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# CENTRAL $*$-DIFFERENTIAL IDENTITIES IN PRIME RINGS

P. H. LEE AND T. L. WONG

ABSTRACT. Let $R$ be a prime ring with involution and $d$, $\delta$ be derivations on $R$. Suppose that $xd(x)-\delta(x)x$ is central for all symmetric $x$ or for all skew $x$. Then $d=\delta=0$ unless $R$ is a commutative integral domain or an order of a 4-dimensional central simple algebra.

It was shown in [1] that if $R$ is a prime ring and $d$, $\delta$ are two derivations of $R$ such that $xd(x)-\delta(x)x$ lies in the center of $R$ for all $x \in R$, then either $d=\delta=0$ or $R$ is commutative. In this paper we are concerned with a similar problem in the setting of rings with involution. Let $R$ be a prime ring with an involution $*$. Suppose that $d$ and $\delta$ are derivations of $R$ such that $xd(x)-\delta(x)x$ is central for all $x=x^*$ or for all $x=-x^*$. Here we show that $d=\delta=0$ unless $R$ is a commutative integral domain or an order of a 4-dimensional central simple algebra. This extends the results in [7] where the same conclusions were proved under the additional assumption $d=\delta$.

In what follows, $R$ will always denote a prime ring with an involution $*$ and $Z$ the center of $R$. $S=\{x \in R \mid x^*=x\}$ is the set of symmetric elements in $R$ and $K=\{x\in R \mid x^*=-x\}$ the set of skew elements. Let $d$ and $\delta$ denote two derivations of $R$. We are going to show that $R$ satisfies the standard identity $s_4=\sum_{\sigma\in S_4}(-1)^\sigma X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)}X_{\sigma(4)}$ provided $d\neq 0$ or $\delta\neq 0$. Let $C$ stand for the extended centroid of $R$ and $\bar{C}$ the algebraic closure of $C$. $RC$ is the central closure of $R$ and $R$ is called *centrally closed* if $RC=R$. For subsets $A$ and $B$, $[A,B]$ will denote the additive subgroup generated by elements of the form $[a,b]=ab-ba$ with $a\in A$ and $b\in B$. The involution $*$ on $R$ can be extended to an involution on $RC$ [4, Lemma 2.4.1] which will also be denoted by $*$. The involution $*$ is said to be *of the first kind* if $\alpha^*=\alpha$ for all $\alpha\in C$ and *of the second kind* otherwise. We begin with a well-known

LEMMA. *If $d(S)\subseteq Z$ or $d(K)\subseteq Z$, then either $d=0$ or $R$ satisfies $s_4$.*

PROOF. Assume that $d\neq 0$. If char $R\neq 2$, then $R$ satisfies $s_4$ by [6, Lemma 5 and Corollary] or [8, Lemma 1.6]. Hence, assume that char $R=2$ and then $K=S$ in this case. For $s\in S$, we have $d(s^2)=2sd(s)=0$. Thus, $0=s^2d(s^2x+x^*s^2)+d(s^2x+x^*s^2)s^2=s^4d(x)+s^2d(x+x^*)s^2+d(x^*)s^4=s^4d(x)+d(x+x^*)s^4+d(x^*)s^4=s^4d(x)+d(x)s^4$ for all $x\in R$. That is, $[s^4,d(R)]=0$ and so $s^8\in Z$ by a theorem due to Herstein [5]. Therefore, $R$ satisfies $s_4$ by [7, Thm.3].

Now we prove a symmetric version of Brešar's Theorem.

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THEOREM 1. *If $sd(s) - \delta(s)s \in Z$ for all $s \in S$, then either $d = \delta = 0$ or $R$ satisfies $s_4$.*

PROOF. Linearize the relation $sd(s)-\delta(s)s \in Z$ to obtain $sd(t)-\delta(t)s+td(s)-\delta(s)t \in Z$ for all $s$, $t$ in $S$. Replacing $t$ with $[s, k]$ for $k \in K$ and using $sd(s) - \delta(s)s \in Z$, we have $s[s, d(k)] - [s, \delta(k)]s \in Z$ for all $s \in S$ and $k \in K$. Thus, for each $k \in K$, the inner derivations $D_k$ and $\Delta_k$, defined by $D_k(x) = [x, d(k)]$ and $\Delta_k(x) = [x, \delta(k)]$, satisfy $sD_k(s) - \Delta_k(s)s \in Z$ for all $s \in S$. Suppose that the theorem has been proved for inner derivations; then we can conclude that either $D_k = \Delta_k = 0$ for each $k \in K$ or $R$ satisfies $s_4$. In the former case, we have $d(K) \subseteq Z$ and $\delta(K) \subseteq Z$ whence either $d = \delta = 0$ or $R$ satisfies $s_4$ by the Lemma. So it suffices to consider the situation when $d(x) = [x, a]$ and $\delta(x) = [x, b]$ for some fixed elements $a$, $b$ in $R$.

Assume first that $Z \cap S \neq 0$, that is, there exists $\alpha \in Z$ with $\alpha^* = \alpha \neq 0$. From $sd(\alpha) - \delta(\alpha)s + \alpha d(s) - \delta(s)\alpha \in Z$, it follows that $d(s) - \delta(s) \in Z$ for all $s \in S$ since $d(\alpha) = \delta(\alpha) = 0$. Again, by the Lemma, either $d = \delta$ or $R$ satisfies $s_4$. But if $d = \delta$, we are done by [7, Thm.1 and Thm.5]. So assume that $Z \cap S = 0$ from which $Z = 0$ follows. Thus $s[s, a] - [s, b]s = 0$ for all $s \in S$. We assume that $a$ and $b$ are not both zero and proceed to show that $R$ satisfies $s_4$. Applying $*$ to $s[s, a] - [s, b]s = 0$, we obtain that $s[s, b^*] - [s, a^*]s = 0$ and so both $s[s, a + b^*] - [s, b + a^*]s = 0$ and $s[s, a - b^*] - [s, b - a^*]s = 0$ for all $s \in S$. Since $a \neq 0$ or $b \neq 0$, $a + b^*$ and $a - b^*$ cannot be both zero in case char $R \neq 2$, and so we may replace $a$ with $a + b^*$ or $a - b^*$ and assume that $b = a^*$ or $b = -a^*$ respectively. In case char $R = 2$, we may still replace $a$ with $a + b^*$ if $a + b^* \neq 0$, while if $a + b^* = 0$, we have $b = a^*$ already. Hence, we assume that $b = a^*$ or $b = -a^*$. Also, we may assume that $b \neq a$.

Let $f(X, Y) = (X + Y)[X + Y, a] - [X + Y, b](X + Y)$. Then $f(X, Y)$ is a nontrivial generalized polynomial identity (GPI) and $R$ satisfies the $*$-GPI $f(X, X^*) = 0$. Since $sd(t) - \delta(t)s + td(s) - \delta(s)t = 0$ for all $s$, $t \in S$, replacing $t$ with $s^2$ yields $2s^2d(s) + sd(s)s - s\delta(s)s - 2\delta(s)s^2 = 0$. But $s^2d(s) = s\delta(s)s$ and $\delta(s)s^2 = sd(s)s$, so we have $sd(s)s = s\delta(s)s$ or, equivalently, $s[s, c]s = 0$ for all $s \in S$ where $c = a - b \neq 0$. Set $g(X, Y) = (X + Y)[X + Y, c](X + Y)$. Then $R$ satisfies the nontrivial $*$-GPI $g(X, X^*) = 0$. In light of [2, Prop.4], $RC$ also satisfies both $*$-GPIs $f(X, X^*) = 0$ and $g(X, X^*) = 0$. If $C$ is infinite and $*$ is of the second kind, $R$ satisfies $f(X, Y) = 0$ by [2, Prop.1]. In particular, $x[x, a] - [x, b]x = 0$ for all $x \in R$ and so $a \in Z = 0$ by Brešar's Theorem [1, Thm.4.1], a contradiction. If $C$ is infinite and $*$ is of the first kind, $*$ can be extended to $RC \otimes_C \bar{C}$ and standard arguments show that both $f(X, X^*) = 0$ and $g(X, X^*) = 0$ hold in $RC \otimes_C \bar{C}$. Since both $RC$ and $RC \otimes_C \bar{C}$ are prime and centrally closed [3, Thm.2.5 and Thm.3.5], we may replace $R$ with $RC$ or $RC \otimes_C \bar{C}$ and assume that $R$ is centrally closed over $C$ and that $C$ is either finite or algebraically closed.

By Martindale's Theorem [9], $R$ is then a primitive ring having a nonzero socle $H$ and with $C$ as the associated division ring. In light of Kaplansky's Theorem [4, Thm.1.2.2], there exists a vector space $V$ over $C$, equipped with a Hermitian or alternate form, such that $R$ acts faithfully and densely on ${}_CV$ and that $r^*$ is the adjoint of $r$ for each $r \in R$. Moreover, $H$ consists of the finite-rank linear transformations having adjoints on ${}_CV$.

If $V$ is finite-dimensional over $C$, the density of $R$ on ${}_CV$ implies that $R \cong M_n(C)$ for some $n > 1$ with symplectic or transpose type involution [4, p.19]. We want to show that $n = 2$. Assume the contrary and we will proceed to arrive at a contradiction that either $c \in Z$ or $a \in Z$.

Suppose that $*$ is symplectic on $M_n(C)$, that is, $n$ is even and $*$ is given by $(a_{ij})^* = (a_{ji}^\sigma)$ where $a_{ij}$ is the $2 \times 2$ matrix block at the $(i, j)$-position and $\sigma$ is the involution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\sigma = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$ on $M_2(C)$. Consider first the case when $n = 4$ and write $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. Setting $X = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ in $g(X, X^*) = 0$ where $I$ is the $2 \times 2$ identity matrix, we have $c_{11} = c_{22}$ and $c_{12} = c_{21}$. Then set $X = \begin{pmatrix} e_{11} & e_{11} \\ 0 & e_{11} \end{pmatrix}$, $\begin{pmatrix} e_{11} & e_{12} \\ 0 & e_{11} \end{pmatrix}$ and $\begin{pmatrix} e_{11} & e_{21} \\ 0 & e_{11} \end{pmatrix}$ successively in $g(X, X^*) = 0$ where $\{e_{ij}\}$ are the usual $2 \times 2$ matrix units, and we obtain that $c_{12} = 0$ and $c_{11}$ is a scalar matrix in $M_2(C)$ and hence $c \in Z$. Now assume that $n > 4$. For $h \neq k$, let $e = (a_{ij})$ with $a_{hh} = a_{kk} = I$ and $a_{ij} = 0$ otherwise. Then $e^2 = e = e^*$ and $eRe \cong M_4(C)$. Proceeding as above, we will get $c \in Z$.

Suppose next that $*$ is of the transpose type, namely, $(\gamma_{ij})^* = (\pi_i \pi_j^{-1} \gamma_{ji}^*)$ where $\pi_1, \ldots, \pi_n$ are $n$ fixed nonzero symmetric elements in $C$. Write $a = \sum \alpha_{ij} e_{ij}$ where $\alpha_{ij} \in C$ and $\{e_{ij}\}$ are the usual matrix units. By setting $X = \pi_i e_{ij}$ with $i \neq j$ in $f(X, X^*) = 0$, we have, for $k \neq i, j$, $\alpha_{ik} = \alpha_{jk} = 0$ and $\alpha_{ii}^* + \alpha_{ii} = \alpha_{jj}^* + \alpha_{jj}$ or $\alpha_{ii}^* - \alpha_{ii} = \alpha_{jj}^* - \alpha_{jj}$ according as $b = a^*$ or $b = -a^*$ respectively. In other words, $a$ is a diagonal matrix and $a^* + a$ or $a^* - a$ lies in $Z$ if $n \geq 3$. In any case $b + a \in Z$ and so $\delta = -d$. Thus we have $sd(s) + d(s)s \in Z$ for all $s \in S$. Hence, $a \in Z$ follows from [7, Thm.6].

It remains to consider the case when $V$ is infinite-dimensional over $C$. For any $e = e^2 = e^* \in H$, we have $eRe \cong M_n(C)$ for some $n = \dim_C Ve$. Since $R$ satisfies $ef(eXe, eX^*e)e = 0$ and $g(eXe, eX^*e) = 0$, the subring $eRe$ satisfies $f_e(X, X^*) = (X+X^*)[X+X^*, eae] - [X+X^*, ebe](X+X^*) = 0$ and $g_e(X, X^*) = (X+X^*)[X+X^*, ece](X+X^*) = 0$. As we have shown above, $eae$ (or $ece$) is central in $eRe$ if $n \geq 3$. Given any $h \in H$, there is a symmetric idempotent $e \in H$ such that $h$, $ha$ and $ah$ (or $hc$ and $ch$) are all in $eRe$ by the $*$-version of Litoff's Theorem. Since $V$ is infinite-dimensional over $C$, we may choose $e$ so that $n = \dim_C Ve \geq 3$. Then $eae$ (or $ece$) is central in $eRe$. Hence $ah = eah = eaeh = heae = hae = ha$ (similarly $ch = hc$). Thus, $a$ (or $c$) centralizes the nonzero ideal $H$ of the prime ring $R$ and hence lies in $Z$. This completes the proof of the theorem.

One might wonder why we use the identity $f(X, X^*) = 0$ instead of $g(X, X^*) = 0$ in the transpose case. Indeed $g(X, X^*) = 0$ implies $c \in Z$ as in the symplectic case provided the characteristic is not 2. However, one can verify, for instance, that $(x + x^*)[x + x^*, y + y^*](x + x^*) = 0$ for all $x, y$ in $M_3(C)$ if char $C = 2$ and $*$ is of the first kind and of the transpose type.

Finally, we give a skew version of Brešar's Theorem.

THEOREM 2. *If $kd(k) - \delta(k)k \in Z$ for all $k \in K$, then either $d = \delta = 0$ or $R$ satisfies $s_4$.*

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PROOF. In light of Theorem 1, it suffices to prove the theorem in the situation when char $R \neq 2$.

Linearize the relation $kd(k)-\delta(k)k \in \mathcal{Z}$ to obtain $kd(h)-\delta(h)k+hd(k)-\delta(k)h \in \mathcal{Z}$ for all $h, k$ in $K$. Replacing $h$ with $[k, h]$ we obtain $k[k, d(h)] - [k, \delta(h)]k \in \mathcal{Z}$. Thus for each $h$, we have $kD_h(k) - \Delta_h(k)k \in \mathcal{Z}$ for all $k \in K$ where $D_h$ and $\Delta_h$ are the inner derivations defined by $d(h)$ and $\delta(h)$ respectively. As before, we need only consider the inner case because of the Lemma. So assume that $d(x) = [x, a]$ and $\delta(x) = [x, b]$ for some fixed elements $a$ and $b$ in $R$. Applying $*$ to $k[k, a] - [k, b]k \in \mathcal{Z}$, we get $k[k, b^*] - [k, a^*]k \in \mathcal{Z}$ and hence $k[k, a + b^*] - [k, b + a^*]k \in \mathcal{Z}$ and $k[k, a - b^*] - [k, b - a^*]k \in \mathcal{Z}$ for all $k \in K$. So we may assume further that $b = a^*$ or $b = -a^*$ and proceed to show that either $a \in \mathcal{Z}$ or $R$ satisfies $s_4$.

If $a \notin \mathcal{Z}$, then $R$ satisfies the nontrivial $*$-GPI $h(X, X^*, Y) = \big[(X - X^*)[X - X^*, a] - [X - X^*, b](X - X^*), Y\big] = 0$. A reduction as in the proof of Theorem 1 enables us to consider only the case when $R = M_n(C)$ for some $n > 2$ with symplectic or transpose type involution. We are going to show that $a \in \mathcal{Z}$ which contradicts our hypothesis.

Assume that $*$ is symplectic on $M_n(C)$. As before, it suffices to prove in the case when $n = 4$. Write $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Suppose first that $b = a^*$. Set $X = e_{11}$ in $h(X, X^*, Y) = 0$; then $a_{12} = 0$. Similarly, $a_{21} = 0$ follows from setting $X = e_{33}$ in $h(X, X^*, Y) = 0$. Next, by setting $X = e_{13} + e_{31}$ in $h(X, X^*, Y) = 0$, we get $a_{11} + a_{11}^{\sigma} = a_{22} + a_{22}^{\sigma}$. Thus $a+a^* \in \mathcal{Z}$ and hence $\delta(x) = [x, a^*] = -[x, a] = -d(x)$ for all $x \in R$. Then we have $kd(k)+d(k)k \in \mathcal{Z}$ for all $k \in K$ and so $a \in \mathcal{Z}$ by [7, Thm.7]. Suppose next that $b = -a^*$. Set $X = e_{11}$ in $h(X, X^*, Y) = 0$; then $a_{12} = 0$ and $a_{11}$ is a diagonal matrix. Similarly, $a_{21} = 0$ and $a_{22}$ being diagonal follow from setting $X = e_{33}$ in $h(X, X^*, Y) = 0$. Next, by setting $X = e_{11} + e_{12}$ in $h(X, X^*, Y) = 0$ we obtain that $a_{11}$ is a scalar matrix. Similarly, $a_{22}$ is also scalar by setting $X = e_{33} + e_{34}$ in $h(X, X^*, Y) = 0$. Thus $a^* = a$ and so $\delta = -d$. Then $a \in \mathcal{Z}$ follows again from [7, Thm.7].

Finally assume that $*$ is of the transpose type, say $(\gamma_{ij})^* = (\pi_i \pi_j^{-1} \gamma_{ji}^*)$. Write $a = \sum \alpha_{ij} e_{ij}$ where $\alpha_{ij} \in C$. By setting $X = \pi_i e_{ij}$ with $i \neq j$ in $h(X, X^*, Y) = 0$, we obtain that $a$ is a diagonal and $a^*+a \in \mathcal{Z}$ or $a^*-a \in \mathcal{Z}$ according as $b = a^*$ or $b = -a^*$ respectively. Hence, we get $b + a \in \mathcal{Z}$ in any case and so $\delta = -d$. Thus, we have $kd(k) + d(k)k \in \mathcal{Z}$ for all $k \in K$ and the proof of the theorem is completed by [7, Thm.7].

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*Department of Mathematics*
*National Taiwan University*
*Taipei, Taiwan*