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QUASI-CODIVISIBLE COVERS

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In this paper quasi-codivisible covers are defined and investigated relative to a torsion theory (T, F) on Mod R. It is shown that if (T, F) is cohereditary, then a right R-module M has a quasi-codivisible cover whenever it has a codivisible cover. Moreover, it is shown that if (T, F) is cohereditary, then the universal existence of quasi-codivisible covers implies that the ring $R/T(R)$ must be right perfect. The converse holds when (T, F) is pseudo-hereditary.

In [1], Bass has shown that a ring R is right perfect if and only if every right R-module has a projective cover. Shortly thereafter, Wu and Jans [10] introduced the notion of a quasi-projective cover and showed that if a module has a projective cover, then it has a quasi-projective cover which is unique up to an isomorphism. The dual implication was investigated by Fuller and Hill [4]. They showed that the universal existence of quasi-projective covers implies that of projective covers, and hence that the ring must be right perfect.

In [3], the concept of a codivisible cover relative to a torsion theory (T, F) on Mod R was introduced and studied. Codivisible covers become projective covers when the torsion theory (T, F) is selected to be the torsion theory in which every module is torsionfree. Rangaswamy [8] proved that if the torsion theory (T, F) is pseudo-hereditary, then every right R-module has a codivisible cover if and only if $R/T(R)$ is a right perfect

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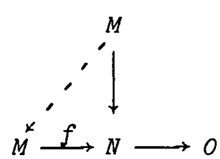
ring where $T(R)$ denotes the torsion ideal of R with respect to $(\mathcal{T},\mathcal{F})$.

The purpose of this paper will be to define quasi-codivisible covers in such a fashion that when $(\mathcal{T},\mathcal{F})$ is the torsion theory in which every module is torsionfree, quasi-codivisible covers become the quasi-projective covers of Wu and Jans. It is then shown that if the torsion theory $(\mathcal{T},\mathcal{F})$ is cohereditary, that is, if $\mathcal{F}$ is closed under taking factor modules, then a module with a codivisible cover has a quasi-codivisible cover which is unique up to an isomorphism. It is also shown that the universal existence of quasi-codivisible covers implies that $R/T(R)$ is right perfect. Hence, the work of Wu and Jans, and Fuller and Hill can be obtained as a special case by selecting the torsion theory $(\mathcal{T},\mathcal{F})$ to be the torsion theory in which every module is torsionfree.

Throughout this paper, R will denote an associative ring with identity and Mod R will be the category of unitary right R-modules. The reader can consult [5], [6], or [9] for the terminology and standard results on torsion theories. $(\mathcal{T},\mathcal{F})$ will be a fixed torsion theory on Mod R and $T(M)$ will denote the torsion submodule of M with respect to $(\mathcal{T},\mathcal{F})$.

DEFINITION 1. A right R-module M is quasi-codivisible if every row exact diagram of the form



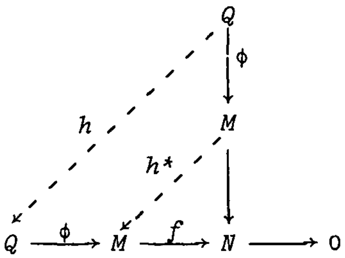
where $\ker(f) \in \mathcal{F}$ can be completed commutatively.

DEFINITION 2. If M is a right R-module, then a quasi-codivisible module Q together with an R-linear epimorphism $\phi : Q \longrightarrow M$ is a quasi-codivisible cover of M if (i) $\ker(\phi)$ is small and torsionfree, and (ii) whenever $0 \neq T \subseteq \ker(\phi)$, then Q/T is not quasi-codivisible.

The following lemma will prove useful.

LEMMA 3. *Let* $\phi : Q \longrightarrow M$ *be an epimorphism where* Q *is quasi-codivisible and* $K = \ker(\phi) \in \mathcal{F}$. *If* K *is stable under endomorphisms of* Q *, then* M *is quasi-codivisible.*

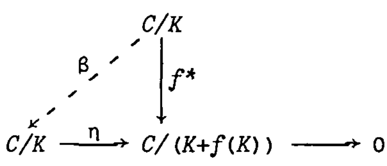
Proof. Consider the row exact diagram



where $\ker(f) \in F$. If $H = \ker(f \circ \phi)$, then $f \circ \phi(T(H)) = 0$, and so $\phi(T(H)) \subseteq \ker(f)$. Hence, it follows that $\phi(T(H)) = 0$, and so $T(H) \subseteq \ker(\phi)$. Consequently, $T(H) = 0$, and so H is torsionfree. Thus, the outer diagram can be completed commutatively by an R-linear map h . But $h(K) \subseteq K$, and so we have an induced map h^* which makes the inner diagram commutative. □

LEMMA 4. *If* (T,F) *is cohereditary and* $\phi : C \longrightarrow C/K : x \longrightarrow x + K$ *is a codivisible cover of the quasi-codivisible module* C/K *, then* K *is stable under endomorphisms of* C .

Proof. Let f be an endomorphism of C . Since the mapping $K \longrightarrow f(K) : x \longrightarrow f(x)$ is an epimorphism and $K \in F$, it follows that $f(K) \in F$. Thus, since $K \oplus f(K) \longrightarrow K + f(K) : (x,y) \longrightarrow x + y$ is an epimorphism, $K + f(K) \in F$. Now f induces a map $f^* : C/K \longrightarrow C/(K+f(K)) : x + K \longrightarrow f(x) + K + f(K)$, and so consider the diagram



where $\eta(x+K) = x + K + f(K)$. The diagram can be completed commutatively by a map β , since C/K is quasi-codivisible and $\ker(\eta) \in F$. Thus, using the codivisibility of C , we have a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\phi} & C/K & & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\ C & \xrightarrow{\phi} & C/K & \longrightarrow & 0 \end{array} .$$

Now let $X = \{x \in C \mid f(x) - \alpha(x) \in K\}$. We claim that $X = C$. Since $\phi \circ \alpha(K) = \beta \circ \phi(K) = 0$, $\alpha(K) \subseteq \ker(\phi) = K$, and so α induces a map $\alpha^* : C/K \longrightarrow C/(K+f(K)) : x + K \longrightarrow \alpha(x) + K + f(K)$. Hence,

$$\begin{aligned} (f^*-\alpha^*)(x+K) &= f^*(x+K) - \alpha^*(x+K) \\ &= \eta \circ \beta(x+K) - (\alpha(x)+K+f(K)) \\ &= \eta \circ \beta(x+K) - \eta \circ \phi \circ \alpha(x) \\ &= \eta \circ \beta(x+K) - \eta \circ \beta \circ \phi(x) \\ &= \eta \circ \beta(x+K) - \eta \circ \beta(x+K) \\ &= 0 . \end{aligned}$$

Thus, $f^*(x+K) - \alpha^*(x+K) = 0$, and therefore $f(x) + K + f(K) - (\alpha(x)+K+f(K)) = 0$. Consequently, $f(x) - \alpha(x) \in K + f(K)$. Now let $f(x) - \alpha(x) = k_1 + f(k_2)$, $k_1, k_2 \in K$. Then $f(x-k_2) - \alpha(x-k_2) = k_1 + \alpha(k_2) \in K$, since $\alpha(k_2) \in \alpha(K) \subseteq K$. Thus, $x - k_2 \in X$, and so $C = K + X$. But K is small, and so $C = X$. Hence, if $x \in K$, then $x \in X$, and so $f(x) - \alpha(x) \in K$. But $\alpha(x) \in K$, and therefore $f(x) \in K$. This shows that $f(K) \subseteq K$. □

PROPOSITION 5. *If* (T,F) *is cohereditary and if* M *has a codivisible cover* $\phi : C \longrightarrow M$ *, then* M *has a quasi-codivisible cover* $\phi^* : Q \longrightarrow M$ *which is unique up to an isomorphism.*

Proof. Use Zorn's Lemma and find the unique maximal submodule X of $K = \ker(\phi)$ which is stable under endormorphisms of C , and set $Q = C/X$. If $\phi^* : Q \longrightarrow M$ is the epimorphism induced by ϕ , then $\ker(\phi^*) = K/X$. Now K is small in C , and so $\ker(\phi^*)$ is small in Q [7]. Note also that $\ker(\phi^*)$ is torsionfree, since (T,F) is cohereditary. Hence, we have a map $C \longrightarrow C/X$ with C codivisible and X torsionfree and stable under endomorphisms of C . Hence, by Lemma 3, Q is quasi-codivisible. Next, let $Y/X \subseteq \ker(\phi^*)$ be such that $(C/X)(Y/X) \cong C/Y$ is quasi-codivisible where $X \subseteq Y \subseteq K$. Then

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$0 \longrightarrow Y \longrightarrow C \longrightarrow C/Y \longrightarrow 0$ is a codivisible cover of C/Y . Hence, it follows from Lemma 4 that Y is stable under endomorphisms of C , and so it must be the case that $X = Y$. Thus, $\phi^* : Q \longrightarrow M$ is a quasi-codivisible cover of M . Now let us show uniqueness. Suppose $\Phi : Q^* \longrightarrow M$ is a quasi-codivisible cover of M and the consider the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{\mu} & \downarrow{\scriptstyle\phi} & & \\ Q^* & \xrightarrow{\Phi} & M & \longrightarrow & 0 \end{array} .$$

Since $\ker(\Phi)$ is torsionfree and C is codivisible, the diagram can be completed commutatively, and so μ must be an epimorphism because $\ker(\Phi)$ is small in Q^* . Now $\ker(\mu) \subseteq K$, and so it follows that $\mu : C \longrightarrow Q^*$ is a codivisible cover of Q^* . Hence, by Lemma 4, $\ker(\mu)$ is stable under endomorphisms of C . Thus, if X is as above, then $\ker(\mu) \subseteq X$. Suppose $\ker(\mu) \neq X$, then $\mu(X) \neq 0$. Now $\Phi \circ \mu(X) = \phi(X) \subseteq \phi(K) = 0$, and so $0 \neq \mu(X) \subsetneq \ker(\Phi)$. Now the map $\mu^* : C \longrightarrow Q^*/\mu(X) : y \longrightarrow \mu(y) + \mu(X)$ is clearly an epimorphism and we claim that $\ker(\mu^*) = X$. If $y \in X$, then $\mu(y) + \mu(X) = 0$, and so $\mu^*(y) = 0$. Hence, $X \subseteq \ker(\mu^*)$. Now if $y \in \ker(\ *)$, then $\mu(y) + \mu(X) = 0$. Let $\mu(y) = \mu(x)$, so that $\mu(y-x) = 0$. Then $y - x \in \ker(\mu) \subset X$, and so we see that $y \in X$, and therefore that $\ker(\mu^*) \subseteq X$. Hence, $\ker(\mu^*) = X$. But this implies that $Q = C/X \cong Q^*/\mu(X)$ which contradicts Definition 2, since $\Phi : Q^* \longrightarrow M$ is a quasi-codivisible cover of M . Consequently, we must have $\ker(\mu) = X$, and so $\mu(X) = 0$. But this yeidls $Q = C/X \cong Q^*$, and therefore Q is unique up to an isomorphism. □

PROPOSITION 6. *If* (T,F) *is cohereditary and every right* R*-module has a quasi-codivisible cover, then* $R/T(R)$ *is a right perfect ring. Moreover, if* (T,F) *is pseudo-hereditary, then the converse holds.*

Proof. Let M be a right $R/T(R)$-module and suppose that $\theta : F \longrightarrow M$ is a free $R/T(R)$-module on M . Next, suppose that $\phi : Q \longrightarrow F \oplus M$ is a quasi-codivisible cover of $F \oplus M$. Since $(F \oplus M)T(R) = 0$, $QT(R) \subseteq \ker(\phi)$. Now Beachy [2] has shown that $QT(R) = T(Q)$ when (T,F) is cohereditary, and so $QT(R) = 0$, since $\ker(\phi)$ is torsionfree. Thus, Q is an $R/T(R)$-module. Now consider the diagram

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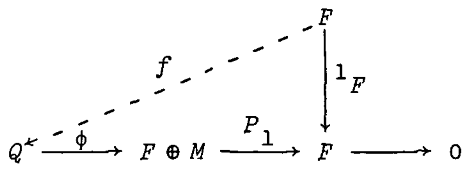
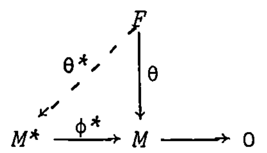
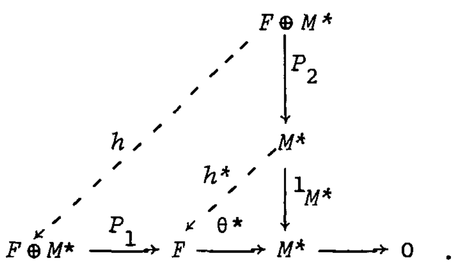


where p_1 is the first projection map and f is the completing map given by the projectivity of F. If $M^* = \ker(p_1 \circ \phi)$, then we can assume that $Q = F \oplus M^*$. We claim that if $\phi^* = \phi|_{M^*}$, then $\phi^* : M^* \longrightarrow M$ is an $R/T(R)$-projective cover of M. Clearly ϕ^* is an epimorphism with small kernel, and so we consider the diagram



which completes commutativity by the projectivity of F. Note that θ^* is an epimorphism, since $\ker(\phi^*)$ is small in M^*. Hence, we have a diagram



Since $0 \longrightarrow \ker(\phi^*) \longrightarrow M^* \xrightarrow{\phi^*} M \longrightarrow 0$ is exact and $\ker(\phi^*)$ and M are torsionfree R-modules, it follows that M^* is torsionfree because $\mathcal{F}$ is closed under extensions. Consequently, $\ker(\theta^* \circ p_1)$ is torsionfree, and so we have a completing map h. If $j_2 : M^* \longrightarrow F \oplus M^*$ is the canonical injection and $h^* = p_1 \circ h \circ j_2$, then the inner diagram is commutative, and so M^* is a projective $R/T(R)$-module. Thus, $\phi^* : M^* \longrightarrow M$ is an $R/T(R)$-projective cover of M, and thus $R/T(R)$ is right-perfect.

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For the converse, if (T,F) is pseudo-hereditary, Rangaswamy [8] has shown that when $R/T(R)$ is right perfect, every right R-module has a codivisible cover. Hence, the result follows from Proposition 5. □

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