DISK-LIKE FUNCTIONS

RICHARD J. LIBERA *

(Received 4 November 1968)

Communicated by E. Strzelecki

1. Introduction

The class $\mathscr{S}$ of functions $f(z)$ which are regular and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ each normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1 \tag{1.1}$$

has been studied intensively for over fifty years. A large and very successful portion of this work has dealt with subclasses of $\mathscr{S}$ characterized by some geometric property of $f[\Delta]$, the image of Δ under $f(z)$, which is expressible in analytic terms. The class of starlike functions in $\mathscr{S}$ is one of these [3]; $f(z)$ is starlike with respect to the origin if the segment $[0, f(z)]$ is in $f[\Delta]$ for every z in Δ and this condition is equivalent to requiring that

$$zf'(z)/f(z) \tag{1.2}$$

have a positive real part in Δ.

In this note the class of 'disk-like' functions is introduced by placing restrictions on the behavior of the imaginary part of (1.2) and a representation formula for these functions is given in terms of Robertson's functions which are starlike in one direction [4].

2. The class $\mathscr{D}$

In the definition which follows the notation $g(t) \in \uparrow (a, b)$ means that $g(t)$ is strictly increasing in the interval $a < t < b$. $g(t) \in \downarrow (a, b)$ has a similar meaning.

Definition 1: $f(z)$ is regular in Δ, satisfies (1.1) and $f(z) \neq 0$ for z in Δ unless $z = 0$. $f(z)$ is disk like with respect to the origin in Δ, or $f(z) \in \mathscr{D}$, if and only if one of the following conditions is satisfied:

i) There exists a constant $\rho = \rho(f(z)) > 0$ and two functions $\theta_k(r) = \theta_k(r; f(z))$, $k = 1, 2$; $0 < \theta_2(r) - \theta_1(r) < 2\pi$; such that for $\rho < r < 1$

$$|f(re^{i\theta})| \in \downarrow(\theta_1(r),\ \theta_2(r)) \quad \text{and} \quad |f(re^{i\theta})| \in \uparrow(\theta_2(r),\ \theta_1(r)+2\pi). \tag{2.1}$$

* This work was supported by National Science Foundation Grant NSF-GP-7439.

https://doi.org/10.1017/S1446788700006431 Published online by Cambridge University Press

ii) $f(z)$ is regular in $\bar{\Delta}$, the closure of Δ, and there exist real numbers θ_1 and θ_2, $0 < \theta_2 - \theta_1 < 2\pi$, such that

$$|f(e^{i\theta})| \in \downarrow(\theta_1, \theta_2) \quad \text{and} \quad |f(e^{i\theta})| \in \uparrow(\theta_2, \theta_1 + 2\pi). \tag{2.2}$$

Functions in $\mathscr{D}$ are not necessarily univalent; $\mathscr{D}'$ will denote the class of univalent members of $\mathscr{D}$, i.e., $\mathscr{D}' = \mathscr{D} \cap \mathscr{S}$. $\theta_1(r)$ may be chosen so that $0 \leqq \theta_1(r) < 2\pi$ and a similar choice can be made for θ_1 in (ii). Hereafter, unless otherwise implied, $f(z)$ in $\mathscr{D}$ will satisfy part (i) of the definition as this is no restriction on the development.

A subclass of $\mathscr{D}$, the circularly symmetric functions, was introduced earlier by Jenkins [1] and has appeared in the recent investigations of Krzyż and Reade [2]. Tammi [5] has obtained distortion theorems and coefficient bounds for functions defined in terms of restrictions on the quotient (1.2).

A geometric interpretation of the conditions of Definition 1 may be given. Let $f(z)$, ρ, $\theta_1(r)$ and $\theta_2(r)$ satisfy (i), let $|f(re^{i\theta_1(r)})| = R_1(r)$ and $|f(re^{i\theta_2(r)})| = R_2(r)$ and let C_r, the image of $|z| = r$ under $f(z)$, enclose a domain D_r. Then C_r is contained entirely in the annulus $R_2(r) \leqq |w| \leqq R_1(r)$ and C_r intersects every circle $|w| = R$, $R_2(r) < R < R_1(r)$, exactly twice (perhaps in the same point) for $\rho < r < 1$. If for $R_2(r) < R < R_1(r)$ we let $\Phi_1(r; R)$ and $\Phi_2(r; R)$ be the arguments, chosen to be unique by continuity, of the intersection of $|w| = R$ with the arcs $\{f(re^{i\theta}) | \theta_1(r) < \theta < \theta_2(r)\}$ and $\{f(re^{i\theta}) | \theta_2(r) < \theta < \theta_1(r) + 2\pi\}$, respectively, then D_r contains the arc $\mathrm{Re}^{i\Phi}$, $\Phi_2(r; R) < \Phi < \Phi_1(r; R)$. It is clear that $f(z)$ is not univalent in Δ if there exist r and R such that $\Phi_1(r; R) - \Phi_2(r, R) \geqq 2\pi$ for all choices of arguments. On the other hand $f(z)$ is univalent in Δ whenever $0 < \Phi_1(r; R) - \Phi_2(r; R) < 2\pi$ for all admissible r and R and appropriate $\Phi_1(r; R)$ and $\Phi_2(r; R)$, and conversely; hence $f(z) \in \mathscr{D}'$, or $f(z) = z$, if and only if every circle centered at the origin meets $f[\Delta]$ in a single, non-overlapping arc or not at all. This gives rise to the following observation which we will use: $f(z)$ is univalent for $|z| \leqq r$ if the plane can be cut from $f(re^{i\theta_2(r)})$ to ∞ by a curve which does not meet C_r in any point other than $f(re^{i\theta_2(r)})$.

Returning now to the definition, we see that for $\rho < r < 1$, (2.1) can be written as

$$\frac{\partial}{\partial\theta} |f(re^{i\theta})| \begin{matrix} < 0 \\ > 0 \end{matrix} \quad \begin{matrix} \text{for } \theta_1(r) < \theta < \theta_2(r) \\ \text{for } \theta_2(r) < \theta < \theta_1(r) + 2\pi, \end{matrix} \tag{2.3}$$

or as

$$\frac{\partial}{\partial\theta} \log |f(re^{i\theta})| \begin{matrix} < 0 \\ > 0 \end{matrix} \quad \begin{matrix} \text{for } \theta_1(r) < \theta < \theta_2(r), \\ \text{for } \theta_2(r) < \theta < \theta_1(r) + 2\pi. \end{matrix} \tag{2.4}$$

Because

$$\mathrm{Re}\left\{\frac{\partial}{\partial\theta} \log f(z)\right\} = \mathrm{Re}\left\{\frac{d}{dz} \log f(z) \cdot \frac{dz}{d\theta}\right\} = \mathrm{Re}\left\{i\, \frac{zf'(z)}{f(z)}\right\} = -\mathrm{Im}\left\{\frac{zf'(z)}{f(z)}\right\}$$

for $z = re^{i\theta}$, (2.2) is equivalent to

https://doi.org/10.1017/S1446788700006431 Published online by Cambridge University Press

$$\operatorname{Im}\left\{\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} \begin{matrix} > 0 & \text{if } \theta_1(r) < \theta < \theta_2(r), \\ < 0 & \text{if } \theta_2(r) < \theta < \theta_1(r)+2\pi. \end{matrix} \tag{2.5}$$

The last form upon normalization relates to functions starlike in one direction. Letting

$$f(z) = z+\sum_{k=2}^{\infty} a_k z^k, \; z \in \Delta, \tag{2.6}$$

then

$$\frac{zf'(z)}{f(z)}-1 = a_2 z-(2a_3-a_2^2)z^2+\cdots. \tag{2.7}$$

Consequently $a_2 \neq 0$, otherwise (2.7) has a multiple zero which is a contradiction of (2.5). It is, therefore, no restriction to assume that a_2 is real and positive. The above discussion yields the following conclusions.

THEOREM 1: *$f(z)$ has the form (2.6) and $a_2 > 0$. $f(z)$ is in $\mathscr{D}$ if and only if*

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)}-1\right\}$$

is starlike in the direction of the real axis.

It should be noticed that if $a_2 \neq 0$, then $f(z)$ is not odd; this is consistent with the geometrical interpretation given above. Consequently the identity function $f(z) = z$ is not in $\mathscr{D}$. A modification of Definition 1 to admit simple monotonic rather than strictly monotonic functions in (2.1) and (2.2) admits the identity function into $\mathscr{D}$, in which case (2.7) is identically zero.

Making use of the fact that $g(z)$ regular in Δ and normalized by (1.1) is convex in one direction if and only if $zg'(z)$ is starlike in one direction [4] we can write the last theorem in another form.

COROLLARY: *$f(z) = z+\sum_{k=2}^{\infty} a_k z^k$, $a_2 > 0$, is in $\mathscr{D}$ if and only if $f(z) = ze^{a_2 g(z)}$ and $g(z)$ is convex in the direction of the imaginary axis.*

This can be obtained directly from Theorem 1 or by observing that for $z = re^{i\theta}$, $\rho < r < 1$, (2.4) is equivalent to

$$\operatorname{Re}\frac{\partial}{\partial\theta}\log\left(\frac{f(z)}{z}\right) \begin{matrix} < 0 & \text{for } \theta_1(r) < \theta < \theta_2(r), \\ > 0 & \text{for } \theta_2(r) < \theta < \theta_1(r)+2\pi. \end{matrix} \tag{2.8}$$

Hence,

$$\log\left(\frac{f(z)}{z}\right) = a_2 z+\left(a_3-\frac{a_2^2}{2}\right)z^2+\cdots$$

is convex in one direction.

3. A univalent subclass of $\mathscr{D}$

For a fixed ϕ, $-\pi < \phi \leqq \pi$, we say that $f(z)$ regular in Δ, or in $\bar{\Delta}$, and normalized by (1.1) is in $\mathscr{R}_\phi$ if and only if $f(z)$ is starlike in the direction with inclination ϕ. That is, the line $w = te^{i\phi}$, t real, intersects C_r, the image of $|z| = r$ under $f(z)$, for r near or equal to 1 exactly twice. Evidently $f(z) \in \mathscr{R}_\phi$ whenever $e^{-i\phi}f(ze^{i\phi})$ is starlike in the direction of the real axis, and conversely. Consequently, there exist functions $\tau_1(r;f(z))$ and $\tau_2(r;f(z))$ such that for $\sigma < r < 1$, or for $r = 1$, and suitable choice of arguments, $\phi < \arg\{f(re^{i\phi})\} < \phi+\pi$ whenever

$$\tau_1(r;f(z)) < \theta < \tau_2(r;f(z))$$

and

$$\phi+\pi < \arg\{f(re^{i\phi})\} < \phi+2\pi$$

whenever

$$\tau_2(r;f(z)) < \theta < \tau_1(r;f(z))+2\pi.$$

Clearly σ depends on $f(z)$, $\sigma = \sigma(f(z))$, and in the case $r = 1$ it is assumed that $f(z)$ is holomorphic in $\bar{\Delta}$.

Using these ideas we may restate Theorem 1 in the following useful form.

THEOREM 2: *$f(z)$ has series representation* (2.6) *and $a_2 \neq 0$. $f(z)$ is in $\mathscr{D}$ if and only if*

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)}-1\right\} \in \mathscr{R}_\phi$$

for $\phi = \operatorname{Arg}\{a_2^{-1}\}$.

Choose $\alpha = \operatorname{Arg} a_2$, where Arg denotes the principal argument, then

$$f_0(z) = e^{i\alpha}f(ze^{-i\alpha}) = z+|a_2|z^2+\cdots$$

is in $\mathscr{D}$, since membership in $\mathscr{D}$ is preserved under rotation. Therefore writing

$$|a_2|^{-1}\left\{\frac{zf_0'(z)}{f_0(z)}-1\right\} = g(z),$$

where $g(z)$ is starlike in the direction of the real axis, yields, upon substitution, the relation

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)}-1\right\} = e^{-i\alpha}g(ze^{i\alpha}).$$

The last function is in $\mathscr{R}_{-\alpha}$ and $-\alpha = \phi$.

DEFINITION 2: $f(z)$ and $g(z)$, both in $\mathscr{R}_\phi$, are similar if and only if

$$\tau_k(r;f(z)) = \tau_k(r;g(z)),\ k = 1, 2$$

for $r = 1$ when $f(z)$ and $g(z)$ are regular in $\bar{\Delta}$ and for $\sigma < r < 1$,

$$\sigma = \sigma(f(z); g(z)),$$

otherwise.

Let $\tau_k(r) = \tau_k(r; f(z)) = \tau_k(r; g(z))$, $k = 1, 2$, then a geometric interpretation of similarity is that for r near or equal to 1, $f(re^{i\tau_1(r)})$ and $g(re^{i\tau_1(r)})$ both lie on one ray of the line $w = te^{i\phi}$, t real, whereas $f(re^{i\tau_2(r)})$ and $g(re^{i\tau_2(r)})$ both lie on the complementary ray. $\mathscr{R}_0$ is the class of functions starlike in the direction of the real axis and any two typically-real functions, all of which are in $\mathscr{R}_0$, are similar.

THEOREM 3: *If*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ a_2 \neq 0,\ \textit{and}\ g(z) = a_2^{-1}\left\{\frac{zf'(z)}{f(z)} - 1\right\}$$

are similar and in $\mathscr{R}_\phi$, $\phi = \operatorname{Arg}\{a_2^{-1}\}$, *then* $f(z)$ *is univalent in* Δ.

To give a proof let $\tau_k(r)$, $k = 1, 2$, and σ be as in Definition 2 and the above paragraph. Suppose furthermore that $f(re^{i\tau_1(r)})$ and $g(re^{i\tau_1(r)})$ fall on the ray $w = te^{i\phi}$, $t > 0$, for $\sigma < r < 1$. Then for $z = re^{i\phi}$, $\sigma < r < 1$ and appropriate choice of arguments

$$\phi < \arg\{g(z)\} < \phi + \pi \quad \text{for } \tau_1(r) < \theta < \tau_2(r)$$

and

$$\phi + \pi < \arg\{g(z)\} < \phi + 2\pi \text{ for } \tau_2(r) < \theta < \tau_1(r) + 2\pi;$$

or

$$0 < \arg\{e^{-i\phi} g(z)\} < \pi \text{ when } \tau_1(r) < \theta < \tau_2(r)$$

and

$$\pi < \arg\{e^{-i\phi} g(z)\} < 2\pi \text{ when } \tau_2(r) < \theta < \tau_1(r) + 2\pi.$$

This is equivalent to

$$\operatorname{Im}\left\{\frac{zf'(z)}{f(z)}\right\} \begin{matrix} > 0, & \tau_1(r) < \theta < \tau_2(r) \\ < 0, & \tau_2(r) < \theta < \tau_1(r) + 2\pi. \end{matrix}$$

Consequently, $|f(re^{i\phi})|$ is strictly decreasing for $\tau_1(r) < \theta < \tau_2(r)$ and for a fixed r, $\rho < r < 1$, hence $f(re^{i\phi})$ cuts every circle $|w| = R$, $|f(re^{i\tau_2(r)})| < R < |f(re^{i\tau_1(r)})|$ exactly once at a point $\mathrm{Re}^{i\mu}$, $\phi < \mu < \phi + \pi$ because $f(z) \in \mathscr{R}_\phi$. We see in the same way that every semicircle $\mathrm{Re}^{i\mu}$, $\phi + \pi < \mu < \phi + 2\pi$ is intercepted only once. Therefore $f(z)$ is univalent on $|z| = r$ and is, for that reason [3], univalent for $|z| \leqq r$. A similar argument covers the remaining cases.

References

[1] J. A. Jenkins, 'On circularly symmetric functions', *Proc. Amer. Math. Soc.*, 6 (1955), 620–624.

[2] J. Krzyż and M. O. Reade, 'Koebe domains for certain classes of analytic functions', *J. d'Analyse Math.*, 18 (1967), 185–195.

[3] Z. Nehari, *Conformal Mapping*, (McGraw-Hill, New York, 1952).
[4] M. S. Robertson, 'Analytic functions starlike in one direction', *Amer. J. Math.*, 58 (1936), 465—472.
[5] O. Tammi, 'On certain combinations of the coefficients of schlicht functions', *Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys.*, No. 140, (1952) 13 pp.,

The University of Delaware
Newark, Delaware