GROTHENDIECK GROUPS OF TWISTED FREE ASSOCIATIVE ALGEBRAS

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Let R be an associative ring with identity, X a set of noncommuting variables, $\mathscr{A} = \{\alpha_x\}_{x \in X}$ a set of automorphisms α_x of R and $R_{\mathscr{A}}\{X\}$ the $\mathscr{A}$-twisted free associative algebra on X over R. Let Y be another set of noncommuting variables, $\mathscr{B} = \{\beta_y\}_{y \in Y}$ a set of automorphisms β_y of $R_{\mathscr{A}}\{X\}$ and $S = (R_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\}$ the $\mathscr{B}$-twisted free associative algebra on Y over $R_{\mathscr{A}}\{X\}$. Next, let X_l be a set of noncommuting variables, for each $l = 1, 2, \ldots$. We form the free associative algebra $S_1 = S\{X_1\}$ on X_1 over S and inductively, we form the free associative algebra $S_{l+1} = S_l\{X_{l+1}\}$ on X_{l+1} over S_l, $l = 1, 2, \ldots$. The main purpose of the paper is to prove that if R is a right Noetherian ring of finite right global dimension, then (a) K_0R and $K_0R_{\mathscr{A}}\{X\}$ are isomorphic; (b) K_nR and K_nS are isomorphic, $n = 0, 1$; and (c) K_nR and K_nS_l ($n = 0, 1$) are isomorphic, for each $l = 1, 2, \ldots$.

1. Statements of main theorems. Let R be an associative ring with identity. We denote the Grothendieck group of R by K_0R and the Whitehead group of R by K_1R.

We recall the definition of twisted free associative algebras. For undefined terminologies, we refer to [**4**] and [**2**].

Let R be an associative ring with identity. Let X be a set of noncommuting variables and $\mathscr{A} = \{\alpha_x\}_{x \in X}$ a set of automorphisms α_x of R. The $\mathscr{A}$-twisted free associative algebra on X over R, denoted by $R_{\mathscr{A}}\{X\}$, is defined as follows: additively, $R_{\mathscr{A}}\{X\} = R\{X\}$ so that its elements are finite linear combinations of words $w(x)$ in $x \in X$ with coefficients in R; if $w(x) = x_1 \ldots x_n$ is a word in $x_1, \ldots, x_n$, we denote the automorphism $\alpha_{x_1} \ldots \alpha_{x_n}$ by $w(\alpha)$; multiplication in $R_{\mathscr{A}}\{X\}$ is given by

$$(rw(x))(r'w'(x)) = rw(\alpha)^{-1}(r')w(x)w'(x),$$

for any $rw(x)$, $r'w'(x)$ in $R_{\mathscr{A}}\{X\}$.

In particular, if $X = \{t\}$ and $\mathscr{A} = \{\alpha\}$, then $R_{\mathscr{A}}\{X\}$ is just the α-twisted polynomial ring $R_\alpha[t]$.

We shall consider $R_{\mathscr{A}}\{X\}$ as an R-ring with augmentation $\varepsilon_X : R_{\mathscr{A}}\{X\} \to R$ defined by $\varepsilon_X(x) = 0$ for each $x \in X$. Then the inclusion map $i : R \to R_{\mathscr{A}}\{X\}$ induces a one-to-one homomorphism $i_* : K_nR \to K_nR_{\mathscr{A}}\{X\}$, $n = 0, 1$.

In [**1**, Theorem 2], we have shown that if $K_1R \to K_1R_\alpha[t]$ is an isomorphism for certain automorphisms α of R, then $K_1R \to K_1R_{\mathscr{A}}\{X\}$ is an isomorphism. Farrell has shown in [**3**, Theorem 1.6] that if R is a right Noetherian ring of finite right global dimension, then $K_1R \to K_1R_\alpha[t]$ is an isomorphism, for any automorphism α of R. Hence if R is a right Noetherian ring of finite right global dimension, then $K_1R \to K_1R_{\mathscr{A}}\{X\}$ is an isomorphism. The first purpose of this paper is to show the analogous result for K_0.

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THEOREM 1. *Let R be a right Noetherian ring of finite right global dimension. Then the inclusion map* $i : R \to R_{\mathscr{A}}\{X\}$ *induces an isomorphism* $i_* : K_0R \to K_0R_{\mathscr{A}}\{X\}$.

We remark that for a (non-twisted) free associative algebra, this is contained in [**5**, Corollary 3.9].

Now let Y be another set of noncommuting variables, $\mathscr{B} = \{\beta_y\}_{y \in Y}$ a set of automorphisms β_y of $R_{\mathscr{A}}\{X\}$, and $S = (R_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\}$ the $\mathscr{B}$-twisted free associative algebra on Y over $R_{\mathscr{A}}\{X\}$. We have the natural inclusion maps $j : R_{\mathscr{A}}\{X\} \to S$ and $k : R \to S$. Then we show:

THEOREM 2. *Let R be a right Noetherian ring of finite right global dimension. Then the inclusion map* $k : R \to S$ *induces an isomorphism* $k_* : K_nR \to K_nS$, $n = 0, 1$.

Next, let X_l be a set of noncommuting variables, for each $l = 1, 2, \ldots$. We form the free associative algebra $S_1 = S\{X_1\}$ on X_1 over S, where S is defined as above, and inductively, we form the free associative algebra $S_{l+1} = S_l\{X_{l+1}\}$ on X_{l+1} over S_l, $l = 1, 2, \ldots$. That is, for each $l = 1, 2, \ldots, S_l$ is the ring of the form

$$S_l = (\ldots(((R_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\})\{X_1\})\ldots)\{X_l\}.$$

We note that the polynomial ring $S_l[t]$ is canonically isomorphic to

$$S[t]_l = (\ldots(((R[t]_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\})\{X_1\})\ldots)\{X_l\}, \tag{1}$$

$(l = 1, 2, \ldots)$. Then we extend the results of Theorem 2 to:

THEOREM 3. *Let R be a right Noetherian ring of finite right global dimension. Then* K_nR *and* K_nS_l $(n = 0, 1)$ *are isomorphic for* $l = 1, 2, \ldots$.

2. Some known results. In this section, we collect some results which will be used in the proof of the theorems. First, we recall the following result of Farrell and Hsiang [**4**, Lemmas 23 and 24].

LEMMA 4. *If R is a right Noetherian ring of finite right global dimension, then the twisted polynomial ring* $R_\alpha[t]$ *and the twisted group ring* $R_\alpha[T]$ *are right Noetherian and of finite right global dimension, where T denotes an infinite cyclic group.*

We have observed in [**2**] that a modification of the proof of Farrell's result [**3**, Theorem 1.6] gives:

LEMMA 5. *If R is a right coherent ring of finite right global dimension, then the inclusion map* $R \to R_\alpha[t]$ *induces an isomorphism* $K_1R \to K_1R_\alpha[t]$.

Hence it is immediate from this lemma and [**1**, Theorem 2] that:

PROPOSITION 6. *Let R be a right coherent ring of finite right global dimension. Then the inclusion map* $R \to R_{\mathscr{A}}\{X\}$ *induces an isomorphism* $K_1R \to K_1R_{\mathscr{A}}\{X\}$.

Also, it is clear from [**4**, Theorems 13 and 19] that:

PROPOSITION 7. *Let R be a ring such that* $K_1R \cong K_1R[t]$ *(in particular, let R be a right coherent ring of finite right global dimension). Let T be an infinite cyclic group and* $R[T]$ *the group ring of T over R. Then* $K_1R[T] \cong K_1R \oplus K_0R$.

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It was proved in [2] and [5] that if R is a right Noetherian ring of finite right global dimension, then the free associative algebra $R\{X\}$ is right coherent and of finite right global dimension. In fact, using Lemma 4 and [2, Theorem 2.1] (cf. [5, Proposition 1.9]), we have:

PROPOSITION 8. *Let R be a right Noetherian ring of finite right global dimension. Then the $\mathscr{A}$-twisted free associative algebra $R_{\mathscr{A}}\{X\}$ is right coherent and of finite right global dimension.*

3. Proofs of main theorems. Now we give the proof of our theorems.

Proof of Theorem 1. Since R is right Noetherian and of finite right global dimension, $R_{\mathscr{A}}\{X\}$ is right coherent and of finite right global dimension by Proposition 8. Thus, by Proposition 7, $K_1(R_{\mathscr{A}}\{X\})[T] \cong K_1R_{\mathscr{A}}\{X\} \oplus K_0R_{\mathscr{A}}\{X\}$. Now, we note that $(R_{\mathscr{A}}\{X\})[T]$ is canonically isomorphic to $(R[T])_{\mathscr{A}}\{X\}$ and since $R[T]$ is right Noetherian and of finite right global dimension, by Lemma 4, it follows from Proposition 6 that $K_1R[T] \cong K_1(R[T])_{\mathscr{A}}\{X\} = K_1(R_{\mathscr{A}}\{X\})[T]$. Thus $K_1R[T] \cong K_1R_{\mathscr{A}}\{X\} \oplus K_0R_{\mathscr{A}}\{X\}$. But $K_1R[T] \cong K_1R \oplus K_0R$ by Proposition 7 and $K_1R_{\mathscr{A}}\{X\} \cong K_1R$ by Proposition 6. Hence $K_1R \oplus K_0R \cong K_1R \oplus K_0R_{\mathscr{A}}\{X\}$. Since the composite isomorphism carries K_1 terms to K_1 terms and K_0 terms to K_0 terms, we deduce that $K_0R \cong K_0R_{\mathscr{A}}\{X\}$. This completes the proof.

Proof of Theorem 2. Since R is right Noetherian and of finite right global dimension, the inclusion map $i : R \to R_{\mathscr{A}}\{X\}$ induces an isomorphism $i_* : K_1R \to K_1R_{\mathscr{A}}\{X\}$ by Proposition 6. Now $R_{\mathscr{A}}\{X\}$ is right coherent and of finite right global dimension by Proposition 8, so that the inclusion map $j : R_{\mathscr{A}}\{X\} \to S$ induces an isomorphism $j_* : K_1R_{\mathscr{A}}\{X\} \to K_1S$, again by Proposition 6. Hence the inclusion map $k : R \to S$ induces an isomorphism $k_* : K_1R \to K_1S$.

Next, we note that $S[t]$ is canonically isomorphic to $(R[t]_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\}$. Since $R[t]$ is right Noetherian and of finite right global dimension by Lemma 4, it follows from the first part of the proof that $K_1S[t] \cong K_1(R[t]_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\} \cong K_1R[t]$. But $K_1R[t] \cong K_1R$ by Lemma 5 and $K_1S \cong K_1R$, thus $K_1S \cong K_1S[t]$. Hence, by Proposition 7, $K_1S[T] \cong K_1S \oplus K_0S$, where $S[T]$ is the group ring of an infinite cyclic group T over S. Finally, as in the proof of Theorem 1, we have

$$\begin{aligned} K_1S[T] &\cong K_1(R[T]_{\mathscr{A}}\{X\})_{\mathscr{B}}\{Y\} \\ &\cong K_1R[T] \quad \text{(first part of the proof)} \\ &\cong K_1R \oplus K_0R \quad \text{(Proposition 7).} \end{aligned}$$

Hence $K_1S \oplus K_0S \cong K_1R \oplus K_0R$. Since $K_1S \cong K_1R$, and since the composite isomorphism carries K_1 terms to K_1 terms and K_0 terms to K_0 terms, therefore $K_0S \cong K_0R$. This completes the proof.

Proof of Theorem 3. We prove the result by induction on l for K_1.

As contained in the proof of Theorem 2, we have shown that K_1S and $K_1S[t]$ are isomorphic, where $S[t]$ is the polynomial ring in t over S. Thus, it follows immediately from this fact and the Gersten theorem on (non-twisted) free associative algebra (cf. [1, Theorem 2]) that K_1S and K_1S_1 are isomorphic. Hence, by Theorem 2, K_1R and K_1S_1 are isomorphic. This starts the induction.

Now suppose that, for a right Noetherian ring R of finite right global dimension, $K_1R \cong K_1S_m$ for some $l = m$. Since $R[t]$ is right Noetherian and of finite right global dimension, by the inductive hypothesis,

$$K_1R[t] \cong K_1S[t]_m,$$

where $S[t]_m$ is given by (1). Since $S_m[t] \cong S[t]_m$ and $K_1R \cong K_1R[t]$, therefore $K_1R \cong K_1S_m[t]$ so that $K_1S_m \cong K_1S_m[t]$. Again, by using the Gersten theorem on free associative algebra, we conclude that $K_1S_m \cong K_1S_m\{X_{m+1}\} = K_1S_{m+1}$. Hence $K_1R \cong K_1S_{m+1}$. This finishes the proof that $K_1R \cong K_1S_l$ for $l = 1, 2, \ldots$.

A similar argument as in the proof of Theorem 1 gives $K_0R \cong K_0S_l$ for $l = 1, 2, \ldots$ and this completes the proof.

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