assume that conditions (a) and (b) are satisfied we have $\alpha_1 \leqslant a_1$. We write

$$s = \begin{cases} \rho_1 & \text{if } \alpha_1 < a_1, \\ \max\,[\rho_1, t] & \text{if } \alpha_1 = a_1. \end{cases} \tag{6.1}$$

We see then easily from (5) and (5.1) that for large positive values of x,

$$f(x) = e^{-x\alpha_1} x^{s-1}\,[g(x) + o(1)], \tag{7}$$

where

$$g(x) = \begin{cases} \dfrac{A_{1,\rho_1}\alpha_1^{\rho_1}}{\Gamma(\rho_1)} & \text{if } \alpha_1 < a_1 \text{ or if } \alpha_1 = a_1 \text{ and } t < \rho_1 = s, \\ h(x) & \text{if } \alpha_1 = a_1 \text{ and } \rho_1 < t = s, \\ \dfrac{A_{1,s}\alpha_1^{s}}{\Gamma(s)} + h(x) & \text{if } \alpha_1 = a_1 \text{ and } \rho_1 = t = s. \end{cases} \tag{7.1}$$

We are now ready to establish condition (c). Let us first assume that $a_1 = \alpha_1$ and $\rho_1 < t$ then

$$f(x) = e^{-x\alpha_1} x^{s-1}\,[h(x) + o(1)].$$

It is seen from (5.1) that $h(x)$ is an almost periodic function without constant term; according to the lemma given in **(1)**, $h(x)$ assumes also negative values. From the almost periodicity of $h(x)$ it follows easily that $f(x)$ must assume negative values for some sufficiently large values of x. This proves part (i) of condition (c).

We next consider the case $a_1 = \alpha_1$ and $\rho_1 = t = s$. Since the function

$$\frac{A_{1,s}\alpha_1^{s}}{\Gamma(s)} + h(x)$$

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is almost periodic we conclude from (5.1), (7) and (7.1) that $f(x)$ is nonnegative for positive x if and only if

$$A_{1,s}\alpha_1^s + \sum_{m=1}^{N} [C_{m,s} v_m^s e^{-ixb_m} + \bar{C}_{m,s} \bar{v}_m^s e^{ixb_m}] \geqslant 0 \tag{8.1}$$

for all $x \geqslant 0$ (summation extended over integers m for which $q_m = s$).

If we write $C_{m,s} v_m^s = R_{m,s} [\cos \theta_{m,s} + i \sin \theta_{m,s}]$ we obtain easily from (8.1)

$$A_{1,s}\alpha_1^s + \sum_{m=1}^{N} 2 R_{m,s} \cos (\theta_{m,s} - b_m x) \geqslant 0 \qquad \text{for } x \geqslant 0. \tag{8.2}$$

If in particular $N = 1$ this reduces to

$$A_{1,s}\alpha_1^s + 2 R_{1,s} \cos (\theta_{1,s} - b_1 x) \geqslant 0 \qquad \text{for } x \geqslant 0,$$

and therefore

$$A_{1,s}\alpha_1^s - 2 R_{1,s} \geqslant 0. \tag{8.3}$$

If we write here for $R_{1,s} = |C_{1,s} v_1^s|$ and for $\alpha_1 = a$ and for $v_1 = ia + b$ we obtain the statement (ii) of condition (c).

It is worthwhile to remark that the above reasoning would still hold for certain irrational meromorphic functions.

References

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