REMARKS ABOUT THE DIGITS OF INTEGERS

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Introduction

Let $n_1 < n_2 < \cdots$ be the sequence of positive integers whose base b representations involve the digit $t \leqq b-1$ at most $d-1$ times. B. D. Craven [2] shows that $\sum_i 1/n_i$ converges by giving an upper bound for this sum as a function of b and d.

The method used by Professor Craven gives an upper bound on the sum $\sum_i 1/0_i$, where $0_1 < 0_2 < \cdots$ is the much larger positive integer sequence whose members have the property that the digit t occurs at most $d-1$ times *in succession* in their base b representations. The first theorem is a corollary to the main theorem in [2].

THEOREM 1. *Let* $0_1 < 0_2 < \cdots$ *be the sequence of positive integers whose base b representations involve the digit* $t \leqq b-1$ *at most* $d-1$ *times in succession. Then* $\sum_i 1/0_i$ *is less than* $b^d(1+d\log b)$.

Using a variation of one of the methods in our article [1], we will give estimates of both types of sum mentioned above.

A method of estimating certain harmonic sums

Let $n = \sum_{r=0}^{\infty} a(n, r)b^r$ be the representation of n to the base b. We define $L(n)$ to be $k+1$, where k is the largest integer i for which $a(n, i) \neq 0$. Clearly $L(n)$ is the number of significant digits in the base b representation of n. For any positive integer n,

$$n < b^{L(n)} \leqq bn. \tag{1}$$

If $\{a_i+jc_i : j = 0, 1, \cdots\}$, $i = 1, 2, \cdots$, is a pairwise disjoint sequence of arithmetic progressions in the positive integers, then

$$\sum_i 1/c_i \leqq 1. \tag{2}$$

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This follows at once from the observation that the natural density of the i-th progression is $1/c_i$.

THEOREM 2. *Let* $n_1 < n_2 < \cdots$ *be the sequence of positive integers whose base b representations involve the digit* $t \leqq b-1$ *at most* $d-1$ *times. Then* $\sum_i 1/n_i$ *is at most* db^2.

PROOF. Let $m_1 < m_2 < \cdots$ be the sequence of positive integers whose base b representations involve the digit t exactly k times. We claim that the sequence of arithmetic progressions

$$\{[m_i + tb^{L(m_i)}] + jb^{L(m_i)+1} : j = 0, 1, \cdots\}, \qquad i = 1, 2, \cdots$$

is pairwise disjoint. Indeed, given any integer in the union of these progressions, the associated m_i is immediately identified by noting where the $(k+1)$-st t occurs in the base b representation of that integer. For example (using arabic notation with $b = 10$, $t = 0$, and $k = 2$), suppose the integer is 103,020,109; then our method of construction would give m_i as 20,109. Hence, $\sum_i b^{-L(m_i)-1} \leqq 1$ by remark (2) and by remark (1) it follows that $\sum_i 1/m_i \leqq b^2$. Letting $k = 0, 1, \cdots, d-1$ and summing, we obtain $\sum_i 1/n_i \leqq db^2$.

THEOREM 3. *Let* $0_1 < 0_2 < \cdots$ *be the sequence of positive integers whose base b representations do not have the digit* $t \leqq b-1$ *appearing d times in succession. Then*

$$\sum_i 1/0_i \leqq b^{d+2}.$$

PROOF. We consider the sequence of progressions

$$\{[0_i + \sum_{k=1}^{d} tb^{L(0_i)+k}] + jb^{L(0_i)+d+1} : j = 0, 1, \cdots\}, \qquad i = 1, 2, \cdots.$$

As before, these progressions are pairwise disjoint, since an integer in the union of the progressions may be associated with a unique 0_i simply by examining the blocks of t's in the integer's base b representation. For example (using arabic notation with $b = 10$, $t = 9$, and $d = 3$), suppose the integer is 699,909,927; then our method of construction identifies 0_i as 9,927.

Thus

$$\sum_i b^{-L(0_i)-d-1} \leqq 1$$

be remark (2), and

$$\sum_i 1/0_i \leqq b^{d+2}$$

by remark (1).

Closing remarks

By a more careful estimate of the sums occurring in the proof of Theorem 2, we could replace db^2 by $Kdb \log b$, where K is a constant slightly larger than 1.

Also, we note that Theorem 3 improves Theorem 1 only if $d \log b > b^2$. Assuming $t = 0$ and using a more elaborate estimate, we could replace b^{d+2} by $Kb^d \log b$ in Theorem 3.

Finally, we wish to thank the referee for a number of comments which helped improve the accuracy and clarity of our article.

References

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[2] B. D. Craven, 'On digital distribution in some integer sequences', *J. Australian Math. Soc.* 5 (1965), 325—330.

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