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COEFFICIENT ESTIMATES FOR ALPHA-SPIRAL FUNCTIONS

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Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ belong to the class of α-spiral functions of order $\rho (|\alpha| < \pi/2, 0 \le \rho < 1)$. In this paper, we determine sharp coefficient estimates for functions of the form $f(z)^t$, where t is a positive integer. We also study the influence of the second coefficient on the other coefficients for such functions. The results obtained not only generalize the results of MacGregor, Boyd, Srivastava, Silverman and Silvia and others, but also give rise to new results.

1. Introduction

Let $S(\alpha, \rho)$ denote the class of functions $f(z)$ analytic in the unit disc $U = \{z : |z| < 1\}$ such that $f(0) = 0$, $f'(0) = 1$ and satisfying

$$\text{Re}\left\{e^{i\alpha} \frac{zf'(z)}{f(z)}\right\} > \rho \cos \alpha \tag{1.1}$$

where α and ρ are real numbers $(|\alpha| < \pi/2, 0 \le \rho < 1)$. The power series representation for $f(z)$ is

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.2}$$

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$S(\alpha, 0)$ is the class of α-spiral functions and it was shown by Špaček [6] that such functions are univalent in U . In [3] Libera proved that if $f \in S(\alpha, \rho)$, then

$$(1.3) \qquad |a_n| \le \prod_{k=0}^{n-2} \frac{|2(1-\rho)\cos\alpha e^{-i\alpha}+k|}{k+1} \quad \text{for } n = 2, 3, \ldots$$

holds, and that equality occurs for the function

$$f(z) = \frac{z}{(1-z)^{2(1-\rho)\cos\alpha\exp(-i\alpha)}} .$$

$S(\alpha, \rho)$ contains the class of starlike functions also. In particular, $S(0, \rho)$ is the class of functions that are starlike of order ρ in U , denoted by $St(\rho)$, and $S(0, 0)$ is the class of normalized starlike functions, denoted by St .

MacGregor [4] obtained upperbounds for the moduli of the coefficients of a starlike function whose power series representation in U is of the form

$$(1.4) \qquad f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n .$$

Boyd [1] and Srivastava [7] extended MacGregor's result to the class of starlike functions of order ρ $(0 \le \rho < 1)$ and α-spiral functions of order ρ respectively. Recently Silverman and Silvia [5] studied how the modulus of the second coefficient in the power series expansion (1.2) influences the growth of other coefficients for starlike, convex and close-to-convex functions.

In Section 2 of the present paper, we determine sharp coefficient estimates for the class $S(\alpha, \rho)$ of α-spiral functions of order ρ whose power series representation is of the form

$$f(z)^t = z^t + \sum_{n=t+k}^{\infty} a_n^{(t)} z^n .$$

In Section 3, the influence of the modulus of the second coefficient on the other coefficients in the power series

$$f(z)^t = z^t + \sum_{n=t+1}^{\infty} a_n^{(t)} z^n$$

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for functions in the class $S(\alpha, \rho)$ is studied. The results thus obtained refine and generalize the corresponding results of MacGregor [4], Boyd [1], Libera [3], Srivastava [7], Silverman and Silvia [5] and others.

2. Coefficient estimates

THEOREM 1. *Suppose* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S(\alpha, \rho)$ *and for the integral* $t \geq 1$ *let*

$$f(z)^t = z^t + \sum_{n=t+k}^{\infty} a_n^{(t)} z^n ;$$

then

$$\left|a_n^{(t)}\right| \leq \frac{mk}{n-t} \prod_{j=0}^{m-1} \left\{\frac{|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j|}{j+1}\right\} \tag{2.1}$$

where $mk+t \leq n \leq t+(m+1)k-1$, m *a positive integer. The estimates are sharp for* $n = mk + t$.

We need the following lemmas.

LEMMA 1. *If* k, t *and* q *are positive integers, then for* $0 \leq \rho < 1$, *we have*

$$4t(1-\rho)\cos^2\alpha\left\{t(1-\rho) + \sum_{m=1}^{q-1} (mk+t(1-\rho))\left[\prod_{j=0}^{m-1} \frac{|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j|}{j+1}\right]^2\right\} \tag{2.2}$$

$$= (qk)^2\left[\prod_{j=0}^{q-1} \frac{|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j|}{j+1}\right]^2 .$$

This can be easily proved by induction on q .

LEMMA 2. *If* $k = 1, 2, \ldots$, $q = 1, 2, \ldots$, *and* $0 \leq \rho < 1$, *then*

$$(n-t)^2 \geq \frac{(qk)^2(n-\rho t)}{qk+t(1-\rho)} \quad \text{for } n \geq t+qk . \tag{2.3}$$

Proof of Theorem 1. Let

$$g(z) = \frac{e^{i\alpha}}{t} \frac{z[f(z)^t]'}{f(z)^t} = e^{i\alpha} \frac{zf'(z)}{f(z)} .$$

Define

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(2.4)
$$\omega(z) = \frac{g(z)-e^{i\alpha}}{g(z)+e^{-i\alpha}-2\rho\cos\alpha} = \sum_{n=k}^{\infty} \omega_n z^n .$$

Since $\operatorname{Re} g(z) > \rho \cos \alpha$, we have $|\omega(z)| < 1$ in $|z| < 1$. Equating the coefficients of the same powers on both sides of the equation

$$e^{i\alpha}\left[z\left(f(z)^t\right)' - tf(z)^t\right] = \left[e^{i\alpha}z\left(f(z)^t\right)' + \left(e^{-i\alpha}-2\rho\cos\alpha\right)tf(z)^t\right]\omega(z)$$

or

(2.5)
$$e^{i\alpha}\sum_{n=t+k}^{\infty}(n-t)a_n^{(t)}z^n$$
$$= \left[\sum_{n=k}^{\infty}\omega_n z^n\right]\left\{2t(1-\rho)\cos\alpha z^t + \sum_{n=t+k}^{\infty}\left(ne^{i\alpha}+t\left(e^{-i\alpha}-2\rho\cos\alpha\right)\right)a_n^{(t)}z^n\right\},$$

we obtain the relations

(2.6)
$$e^{i\alpha}\nu a_{\nu+t}^{(t)} = 2t(1-\rho)\cos\alpha\omega_\nu \quad \text{for} \quad \nu = k, k+1, \ldots, 2k-1 .$$

Since $|\omega(z)| < 1$, we have $\sum_{n=k}^{\infty}|\omega_n|^2 \le 1$ and therefore

(2.7)
$$\sum_{n=k}^{2k-1}|\omega_n|^2 \le 1 .$$

From (2.6) and (2.7) we obtain

(2.8)
$$\sum_{\nu=k}^{2k-1}\nu^2\left|a_{\nu+t}^{(t)}\right|^2 \le 4t^2(1-\rho)^2\cos^2\alpha .$$

We rewrite (2.5) in the form

(2.9)
$$\sum_{n=t+k}^{p}e^{i\alpha}(n-t)a_n^{(t)}z^n + \sum_{n=p+1}^{\infty}d_n z^n$$
$$= \omega(z)\left\{2t(1-\rho)\cos\alpha z^t + \sum_{n=t+k}^{p-k}\left[ne^{i\alpha}+t\left(e^{-i\alpha}-2\rho\cos\alpha\right)\right]a_n^{(t)}z^n\right\}.$$

Since (2.9) has the form $F(z) = \omega(z)G(z)$, where $|\omega(z)| < 1$, it follows that

(2.10)
$$\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(re^{i\varphi}\right)\right|^2 d\varphi \le \frac{1}{2\pi}\int_0^{2\pi}\left|G\left(re^{i\varphi}\right)\right|^2 d\varphi$$

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for each r $(0 < r < 1)$. Expressing (2.10) in terms of the coefficients in (2.9) we get

$$(2.11)\quad \sum_{n=t+k}^{p} (n-t)^2\left|a_n^{(t)}\right|^2 r^{2n} + \sum_{n=p+1}^{\infty} |d_n|^2 r^{2n}$$
$$\le 4t^2(1-\rho)^2 \cos^2 \alpha r^{2t} + \sum_{n=t+k}^{p-k} \left|ne^{i\alpha}+t\left(e^{-i\alpha}-2\rho \cos \alpha\right)\right|^2 \left|a_n^{(t)}\right|^2 r^{2n} .$$

In particular (2.11) implies

$$(2.12)\quad \sum_{n=t+k}^{p} (n-t)^2\left|a_n^{(t)}\right|^2 r^{2n}$$
$$\le 4t^2(1-\rho)^2 \cos^2 \alpha r^{2t} + \sum_{n=t+k}^{p-k} \left|ne^{i\alpha}+t\left(e^{-i\alpha}-2\rho \cos \alpha\right)\right|^2 \left|a_n^{(t)}\right|^2 r^{2n} .$$

By letting $r \to 1$ in (2.12), we conclude that

$$(2.13)\quad \sum_{n=t+k}^{p} (n-t)^2\left|a_n^{(t)}\right|^2$$
$$\le 4t^2(1-\rho)^2 \cos^2 \alpha + \sum_{n=t+k}^{p-k} \left|ne^{i\alpha}+t\left(e^{-i\alpha}-2\rho \cos \alpha\right)\right|^2 \left|a_n^{(t)}\right|^2 .$$

This inequality is equivalent to

$$(2.14)\quad \sum_{n=p-k+1}^{p} (n-t)^2\left|a_n^{(t)}\right|^2$$
$$\le 4t(1-\rho) \cos^2 \alpha \left\{t(1-\rho) + \sum_{n=t+k}^{p-k} (n-\rho t)\left|a_n^{(t)}\right|^2\right\} .$$

By an inductive argument we will prove the inequalities

$$(2.15a)\quad \sum_{n=t+mk}^{t+(m+1)k-1} (n-t)^2\left|a_n^{(t)}\right|^2 \le (mk)^2\left\{\prod_{j=0}^{m-1} \frac{\left|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j\right|}{j+1}\right\}^2 ,$$

$$(2.15b)\quad \sum_{n=t+mk}^{t+(m+1)k-1} (n-\rho t)\left|a_n^{(t)}\right|^2$$
$$\le \left(mk+t(1-\rho)\right)\left\{\prod_{j=0}^{m-1} \frac{\left|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j\right|}{j+1}\right\}^2$$

for $m = 1, 2, \ldots$.

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For $m = 1$, (2.15a) gives $\sum_{n=t+k}^{t+2k-1} (n-t)^2 \left|a_n^{(t)}\right|^2 \le 4t^2(1-\rho)^2 \cos^2 \alpha$,

which is true by (2.8). Thus (2.15a) is valid for $m = 1$. We can prove (2.15b) for $m = 1$ by using Lemma 2 and (2.8) as follows:

$$\begin{aligned}\sum_{n=t+k}^{t+2k-1} (n-\rho t)\left|a_n^{(t)}\right|^2 &\le \frac{k+t(1-\rho)}{k^2} \sum_{n=t+k}^{t+2k-1} \frac{k^2(n-\rho t)}{k+t(1-\rho)} \left|a_n^{(t)}\right|^2 \\ &\le \frac{k+t(1-\rho)}{k^2} \sum_{n=t+k}^{t+2k-1} (n-t)^2 \left|a_n^{(t)}\right|^2 \\ &\le \left(k+t(1-\rho)\right)\left|\frac{2t}{k}(1-\rho) \cos \alpha e^{-i\alpha}\right|^2 .\end{aligned}$$

Now suppose that (2.15a) and (2.15b) hold for $m = 1, 2, \ldots, q-1$. Using (2.14) with $p = t - 1 + (q+1)k$ and the inductive hypothesis concerning (2.15b), we obtain the inequalities

$$\begin{aligned}&\sum_{n=t+qk}^{t+(q+1)k-1} (n-t)^2\left|a_n^{(t)}\right|^2 \\ &\le 4t(1-\rho) \cos^2 \alpha\left\{t(1-\rho) + \sum_{n=t+k}^{t+qk-1} (n-\rho t)\left|a_n^{(t)}\right|^2\right\} \\ &= 4t(1-\rho) \cos^2 \alpha\left\{t(1-\rho) + \sum_{m=1}^{q-1} \sum_{n=t+mk}^{t+(m+1)k-1} (n-\rho t)\left|a_n^{(t)}\right|^2\right\} \\ &\le 4t(1-\rho) \cos^2 \alpha\left\{t(1-\rho) + \sum_{m=1}^{q-1} \left(mk+t(1-\rho)\right)\left[\prod_{j=0}^{m-1} \frac{\left|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j\right|}{j+1}\right]^2\right\} \\ &= (qk)^2\left[\prod_{j=0}^{q-1} \frac{\left|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j\right|}{j+1}\right]^2 , \text{ by using Lemma 1.}\end{aligned}$$

This last sequence of inequalities implies (2.15a) where $m = q$. Continuing our argument, we use Lemma 2 and (2.15a) with $m = q$ to deduce (2.15b) for $m = q$ as follows:

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$$\sum_{n=t+qk}^{t+(q+1)k-1} (n-\rho t)\left|a_n^{(t)}\right|^2 \le \frac{qk+t(1-\rho)}{(qk)^2} \sum_{n=t+qk}^{t+(q+1)k-1} \frac{(qk)^2(n-\rho t)}{qk+t(1-\rho)} \left|a_n^{(t)}\right|^2$$

$$\le \frac{qk+t(1-\rho)}{(qk)^2} \sum_{n=t+qk}^{t+(q+1)k-1} (n-t)^2\left|a_n^{(t)}\right|^2$$

$$\le \left(qk+t(1-\rho)\right)\left[\prod_{j=0}^{q-1} \frac{\left|(2t/k)(1-\rho)\cos\alpha e^{-i\alpha}+j\right|}{j+1}\right]^2 .$$

This completes the proof of (2.15a) and (2.15b). The theorem follows from (2.15a).

The result is sharp for $n = mk + t$, $m = 1, 2, \ldots$, for the function

$$f(z) = \frac{z}{\left(1-z^k\right)^{2(1-\rho)\cos\alpha\exp(-i\alpha)/k}} .$$

Putting $t = 1$ in the theorem, we get the following result due to Srivastava [7] which in turn leads to Boyd's result [1] for $\alpha = 0$, MacGregor's result [4] for $\alpha = 0$, $\rho = 0$, and Zamorski's result [8] for $\rho = 0$ and $k = 1$.

COROLLARY 1. *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \in S(\alpha, \rho)$, *then*

$$|a_n| \le \frac{k}{(n-1)(m-1)!} \prod_{j=0}^{m-1} \left(j + \frac{2(1-\rho)\cos^2\alpha}{k}\right)\left\{1 + \left(\frac{2(1-\rho)\cos\alpha\sin\alpha}{jk+2(1-\rho)\cos^2\alpha}\right)^2\right\}^{\frac{1}{2}}$$

where $mk+1 \le n \le (m+1)k$, $m = 1, 2, 3, \ldots$.

The following coefficient estimates for k-fold symmetric functions follow immediately from the above theorem. The corresponding result for starlike functions obtained from the following corollary for $\alpha = 0$, was obtained by Goluzin [2].

COROLLARY 2. *If* $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1} \in S(\alpha, \rho)$, *then*

$$|a_{mk+1}| \le \frac{1}{m!} \prod_{j=0}^{m-1} \left|j + \frac{2(1-\rho)\cos\alpha e^{-i\alpha}}{k}\right| . \tag{2.16}$$

Equality in (2.16) *occurs for*

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$$f(z) = \frac{z}{\left(1-z^k\right)^{2(1-\rho)\cos\alpha\exp(-i\alpha)/k}} .$$

For $k = 1$, the estimate in (2.16) is the same as (1.3). For $k = 2$, this result asserts that the coefficients of odd α-spiral functions of order ρ satisfy the inequality

$$|a_{2m+1}| \le \frac{1}{m!} \prod_{j=0}^{m-1} |(1-\rho)\cos\alpha e^{-i\alpha}+j| \quad \text{for } m = 1, 2, \ldots .$$

3. Fixed coefficient results

We denote by $S_p(\alpha, \rho)$ the subclass of $S(\alpha, \rho)$ for which the modulus of the second coefficient is p.

THEOREM 2. *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\alpha, \rho)$ *and for integral* $t \ge 1$ *let*

$$f(z)^t = z^t + \sum_{n=t+1}^{\infty} a_n^{(t)} z^n ;$$

then

$$\text{(3.1)} \quad \left|a_n^{(t)}\right| \le \left(\frac{1+p}{|2t(1-\rho)\cos\alpha e^{-i\alpha}+1|}\right) \prod_{j=0}^{n-1-t} \left(\frac{|2t(1-\rho)\cos\alpha e^{-i\alpha}+j|}{j+1}\right)$$

$(n = t+2, t+3, \ldots)$

where $\left|a_{t+1}^{(t)}\right| = p(t) = p$ *(say).*

Proof. From (2.14), putting $k = 1$, we have

$$\text{(3.2)} \quad \left|a_n^{(t)}\right|^2 \le \frac{4t(1-\rho)\cos^2\alpha}{(n-t)^2} \sum_{k=t}^{n-1} [k-t+t(1-\rho)] \left|a_k^{(t)}\right|^2$$

for $n = t+1, t+2, \ldots$. For $n = t + 1$, we have

$$\left|a_{t+1}^{(t)}\right| = p \le |2t(1-\rho)\cos\alpha e^{-i\alpha}|$$

and hence (3.2) can be rewritten as follows:

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$$(3.3)\quad \left|a_n^{(t)}\right|^2 \le \frac{4t(1-\rho)\cos^2\alpha}{(n-t)^2}\left\{t(1-\rho)+\left(1+t(1-\rho)\right)p^2+\sum_{k=t+2}^{n-1}[k-t+t(1-\rho)]\left|a_k^{(t)}\right|^2\right\}.$$

For $n = t + 2$, (3.3) gives

$$\left|a_{t+2}^{(t)}\right|^2 \le \frac{4t(1-\rho)\cos^2\alpha}{(t+2-t)^2}\left\{t(1-\rho)+\left(1+t(1-\rho)\right)p^2\right\}$$

$$\le \frac{4t^2(1-\rho)^2\cos^2\alpha(1+p)^2}{(t+2-t)^2}$$

if $t(1-\rho) + \left(1+t(1-\rho)\right)p^2 \le t(1-\rho)(1+p)^2$, which is true. Hence (3.1) holds for $n = t + 2$.

Now assume (3.1) is true for $n = t+2, t+3, \ldots, (m-1)$. We can easily prove the following result by induction on m :

$$(3.4)\quad t(1-\rho) + \left(1+t(1-\rho)\right)p^2 + \sum_{k=t+2}^{m-1}[k-t+t(1-\rho)]\left|a_k^{(t)}\right|^2$$

$$\le t(1-\rho)(1+p)^2\left[\prod_{k=t+2}^{m-1}\frac{\left|2t(1-\rho)\cos\alpha e^{-i\alpha}+k-t\right|}{k-t}\right]^2$$

for $m = t+3, t+4, \ldots$.

Writing (3.3) for $n = m$ and using (3.4), we have

$$\left|a_m^{(t)}\right|^2 \le \frac{4t(1-\rho)\cos^2\alpha}{(m-t)^2}\left\{t(1-\rho)+\left(1+t(1-\rho)\right)p^2+\sum_{k=t+2}^{m-1}[k-t+t(1-\rho)]\left|a_k^{(t)}\right|^2\right\}$$

$$\le \frac{4t^2(1-\rho)^2\cos^2\alpha}{(m-t)^2}\left[\prod_{k=t+2}^{m-1}\frac{\left|2t(1-\rho)\cos\alpha e^{-i\alpha}+k-t\right|}{k-t}\right]^2$$

$$= \left\{\frac{(1+p)}{\left|2t(1-\rho)\cos\alpha e^{-i\alpha}+1\right|}\left[\prod_{k=t}^{m-1}\left\{\frac{\left|2t(1-\rho)\cos\alpha e^{-i\alpha}+k-t\right|}{k-t+1}\right\}\right]\right\}^2$$

and the result follows.

Putting $\alpha = 0$ and $t = 1$, we get the following result due to Silverman and Silvia [5].

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COROLLARY 2.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in St_p(\rho)$, *then*

$$|a_n| \le \left(\frac{1+p}{3-2\rho}\right)\left(\prod_{k=2}^{n}(k-2\rho)\right)/(n-1)! \quad (n = 3, 4, \ldots) .$$

The following result for k-fold symmetric functions can be obtained by a proof similar to that used in Theorem 2.

THEOREM 3. *Suppose* $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1} \in S(\alpha, \rho)$ *and for integral* $t \ge 1$ *let*

$$f(z)^t = z^t + \sum_{m=1}^{\infty} a_{mk+t}^{(t)} z^{mk+t}$$

then

$$\left|a_{mk+t}^{(t)}\right| \le \frac{1+p}{\left|\left(2t(1-\rho)\cos\alpha e^{-i\alpha}/k\right)+1\right|}\prod_{j=0}^{m-1}\left(\frac{\left|\left(2t(1-\rho)\cos\alpha e^{-i\alpha}/k\right)+j\right|}{j+1}\right)$$

$(m = 2, 3, \ldots)$

where $\left|a_{k+t}^{(t)}\right| = p$.

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