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HARMONIC SPINORS ON HYPERBOLIC SPACE

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ABSTRACT. The purpose for this short note is to describe the space of harmonic spinors on hyperbolic n-space H^n. This is a natural continuation of the study of harmonic functions on H^n in [Minemura] and [Helgason]—these results were generalized in the form of Helgason's conjecture, proved in [KKMOOT],—and of [Gaillard 1, 2], where harmonic forms on H^n were considered. The connection between invariant differential equations on a Riemannian semisimple symmetric space G/K and homological aspects of the representation theory of G, as exemplified in (8) below, does not seem to have been previously mentioned. This note is divided into three main parts respectively dedicated to the statement of the results, some reminders, and the proofs. I thank the referee for having suggested various improvements.

Results. To each euclidean finite dimensional space E [Chevalley] attached a complex finite dimensional space $S = S(E)$, called the space of *spinors* for E, together with a natural map $E \otimes S \longrightarrow S$, called *Clifford multiplication*. Moreover S is a module for $\mathrm{Spin}(E)$—the two-fold cover of $\mathrm{SO}(E)$—known as the *spin* representation of $\mathrm{Spin}(E)$. (As usual $\mathrm{SO}(n) = \mathrm{SO}(\mathbb{R}^n)$, $\mathrm{Spin}(n) = \mathrm{Spin}(\mathbb{R}^n)$.) It is well known that S is $\mathrm{Spin}(E)$-irreducible if n is odd, and breaks up as $S = S^+ \oplus S^-$ if n is even. In that case S^+ and S^- are the so-called *half-spin* representations. Let M be a riemannian n-manifold. If the principal $\mathrm{SO}(n)$-bundle P of oriented othonormal frames on M admits a cover $Q \longrightarrow P$ which induces, for each $x \in M$, the cover $\mathrm{Spin}(T_xM) \longrightarrow \mathrm{SO}(T_xM)$—for instance if M is simply-connected,— then one can form the associated vector bundle $Q \times_{\mathrm{Spin}(n)} S$, whose sections are called *spinors*. Let $G \approx \mathrm{Spin}(n, 1)$ be the two-fold cover of the group of orientation preserving isometries of H^n. In particular the stabilizer K of any given point of H^n is isomorphic to $\mathrm{Spin}(n)$, the two-fold cover of $\mathrm{SO}(n)$, and $H^n \approx G/K$. Then G acts on the space Σ of spinors (with $\Sigma = \Sigma^+ \oplus \Sigma^-$ when n is even). In addition there is a first order differential operator, the *Dirac operator*, $D: \Sigma \longrightarrow \Sigma$. A spinor σ on H^n is called *harmonic* if $D^2\sigma = 0$. Suppose n is even. It is known that $D\Sigma^+ \subset \Sigma^-$ and $D\Sigma^- \subset \Sigma^+$. Let $e^+ : \Sigma \twoheadrightarrow \Sigma^+$ and $e^- : \Sigma \twoheadrightarrow \Sigma^-$ be the projections, and put $\mathcal{D}_0 = \mathbb{C}e^+ \oplus \mathbb{C}e^-$. Let $\mathcal{D}$ be the algebra of invariant differential operators acting on Σ.

(1) THEOREM. *(a) If n is even, then $\mathcal{D}$ is generated over $\mathcal{D}^0$ by D subject to the sole relation $D = e^+De^- + e^-De^+$.*

(b) If n is odd, then $\mathcal{D} = \mathbb{C}[D] \approx \mathbb{C}[x]$.

The significance of this algebra is discussed in [Guichardet].—Put $\mathcal{H}_k = \{\sigma \in \Sigma \mid D^k\sigma = 0\}$, and, if n is even, set $\mathcal{H}_k^+ = \mathcal{H}_k \cap \Sigma^+$ and $\mathcal{H}_k^- = \mathcal{H}_k \cap \Sigma^-$. Then $\mathcal{H}_2$ is the

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space of *harmonic spinors* on H^n. The sequences

(2) $$0 \longrightarrow \mathcal{H}_1^+ \longrightarrow \mathcal{H}_2^+ \xrightarrow{D} \mathcal{H}_1^- \longrightarrow 0 \quad n \text{ even},$$

(3) $$0 \longrightarrow \mathcal{H}_1^- \longrightarrow \mathcal{H}_2^- \xrightarrow{D} \mathcal{H}_1^+ \longrightarrow 0 \quad n \text{ even},$$

(4) $$0 \longrightarrow \mathcal{H}_1 \longrightarrow \mathcal{H}_2 \xrightarrow{D} \mathcal{H}_1 \longrightarrow 0 \quad n \text{ odd}$$

are obviously left exact.—Let $P \subset G$ be a minimal parabolic subgroup, so G/P is the boundary ∂H^n of H^n. The group $M = \mathrm{Spin}(n-1)$ can be viewed at the same time as $K \cap P$ and as the quotient of P by its radical. Let V be the space of half-density valued hyperfunction sections of the invariant vector bundle over ∂H^n whose fiber above the identity coset is the spin representation of M lifted to P. Technically V is the *maximal globalization* of a unitary principal series. More precisely, recall:

(6) THEOREM [SCHMID]. *There is a category $\mathcal{S}$ of topological G-modules having the following properties:*

- *(a) $\mathcal{S}$ is equivalent to the category of Harish-Chandra modules; in particular $\mathcal{S}$ is abelian and all its objects have finite length;*
- *(b) $\mathcal{H}_k, V \in \mathcal{S}$;*
- *(c) if $X \in \mathcal{S}$ and $Y \subset X$ is a closed invariant subspace, then $Y, X/Y \in \mathcal{S}$;*
- *(d) let $X, Y \in \mathcal{S}$ and $\varphi: X \longrightarrow Y$ be a map; then φ is in $\mathcal{S}$ if and only if it is continuous, linear and equivariant;*
- *(e)* $\mathrm{Hom}_G\big(W, \Gamma(G \times_K X)\big) \approx \mathrm{Hom}_K(W, X)$ *for all $W \in \mathcal{S}$ and all finite dimensional smooth K-modules X (here Γ refers to the C^∞ sections).*

The objects of $\mathcal{S}$ are called *maximal globalizations*. From now on Hom_G and Ext_G will respectively denote Hom in the category of all topological G-modules and Ext in category $\mathcal{S}$. In view of (5.d) this should cause no confusion. The next result is essentially due to Thieleker.

(7) THEOREM. *If n is even, then:*

- *(a) $\mathcal{H}_1^+$ and $\mathcal{H}_1^-$ are (topologically) irreducible,*
- *(b)* $V \approx \mathcal{H}_1^+ \oplus \mathcal{H}_1^-$,
- *(c)* $\mathrm{Hom}_G(\mathcal{H}_1^+, \Sigma^+) \approx \mathbb{C} \approx \mathrm{Hom}_G(\mathcal{H}_1^-, \Sigma^-)$,
- *(d)* $\mathrm{Hom}_G(\mathcal{H}_1^+, \Sigma^-) = 0 = \mathrm{Hom}_G(\mathcal{H}_1^-, \Sigma^+)$.

If n is odd, then

- *(e) $\mathcal{H}_1$ is irreducible,*
- *(f)* $V \approx \mathcal{H}_1 \oplus \mathcal{H}_1$,
- *(g)* $\mathrm{Hom}_G(\mathcal{H}_1, \Sigma) \approx \mathbb{C}$.

The main result of this note is:

(8) THEOREM. *(a) The sequences (2), (3) and (4) are exact, nonsplit, and generate respectively the groups* $\mathrm{Ext}^1_G(\mathcal{H}_1^-, \mathcal{H}_1^+)$, $\mathrm{Ext}^1_G(\mathcal{H}_1^+, \mathcal{H}_1^-)$, *and* $\mathrm{Ext}^1_G(\mathcal{H}_1, \mathcal{H}_1)$.

(b) $\mathrm{Ext}^p_G(\mathcal{H}_k, X) = 0 = \mathrm{Ext}^p_G(X, \mathcal{H}_k)$ *for* $p \geq 2$, $k \geq 1$ *and* $X \in \mathcal{S}$.

REMINDERS. First let me recall more precisely the notion of spin representation (this is taken almost word for word from [Borel & Wallach, § II.6]). Let E be a finite dimensional real vector space equipped with a (positive definite) inner product $(\ |\ )$. An E-module is a (complex) vector space W along with a linear map $c\colon E \longrightarrow \mathrm{End}\, W$, called *Clifford multiplication*, which satisfies

$$c(v)^2 = -(v \mid v) \tag{9}$$

for all v. (Here and in the sequel Hom $=$ $\mathrm{Hom}_{\mathbb{C}}$, End $=$ $\mathrm{End}_{\mathbb{C}}$, $\otimes = \otimes_{\mathbb{C}}$.) Let $\mathbf{o}(E)$ be the Lie algebra of all skew-adjoint operators on E. Identify $\mathbf{o}(E)$ with $\bigwedge^2 E$ (the second exterior power of E) by $(x \wedge y)v = (x \mid v)y - (y \mid v)x$ for all $x, y, v \in E$. Any E-module W can be endowed with an $\mathbf{o}(E)$-module structure by setting

$$(x \wedge y)w = 1/2c(x)c(y)w \text{ for } x \perp y \tag{10}$$

Up to isomorphism there is only one simple E-module $S = S(E)$ [Chevalley, p. 55, 57]. The corresponding $\mathbf{o}(E)$-module is called the *spin* representation, and exponentiates to $\mathrm{Spin}(E)$, the two-fold cover of $\mathrm{SO}(E)$. If $\dim E$ is odd, S is $\mathbf{o}(E)$-simple. If $\dim E$ is even, S is a sum of two simple nonisomorphic representations S^+ and S^- of $\mathbf{o}(E)$, called the *half-spin* representations. In this case, by [Atiyah, § IV.I] for instance, $c(E)S^+ \subset S^-$ and $c(E)S^+ \subset S^-$. Let $E' \subset E$ be a hyperplane. If $\dim E$ is even, then S^+ and S^- are simple E'-modules (and, thus, E'-isomorphic). If $\dim E$ is odd, then S is a simple E'-module. This fact will be referred to as *branching law.*

For any tangent vector $v \in TH^n$ denote the corresponding infinitesimal translation by $\tau(v)$. Then $\tau\colon TH^n \longrightarrow \mathbf{g}$ is a $\mathbf{g}$-valued 1-form on H^n where $\mathbf{g}$ is the Lie algebra of G. Let F be an invariant vector bundle over H^n and denote by $\Omega^p F$ the space of F-valued p-forms on H^n. The action of G on $\Omega^p F$ gives rise, by differentiation, to an action of $\mathbf{g}$. There is a natural operator $\nabla\colon \Omega^0 F \longrightarrow \Omega^1 F$, called *connection*, defined by $(\nabla s)(v) = \big(\tau(v)s\big)(x)$ for $v \in T_x H^n$. It is often convenient to write $\nabla_v s$ for $(\nabla s)(v)$. Note the *Leibniz rule* $\nabla_v(fs) = (vf)s(x) + f(x)\nabla_v s$ for function f on H^n. Suppose F is the spinor bundle, which exists because H^n is simply connected. In particular $F_x = S(T_x H^n)$. Therefore one can view $\nabla\sigma$ as a section of $\mathrm{Hom}(TH^n, F)$ and, identifying TH^n to T^*H^n, thanks to the hyperbolic metric, one can regard the Clifford multiplication as a map $c\colon \mathrm{Hom}(TH^n, F) \longrightarrow F$. Define the *Dirac operator* $D\colon \Sigma \longrightarrow \Sigma$ by $D\sigma = c(\nabla\sigma)$. Clearly D is an order one invariant differential operator.

PROOFS. Denote by S_K and S_M the spin representations of K and M.

PROOF OF (1). Since the proof is very similar to (and easier than) that of 3.1 in [Gaillard 3], I will not insist too heavily on the details, and the odd case will be omitted altogether. So suppose n is even. There is an obvious map from the algebra implicitly presented in (1.a) into $\mathcal{D}$, and, thus, into $\operatorname{gr}\mathcal{D}$, the graded algebra associated to $\mathcal{D}$. It suffices to show that the second map is an isomorphism. Denoting the symmetric algebra of $\mathbb{C}^n$ by Sym, one has, $\operatorname{gr}\mathcal{D} \approx (\mathrm{Sym} \otimes \operatorname{End} S_K)^K$. Using the linear isomorphism $\mathrm{Sym} \approx I \otimes H$, with $I = \{\text{invariants}\}$ and $H = \{\text{harmonics}\} \approx \operatorname{Ind}_M^K \mathbb{C}$ together with Frobenius reciprocity for $M \subset K$, one can write $\operatorname{gr}\mathcal{D} \approx I \otimes (H \otimes \operatorname{End} S_K)^K \approx I \otimes \operatorname{End}_M S_K$. It is well known that I is a polynomial algebra in the obvious generator. In view of the above branching law, one has $\dim \operatorname{End}_M S_K = 4$. Let H^k be the space of degree k homogeneous elements in H. Clearly $\dim(H^0 \otimes \operatorname{End} S_K)^K = 2$. One knows (see for instance [Borel & Wallach, § II.6.5]) $\operatorname{End} S_K \approx \bigwedge \mathbb{C}^n$ (the exterior algebra of $\mathbb{C}^n$). One easily deduces from this that $(H^1 \otimes \operatorname{End} S_K)^K$ has dimension 2, and, thus, is generated by e^+De^- and e^-De^+. This implies $\operatorname{gr}\mathcal{D} = Ie^+De^- \oplus Ie^-De^+$, which, in turn, yields the claim. ■

Let V be the space of half-density valued hyperfunction sections of the invariant vector bundle over ∂H^n whose fiber above the identity coset is the spin representation of M lifted to P, as in (6).

(11) PROPOSITION (THIELEKER). *There are irreducible submodules $V^+, V^- \subset V$ such that*

(a) $V = V^+ \oplus V^-$,

(b) V^+ *and* V^- *are isomorphic if and only if n is odd,*

(c) $\operatorname{Hom}_G(V^+, \Sigma^+) \approx \mathbb{C} \approx \operatorname{Hom}_G(V^-, \Sigma^-)$ *(n even).*

(d) $\operatorname{Hom}_G(V^+, \Sigma^-) = 0 = \operatorname{Hom}_G(V^-, \Sigma^+)$ *(n even),*

(e) $\operatorname{Hom}_G(V^+, \Sigma) \approx \mathbb{C}$ *(n odd).*

One has the following K-module isomorphisms:

(f) $\mathcal{H}_2^+ \approx \mathcal{H}_1^+ \oplus \mathcal{H}_1^- \approx \mathcal{H}_2^-$ *for n even;* $\mathcal{H}_2 \approx \mathcal{H}_1 \oplus \mathcal{H}_1$ *for n odd.*

PROOF. Parts (a)–(e) follow from [Thieleker 1, Theorem 6] coupled with (5.e). By (1), there is a scalar λ such that $D^2 - \lambda$ is a nonzero multiple of the Casimir operator. From this point on, in view of the branching law recalled above, the proof of (f) is similar to that of Lemma 12 in [Gaillard 2]. ■

It is well known since Poincaré that H^n can be realized as the upper half-space $H^n\{x \in \mathbb{R}^n \mid x_n > 0\}$ with the metric $\Sigma\, dx_i^2/x_n$. Set $o = (0, \ldots, 0, 1) \in H^n$. Let (e_j) be the standard basis of $\mathbb{R}^n$ viewed as a basis of T_oH^n. Identify K to the stabilizer of o, and P to that of $\infty \in \partial H^n = \mathbb{R}^{n-1} \cup \{\infty\}$. Then $P \approx \mathbb{R}^+ \times \mathrm{Spin}(n-1)$, $\mathbb{R}^+ = \{t \in R \mid t > 0\}$. Define the groups $A, N \subset P$ by $A = \{x \mapsto ax \mid a > 0\} \approx \mathbb{R}^+$ and $N = \{x \mapsto b + x \mid b \in \mathbb{R}^{n-1}\} \approx \mathbb{R}^{n-1}$ of P. Identify $\bigwedge^2 T_oH^n$ to $\mathbf{k}$ as above. Denote by Σ^N the space of N-invariant spinors on H^n. Recall $F_x = S(T_xH^n)$.

(12) LEMMA. *(a) If $\sigma \in \Sigma^N$ and $j \leq n-1$, then $\nabla_{e_j}\sigma = (e_j \wedge e_n)\sigma(o)$, where the right hand side denotes the action of $e_j \wedge e_n \in \mathbf{k}$ on $\sigma(o) \in F_o = S_K$.*

(b) If σ is A-invariant, then $\nabla_{e_n}\sigma = 0$.

PROOF. To prove (b), note $\nabla_{e_n}\sigma = \big(\tau(e_n)\sigma\big)(o) = 0$, the first equality following from the definition of ∇, and the second one from the facts that $\tau(e_n) \in A$ and that σ is assumed to be A-invariant. To prove (a), use the isomorphism $NA \xrightarrow{\sim} H^n$, $g \mapsto go$ to identify F

with $H^n \times F_o$. Then $\sigma = f(x_n)s$ for some $f \in C^\infty(\mathbb{R}^+)$ and $s \in F_o$. Because of the Leibniz rule, f has no influence on the formula. Since $e_j \wedge e_n$ acts semisimply on F_o one can assume $(e_j \wedge e_n)s = -i\lambda s$ with $2\lambda \in \mathbb{Z}$. The one-parameter subgroup generated by $\tau(e_j)$ preserves the hyperbolic 2-plane through e_j and e_n. Thus I can suppose $n = 2, j = 1$, and identify $\mathbb{R}^2$ to $\mathbb{C}$. Then $o = i \in \mathbb{C}$ and $e_1 = \partial/\partial x \in T_i H^2$. I can also assume $\sigma = (dz)^\lambda$. An easy computation yields successively $\tau(e_1) = 1/2(1 - z^2)\partial/\partial z + 1/2(1 - \bar{z}^2)\partial/\partial \bar{z}$, and $\nabla_{e_1}\sigma = -i\lambda\sigma(i)$, as desired. ∎

(13) LEMMA. *Under the above isomorphism $\Sigma^N \approx C^\infty(\mathbb{R}^+) \otimes S_K$ one has: $D = \left(x_n\partial/\partial x_n - \frac{n-1}{2}\right) \otimes c(e_n)$ on Σ^N.*

PROOF. By definition the Dirac operator satisfies $(D\sigma)(o) = \sum_{1 \le j \le n} c(e_j)\nabla_{e_i}\sigma$. One has:

$$\begin{aligned}
\sum_{1 \le j \le n-1} c(e_j)\nabla_{e_i}\sigma &= \sum_{1 \le j \le n-1} c(e_j)(e_j \wedge e_n)\sigma(e) \quad \text{by (10.a)} \\
&= 1/2 \sum_{1 \le j \le n-1} c(e_j)c(e_j)c(e_n)\sigma(e_n) \quad \text{by (10)} \\
&= -1/2 \sum_{1 \le j \le n-1} c(e_n)\sigma(e_n) \quad \text{by (9)} \\
&= \frac{1-n}{2} c(e_n)\sigma(e_n).
\end{aligned}$$

Suppose now that σ is N-invariant, *i.e.* $\sigma = f(x_n)s$ for some $f \in C^\infty(\mathbb{R}^+)$, $s \in F_o$. By the Leibniz rule and (10.b), one obtains $(D\sigma)(o) = \left(f'(1) - \frac{n-1}{2}\right)c(e_n)s$. Now the statement follows by A-invariance of D. ∎

Recall that a Dirac section of an invariant vector bundle over ∂H^n is just a distribution section of order zero supported on a single point, and that the linear span of the Dirac sections is dense in the space of all hyperfunction sections. Let now E be the vector bundle over ∂H^n whose space of (scalar valued) hyperfunction sections is V, let $v \in V$ be a Dirac section of E supported on the point at infinity, and let φ be in $\mathrm{Hom}_G(V, \Sigma)$.

(14) LEMMA. *(a) $\big(\varphi(v)\big)(x) = x_n^{(n-1)/2}\big(\varphi(v)\big)(e_n)$,*
(b) $\varphi V \subset \mathcal{H}_1$.

PROOF. Part (a) is proved as Lemma 2 in [Gaillard 1], and (b) follows from (a) combined with (13). ∎

(15) LEMMA. *If n is even, then $V^+ \approx \mathcal{H}_1^+$ and $V^- \approx \mathcal{H}_1^-$. If n is odd, then $V^+ \approx \mathcal{H}_1 \approx V^-$. Moreover the sequences (2), (3) and (4) are exact and nonsplit.*

PROOF. Suppose n is even. Let $\varphi \in \mathrm{Hom}_G(V, \Sigma^+)$, $\varphi \neq 0$. Using the above notation, let s be in F_o^+, and put $\sigma = x_n^{(n-1)/2}\log(x_n)s$. Then, in view of (13), σ satisfies $D^2\sigma = 0 \neq D\sigma \in \varphi V^-$. This implies that V^+ and V^- occur both as composition factors oin $\mathcal{H}_2^+$. (Thanks to Schmid's Theorem (6) it makes sense to talk about composition factors.) By (9.f), this means that these are the only composition factors of $\mathcal{H}_2^+$. Then it is clear that $\mathcal{H}_1^+ \approx V^+$, and that (2) is exact and nonsplit. The other cases are similar. ∎

In view of (11) and (15) I am left with proving the second part of (7.a). Let me verify for instance that (2) generates $\mathrm{Ext}^1_G(V^-, V^+)$. Using [Thieleker 2, § 10] and [Collingwood], one easily checks that V^+ and V^- are (up to isomorphism) the only irreducible modules with their (common) infinitesimal character. Combining (1.a) with [Guichardet, 3.2.13], yields the claim. Again, the other cases are analogous. Statement (7.b) can be easily checked by combining the above arguments with Casselman's Theorem in [Borel & Wallach, I.5.5] and the usual long exact sequences for Ext. ■

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