J. Austral. Math. Soc. (*Series A*) **66** (1999), 418–428

β-TRANSFORMATION, INVARIANT MEASURE AND UNIFORM DISTRIBUTION

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(Received 18 December 1997; revised 8 April 1999)

Communicated by A. H. Dooley

Abstract

Let T_β be the β-transformation on [0,1). When β is an integer T_β is ergodic with respect to Lebesgue measure and almost all orbits $\{T_\beta^n x\}$ are uniformly distributed. Here we consider the non-integer case, determine when T_α, T_β have the same invariant measure and when (appropriately normalised) orbits are uniformly distributed.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 28D05, 11K55, 11K06.
Keywords and phrases: β-transformation, invariant measure, uniform distribution..

1. Introduction and results

Let $\beta > 1$ be a real number. The β-transformation is the map $T_\beta : [0, 1) \mapsto [0, 1)$ given by

$$T_\beta x = \beta x - [\beta x], \quad \text{for all } x \in [0, 1)$$

where $[t]$ is the largest integer which is not greater than t. Ergodic properties of β-transformations are studied by many authors (see [1, 3–5, 8]). For each $\beta > 1$, T_β possesses a probability invariant measure, μ_β, which is equivalent to Lebesgue measure, and T_β is ergodic with respect to ν_β. We ask when is $\mu_\alpha = \mu_\beta$? The related map S_β on [0,1), defined by

$$S_\beta x = \begin{cases} T_\beta x & \text{if } x < [\beta]/\beta, \\ (x - [\beta])/(\beta - [\beta]) & \text{if } x \geq [\beta]/\beta, \end{cases}$$

preserves Lebesgue measure and is ergodic (see [2, pp. 168–172]). For $x \in [0, 1)$, we define a sequence

$$y_n(\beta) = \begin{cases} T_\beta^n x & \text{if } T_\beta^{n-1}x < [\beta]/\beta, \\ T_\beta^n x/(\beta - [\beta]) & \text{if } T_\beta^{n-1}x \geq [\beta]/\beta, \end{cases}$$

and ask when is $\{y_n(\beta)\}$ uniformly distributed for almost all $x \in [0, 1)$?

In the case where α, β are integers it is well-known that $\nu_\alpha = \nu_\beta$ is Lebesgue measure and that $\{y_n(\beta)\}$ is uniformly distributed for almost all x, so the interested cases are non-integer cases. Let

$$\mathscr{A} = \{\beta : \beta > 1 \text{ satisfies } x^2 - kx - l = 0,\ k, l \in \mathbb{Z}, k \geq l \geq 1\}.$$

Our results can be stated as follows.

THEOREM 1. *Suppose that $\beta > 1$ is not an integer. If $\beta \in \mathscr{A}$ then we have $\mu_\beta = \mu_{\beta+1}$. For any other $\alpha \neq \beta$ we have $\mu_\alpha \neq \mu_\beta$.*

THEOREM 2. *Suppose that $\beta > 1$ is not an integer. If $\beta \in \mathscr{A}$ then for almost all x, $\{y_n\}$ is uniformly distributed. If $\beta \notin \mathscr{A}$ then for almost all x, $\{y_n\}$ is not uniformly distributed.*

For $x \in [0, 1)$, we define $x_n(\beta) = T_\beta^n x$. By the ergodicity of T_β, for almost all x, the sequence $\{x_n\}$ is μ_β-distributed. When β is an integer, since μ_β is the Lebesgue measure restricted to $[0, 1)$, then $\{x_n(\beta)\}$ is uniformly distributed for almost all $x \in [0, 1)$. We may rewrite the definition of $\{y_n(\beta)\}$ as

$$y_n(\beta) = \begin{cases} x_n & \text{if } x_{n-1} < [\beta]/\beta, \\ x_n/(\beta - [\beta]) & \text{if } x_{n-1} \geq [\beta]/\beta. \end{cases}$$

Clearly, if β is an integer then $\{x_n\}$ and $\{y_n\}$ coincide. We may define $z_n(\beta) = S_\beta^n x$. Since S_β preserves Lebesgue measure, we see that $\{z_n(\beta)\}$ is uniformly distributed for almost all x. Comparing the definitions of $\{y_n(\beta)\}$ and S, it may seem plausible that, for any $\beta > 1$, $\{y_n(\beta)\}$ should be uniformly distributed for almost all x.

It is also interesting to compare Theorem 2 with the results of [6]. Schweiger in 1972 studied sequences similar to our $\{y_n(\beta)\}$ for some special Oppenheim series [6, 7]. The Oppenheim series is defined as follows: Let a_n be a decreasing sequence with $a_1 = 1$ and $\lim_{n\to\infty} a_n = 0$. Let $b_n \geq 1$. The map $T : [0, 1) \mapsto [0, 1)$ is piecewise defined as

$$Tx = \frac{x - a_{n+1}}{b_n(a_n - a_{n+1})}, \quad x \in [a_{n+1}, a_n).$$

We define $T0 = 0$, if necessary.

For $x \in [0, 1)$, let $u_n = T^n x$ and $v_n = b_{k_n} u_n$, if $u_{n-1} \in [a_{k_n+1}, a_{k_n})$. Schweiger [6] showed that in cases

1. $a_n = 1/n$, $b_n = 1$ (Lüroth's series);
2. $a_n = 1/n$, $b_n = n$ (Engel's series) and
3. $a_n = 1/n$, $b_n = n(n+1)$ (Sylvester's series)

$\{v_n\}$ is uniformly distributed for almost all $x \in [0, 1)$.

If, for non-integral $\beta > 1$, we let $a_1 = 1$, $a_n = ([\beta]+2-n)/\beta$, $n = 2, \dots, [\beta]+2$ and $a_n = 0$ for $n > [\beta]+2$, and let $b_1 = 1/(\beta - [\beta])$ and $b_n = 1$ for $n > 1$, then the map T is just the β-transformation T_β, and $\{v_n\}$ is just $\{y_n(\beta)\}$. Again, this may suggest the possibility of uniform distribution of $\{y_n(\beta)\}$.

Before giving the proofs let us develop some background concerning β-expansions. For $x \in [0, 1)$ we have

$$x = \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \cdots \tag{1.1}$$

where $c_n = [\beta T_\beta^{n-1} x]$. Equation (1.1) is called the β-expansion of x. Suppose that the β-expansion of $\beta - [\beta]$ is

$$\beta - [\beta] = \frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \cdots .$$

Then we have

$$1 = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \frac{\varepsilon_3}{\beta^3} + \cdots \tag{1.2}$$

where $\varepsilon_1 = [\beta]$ and (1.2) is called the β-expansion of 1. Notice that to say $\beta \in \mathscr{A}$ is equivalent to say that the β expansion of 1 is

$$1 = \frac{k}{\beta} + \frac{l}{\beta^2}.$$

We also denote $T_\beta^0 1 = 1$, $T_\beta 1 = \beta - [\beta]$ and $T_\beta^n 1 = T_\beta(T_\beta^{n-1} 1)$ for $n \geq 2$. β-expansions have the following properties:

(P) Let (1.2) be the β-expansion of 1. For any $x \in [0, 1)$ with the β-expansion given by (1.1) and any $n \geq 1$ we have

$$(c_n, c_{n+1}, c_{n+2}, \dots) < (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots),$$

where '<' is according to the lexicographical order.

By (P) we get that for any $n \geq 2$

$$(\varepsilon_n, \varepsilon_{n+1}, \varepsilon_{n+2}, \dots) < (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots).$$

The absolutely continuous invariant measure for T_β, μ_β, can be defined as follows (see [4]). Let

$$h_\beta(x) = \sum_{x < T_\beta^n 1,\ n \geq 0} \frac{1}{\beta^n}.$$

Then for any Borel subset E of [0,1),

$$\mu_\beta(E) = \frac{1}{c_\beta} \int_B h_\beta(x)dx,$$

where $c_\beta = \int_0^1 h_\beta(x)dx$ is the normalizing constant.

Theorem 1 will be proved in Section 2 and Theorem 2 in Section 3.

2. Proof of Theorem 1

PROOF. Assume that $\beta > 1$ is not an integer and that

$$1 = \frac{k}{\beta} + \frac{l}{\beta^2}$$

where $k \geq l > 0$. Then

$$T_\beta^0 1 = 1, \quad T_\beta 1 = \frac{l}{\beta}, \quad \text{and} \quad T_\beta^n 1 = 0 \quad \text{for } n \geq 2.$$

Hence

$$h_\beta(x) = \begin{cases} 1 + \dfrac{1}{\beta} & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1. \end{cases}$$

For $\beta + 1$ we have

$$T_{\beta+1}^0 1 = 0, \quad T_{\beta+1} 1 = \beta + 1 - [\beta + 1] = \beta - [\beta] = \frac{l}{\beta}$$

and

$$T_{\beta+1}^2 = (\beta + 1)\frac{l}{\beta} - \left[(\beta + 1)\frac{l}{\beta}\right] = \frac{l}{\beta}.$$

Hence for any $n \geq 1$ we have

$$T^n_{\beta+1} 1 = \frac{l}{\beta}.$$

Therefore

$$h_{\beta+1}(x) = \begin{cases} 1 + \dfrac{1}{\beta+1} + \dfrac{1}{(\beta+1)^2} + \cdots & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1 \end{cases}$$

$$= \begin{cases} 1 + \dfrac{1}{\beta} & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1 \end{cases}$$

$$= h_\beta(x).$$

This shows that $\mu_\beta = \mu_{\beta+1}$.

On the other hand, suppose that $\mu_\alpha = \mu_\beta$ for some $\alpha > 1$. Then we must have

$$\sum_{x < T^n_\alpha 1,\ n \geq 0} \frac{1}{\alpha^n} = h_\alpha(x) = h_\beta(x) = \begin{cases} 1 + \dfrac{1}{\beta} & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1. \end{cases}$$

Therefore α must satisfy one of the following two cases:

(a) $T_\alpha 1 = l/\beta$ and $T^n_\alpha 1 = 0$ for $n \geq 2$, or

(b) $T^n_\alpha 1 = l/\beta$ for all $n \geq 1$.

In case (a) we have

$$h_\alpha(x) = \begin{cases} 1 + \dfrac{1}{\alpha} & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1 \end{cases}$$

which gives $\alpha = \beta$.

In cases (b) we obtain

$$h_\alpha(x) = \begin{cases} 1 + \dfrac{1}{\alpha} + \dfrac{1}{\alpha^2} + \cdots & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1 \end{cases}$$

$$= \begin{cases} 1 + \dfrac{1}{\alpha - 1} & \text{if } 0 \leq x < \dfrac{l}{\beta}, \\ 1 & \text{if } \dfrac{l}{\beta} \leq x < 1 \end{cases}$$

which yields $\alpha = \beta + 1$.

Now we assume that $\beta \notin \mathscr{A}$. First we assume that the β-expansion of 1 is

$$1 = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_m}{\beta^m}, \tag{2.1}$$

where $m \geq 3$ and $\varepsilon_m > 0$. Suppose that there exists $\alpha \neq \beta$ such that $\mu_\alpha = \mu_\beta$. If the α-expansion of 1 is

$$1 = \frac{e_1}{\alpha} + \frac{e_2}{\alpha^2} + \cdots + \frac{e_n}{\alpha^n}, \quad e_n > 0 \tag{2.2}$$

then we would have $\alpha = \beta$. In fact if (2.1) holds, then we have $T_\alpha^i 1 \neq T_\alpha^j 1$ for $0 \leq i < j \leq n$. Since $\mu_\alpha = \mu_\beta$, by (2.1) we see that the density function of μ_α is a step function with m pieces. Hence we must have $n = m$. Thus

$$1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{m-1}} = h_\beta(0) = h_\alpha(0) = 1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{m-1}}$$

which gives $\alpha = \beta$. Thereafter we may assume that the α-expansion of 1 is

$$1 = \frac{e_1}{\alpha} + \frac{e_2}{\alpha^2} + \cdots$$

where there are infinitely many $e_n > 0$. Then we get

$$1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{m-1}} = h_\beta(0) = h_\alpha(0) = 1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots \tag{2.3}$$

which gives

$$\alpha = \frac{\beta^m - 1}{\beta^{m-1} - 1} > \beta.$$

We have $T_\beta^{m-1} 1 = \varepsilon_m/\beta$. Then there must exist $i \leq m$ such that

$$\frac{\varepsilon_m}{\beta} = T_\beta^{m-1} 1 = T_\alpha^{i-1} 1 = \frac{e_i}{\alpha} + \frac{e_{i+1}}{\alpha^2} + \cdots \tag{2.4}$$

Then the right hand side of (2.4) is the α-expansion of ε_m/β. Since $\alpha > \beta$ we get $e_i \geq \varepsilon_m$. By (2.3) and (2.4) we obtain

$$\begin{aligned}
&\frac{\varepsilon_m}{\beta} + \frac{\varepsilon_m}{\beta^2} + \cdots + \frac{\varepsilon_m}{\beta^{m-1}} \\
&\quad = \left(\frac{e_i}{\alpha} + \frac{e_{i+1}}{\alpha^2} + \cdots\right) + \frac{1}{\beta}\left(\frac{e_i}{\alpha} + \frac{e_{i+1}}{\alpha^2} + \cdots\right) + \cdots + \frac{1}{\beta^{m-1}}\left(\frac{e_i}{\alpha} + \frac{e_{i+1}}{\alpha^2} + \cdots\right) \\
&\quad > \frac{e_i}{\alpha} + \frac{e_i + e_{i+1}}{\alpha^2} + \frac{e_i + e_{i+1} + e_{i+2}}{\alpha^3} + \cdots + \frac{e_i + e_{i+1} + \cdots + e_{i+m-2}}{\alpha^{m-1}} \\
&\qquad + \frac{e_{m+1} + e_{m+2} + \cdots + e_{i+m-1}}{\alpha^m} + \frac{e_{i+2} + e_{i+3} + \cdots + e_{i+m}}{\alpha^{m+1}} + \cdots
\end{aligned}$$

We use A to denote the last expression. There are two possibilities:

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(a) There exists j with $i+1 \le j \le i+m-2$ such that $e_j > 0$, or

(b) $e_{i+1} = e_{i+2} = \cdots = e_{i+m-2} = 0$.

In case (a) we have

$$\begin{aligned} A &\ge \frac{\varepsilon_m}{\alpha} + \frac{\varepsilon_m}{\alpha^2} + \cdots + \frac{\varepsilon_m}{\alpha^{m-1}} + \frac{e_j}{\alpha^{j-i+1}} + \frac{e_j}{\alpha^{j-i+2}} + \cdots + \frac{e_j}{\alpha^{j+i+m-1}} \\ &\ge \frac{\varepsilon_m}{\alpha-1}\left(1-\frac{1}{\alpha^{m-1}}\right) + \frac{1}{1-\alpha}\cdot\frac{1}{\alpha^{j-i}}\left(1-\frac{1}{\alpha^{m-1}}\right) \\ &\ge \frac{\varepsilon_m}{\alpha-1}\left(1-\frac{1}{\alpha^{m-1}}\right) + \frac{1}{1-\alpha}\cdot\frac{1}{\alpha^{m-2}}\left(1-\frac{1}{\alpha^{m-1}}\right). \end{aligned}$$

On the other hand, we have

$$\frac{\varepsilon_m}{\beta} + \frac{\varepsilon_m}{\beta^2} + \cdots + \frac{\varepsilon_m}{\beta^{m-1}} = \frac{\varepsilon_m}{\alpha} + \frac{\varepsilon_m}{\alpha^2} + \cdots = \frac{\varepsilon_m}{\alpha-1}.$$

If we can show that

$$\text{(2.5)} \qquad \frac{1}{\alpha^{m-2}}\left(1-\frac{1}{\alpha^{m-1}}\right) \ge \frac{\varepsilon_m}{\alpha^{m-1}}$$

then we get a contradiction:

$$\frac{\varepsilon_m}{\beta} + \frac{\varepsilon_m}{\beta^2} + \cdots + \frac{1}{\beta^{m-1}} > A \ge B.$$

Inequality (2.5) is equivalent to

$$\alpha^{m-1} - 1 \ge \varepsilon_m \alpha^{m-2}.$$

Since $\varepsilon_m \le \varepsilon_1 = [\beta] \le [\alpha]$, it is enough to show

$$\alpha^{m-1} - 1 \ge [\alpha]\alpha^{m-2}$$

which is equivalent to

$$\text{(2.6)} \qquad 1 \ge \frac{[\alpha]}{\alpha} + \frac{1}{\alpha^{m-1}}.$$

Thus if (2.6) holds we have $\mu_\alpha \ne \mu_\beta$.

If (2.6) does not hold, since there are only $m-1$ choices for $T_\alpha^i 1$, $i \ge 1$, then we get

$$1 = \frac{[\alpha]}{\alpha} + \frac{l}{\alpha^m} + \frac{l}{\alpha^{2m-1}} + \cdots,$$

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for some $1 \leq l < [\alpha]$. This is included in case (b).

In case (b) we have

$$1 = \frac{e_1}{\alpha} + \frac{e_2}{\alpha^2} + \cdots + \frac{e_i}{\alpha^i} + \frac{e_{i+m-1}}{\alpha^{i+m-1}} + \cdots .$$

Since there are only $m-1$ possibilities for $T_\alpha^j 1, j \geq 1$, we deduce that

$$1 = \frac{e_1}{\alpha} + \frac{e_i}{\alpha^i} + \frac{e_i}{\alpha^{i+m-1}} + \frac{e_i}{\alpha^{i+2m-2}} + \cdots .$$

Then

$$\begin{aligned} T_\alpha^i 1 &= \frac{e_i}{\alpha^{m-1}} + \frac{e_i}{\alpha^{2m-2}} + \cdots \\ &= \frac{e_i}{\alpha^{m-1}-1} < \frac{1}{\beta^{m-2}} . \end{aligned} \tag{2.7}$$

Since $h_\alpha(x) = h_\beta(x)$ for each $1 \leq l \leq m-1$ there would exist $1 \leq k \leq m-1$ such that $T_\beta^l 1 = T_\alpha^k 1$. Then by (2.7) we get

$$1 = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_m}{\beta^m}. \tag{2.8}$$

In fact if (2.8) is not true then we have $T_\beta^j 1 \geq 1/\beta^{m-2}$ for any $1 \leq j \leq m-1$. Now we have

$$\frac{\varepsilon_m}{\beta} = T_\beta^{m-1} 1 = T_\alpha^{i-1} 1$$

and

$$\begin{aligned} \frac{\varepsilon_m}{\beta^2} = T_\beta^{m-2} 1 &= \begin{cases} T_\alpha^{i-2} 1 & i \geq 3, \\ T_\alpha^{m-1} 1 & i = 2 \end{cases} \\ &= \frac{1}{\alpha} T_\alpha^{i-1} 1 = \frac{1}{\alpha} \cdot \frac{\varepsilon_m}{\beta} \end{aligned}$$

which gives $\alpha = \beta$, a contradiction.

Now we consider those β for which the β-expansion of 1 has infinitely many non-zero terms. By the above discussion, if $\mu_\beta = \mu_\alpha$ for some α then the α expansion of 1 must have infinitely many non-zero terms. Since $\mu_\beta = \mu_\alpha$ we have

$$h_\beta(x) = c \cdot h_\alpha(x)$$

for some constant c. Notice that we have

$$\lim_{x \to 1} h_\beta(x) = 1 = \lim_{x \to 1} h_\alpha(x).$$

Then $c = 1$. We also have

$$\lim_{x\to 0} h_\beta(x) = 1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots = \frac{\beta}{\beta - 1}$$

and

$$\lim_{x\to 0} h_\alpha(x) = 1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots = \frac{\alpha}{\alpha - 1}.$$

Therefore, we have $\alpha = \beta$ and the proof is complete. □

3. Proof of Theorem 2

PROOF. Let $\beta > 1$ be a non-integer. Given a Borel set E, by the ergodicity of T_β, for almost all x we have

$$\lim_{N\to\infty} \frac{\{n : n \geq N,\ x_n(\beta) \in E\}}{N} = \mu_\beta(E). \tag{3.1}$$

For $0 < \alpha \leq 1$, by definition, $y_n(\beta) < \alpha$ if and only if

$$x_{n-1}(\beta) \in \bigcup_{i=0}^{[\beta]-1} \left[\frac{i}{\beta}, \frac{i+\alpha}{\beta}\right) \cup \left[\frac{[\beta]}{\beta}, \frac{[\beta] + (\beta - [\beta])\alpha}{\beta}\right). \tag{3.2}$$

For convenience, we use E_α to denote the right hand side of (3.2). Now for almost all x we have

$$\begin{aligned} &\lim_{N\to\infty} \frac{\{n : n \leq N,\ y_n(\beta) \leq \alpha\}}{N} = \mu_\beta(E_\alpha) \\ &= \sum_{i=0}^{[\beta]-1} \left(F\left(\frac{i+\alpha}{\beta}\right) - F\left(\frac{i}{\beta}\right)\right) + F\left(\frac{[\beta] + (\beta - [\beta])\alpha}{\beta}\right) - F\left(\frac{[\beta]}{\beta}\right) \end{aligned}$$

where $F(t) = \mu_\beta(\{x < t\})$. Let $G(t) = \mu_\beta(E_t)$. Then

$$G'(t) = \frac{1}{\beta}\sum_{i=0}^{[\beta]-1} \rho\left(\frac{i+t}{\beta}\right) + \frac{\beta - [\beta]}{\beta}\rho\left(\frac{[\beta] + (\beta - [\beta])t}{\beta}\right), \tag{3.3}$$

where $\rho(x) = h_\beta(x)/c_\beta$ is the density function of μ_β. In order that $\{y_n(\beta)\}$ be uniformly distributed, we need that $G'(t) \equiv 1$. Noting that $\rho(t)$ is a decreasing step function and $G(t)$ is a distribution function, we obtain that $G'(t) \equiv 1$ if and only if each term in the sum of the right hand side of (3.3) is a constant.

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If $\beta \in \mathscr{A}$ then the β-expansion of 1 is

$$1 = \frac{k}{\beta} + \frac{l}{\beta^2}$$

and

$$T_\beta^0 1 = 1, \quad T_\beta^1 \beta = \frac{l}{\beta}, \quad \text{and} \quad T_\beta^n 1 = 0, \quad n \geq 2.$$

Hence

$$h_\beta(x) = \begin{cases} 1 + 1/\beta & \text{if } x < l/\beta, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, each term of the right hand side of (3.3) is a constant. Therefore, $G'(t) \equiv 1$ which implies that $\{y_n(\beta)\}$ is uniformly distributed for almost all x.

Now assume that $\beta \notin \mathscr{A}$. Then we have $T_\beta 1 \neq i/\beta$ for any $0 \leq i \leq [\beta]$. If $i/\beta < T1 < (i+1)/\beta$ where $0 \leq i \leq [\beta - 1]$ then $h_\beta(i+t)/\beta$ is not a constant for $t \in [0, 1)$. In fact if t_1, t_2 satisfies $(i+t_1)/\beta < T_\beta 1 < (i+t_2)/\beta$ then

$$h_\beta\left(\frac{i+t_1}{\beta}\right) - h_\beta\left(\frac{i+t_2}{\beta}\right) \geq \frac{1}{\beta}.$$

If $[\beta]/\beta < T_\beta < 1$ then for $t_1, t_2 \in (0, 1)$ with $([\beta] + (\beta - [\beta])t_1)/\beta < T_\beta 1 < ([\beta] + (\beta - [\beta])t_2)/\beta$ we have

$$h_\beta\left(\frac{[\beta] + (\beta - [\beta])t_1}{\beta}\right) - h_\beta\left(\frac{[\beta] + (\beta - [\beta])t_2}{\beta}\right) \geq \frac{1}{\beta}.$$

In either case we have $G'(t) \not\equiv 1$. This completes the proof. □

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https://doi.org/10.1017/S1446788700036703 Published online by Cambridge University Press