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ON DIFFERENTIAL EQUATIONS OF VON GEHLEN AND ROAN

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Abstract. Polynomials appearing in the description of ground states of superintegrable chiral Potts models are shown to satisfy a special class of generalised hypergeometric differential equations after a simple modification. This proves a conjecture of von-Gehlen and Roan.

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1. Introduction. Let $N \geq 2$ be a positive integer and $\omega = \exp(2\pi i/N)$ a primitive Nth root of unity. Take a pair of linear operators $X, Z \in \mathrm{End}(\mathbf{C}^N)$ that satisfies the following commutation relation and the normalisation condition:

$$ZX = \omega XZ, \quad X^N = Z^N = id.$$

The superintegrable chiral Potts Hamiltonian (see for example [**1**], [**2**]) on a chain of length L is a linear operator on $(\mathbf{C}^N)^{\otimes L}$ defined by

$$H(k') = -\sum_{l=1}^{L}\sum_{n=1}^{N-1} \frac{2}{1-\omega^{-n}} \left(X_l^n + k' Z_l^n Z_{l+1}^{N-n}\right),$$

where k' is a real parameter and X_l denotes the operator acting on the lth component as X and for other components as identity.

Note that if we write

$$H(k') = H_0 + k' H_1,$$

H_0 and H_1 satisfy the Dolan–Grady relation

$$[H_i, [H_i, [H_i, H_j]]] = 4N^2 [H_i, H_j], \quad i, j = 0, 1$$

and give a representation of the so-called Onsager algebra, which can also be viewed as either a deformation of the nilpotent part of the affine Lie algebra of type $A_1^{(1)}$ or a quotient of the loop algebra of $\mathfrak{sl}_2$.

The principal problem in statistical mechanics defined by this operator is to find eigenvalues and eigenvectors. Bethe Ansatz affords us a method for such purpose.

It is known that ground state eigenvalues and eigenvectors are described by zeroes of polynomials F_j (cf. [**1**, **2**, **3**, **4**, **6**]) defined by the relation

$$\left(\frac{t^N-1}{t-1}\right)^L = \sum_{j=0}^{N-1} t^j F_{j+1}(s), \qquad s = t^N.$$

In [**3**, **4**, **6**] von Gehlen and Roan derived a system of first-order differential equation for F_j.

The vector of polynomials

$$F = {}^t(F_1, F_2, \ldots, F_N)$$

satisfy

$$Ns(s-1)\frac{dF}{ds} = BF,$$

$$B = \begin{pmatrix} d_0 & -Ls & \cdots & -Ls \\ -L & d_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -Ls \\ -L & \cdots & -L & d_{N-1} \end{pmatrix},$$

$$d_j = L(N-1)s - j(s-1).$$

When $N = 2$ (the Ising case) each polynomial satisfies Gauß hypergeometric differential equation. Further by a suitable change of variable the polynomials turn out to be Chebyshev polynomials, and this was convenient for the description of eigenvalues.

Proceeding further they also derived third-order differential equations for the case $N = 3$. One of them takes the following form:

$$\begin{aligned} &27s^2(s-1)^2F_1''' - 27s(s-1)((2L-4)s+2)F_1'' \\ &\quad + 3(3L^2s(4s-1) - 3Ls(10s-7) + 2(s-1)(10s-1))F_1' \\ &\quad - (L-1)(L(L(8s+1) - 4(s-1))F_1 = 0. \end{aligned}$$

These equations have regular singular points only at $s = 0, 1, \infty$, although this in not explicitly mentioned in [**3**, **4**, **6**]. They also studied the zeroes of polynomials in the case of $N = 3$ numerically.

Based on such calculations they conjectured that each of F_j satisfies an Nth-order ordinary differential equations of the form that follows.

CONJECTURE 1.

$$\begin{aligned} &N^N s^{N-1}(s-1)^{N-1}\frac{d^N F_j}{ds^N} \\ &\quad + \sum_{k=1}^{N-1} N^k s^{k-1}(s-1)^{k-1} D_{jk}(s)\frac{d^k F_j}{ds^k} + D_{j0}(s)F_j = 0, \end{aligned}$$

where D_{jk} are polynomials in s.

In this paper we show that after a simple transformation the scalar differential equations in question are generalised hypergeometric differential equations, which form a special class of Fuchsian differential equations which have regular singular points only at three points 0, 1, ∞ and no accessory parameters (rigid system).

We find that defining G by $G(s) = (s-1)^{-L}F(s)$ the differential equations for G_j become a special kind of generalised hypergeometric differential equations.

For a given generalised differential equation, there corresponds a system of first-order differential equations of Okuba type. The explicit relationship is given for example in [**5**]. However the converse direction seems not to be known. In fact as we will see in our case each G_j satisfies different generalised differential equations.

The detailed proof will appear elsewhere.

2. A normal form of differential equations. The differential equations for G takes the following form:

$$N\frac{dG}{ds} = \left(-\frac{L}{s-1}A_1 + \frac{1}{s}A_0\right)G,$$

$$A_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ L & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ L & \cdots & L & -N+1 \end{pmatrix}.$$

First we look for an Nth-order matrix P and numbers a_j, b_j which satisfy the following relations:

$$\frac{1}{N}LPA_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_0 & a_1 - b_1 \cdots & a_{N-2} - b_{N-2} & a_{N-1} - b_{N-1} \end{pmatrix} P, \tag{1}$$

$$\frac{1}{N}PA_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 0 \\ 0 & & \cdots & 0 & 1 \\ 0 & -b_1 & \cdots & -b_{N-2} & -b_{N-1} \end{pmatrix} P. \tag{2}$$

If we can find such non-singular matrix P, then the first component $(PG)_1$ of PG is annhilated by the generalised hypergeometric differential operator with the parameters a_j, b_j:

$$s\left(\sum_{j=0}^{N} a_j\vartheta^j\right) - \sum_{j=1}^{N} b_j\vartheta^j, \quad \vartheta = s\frac{d}{ds}, \quad a_N = 1,\ b_N = 1. \tag{3}$$

Factorising as

$$\sum_{j=0}^{N} a_j\vartheta^j = \prod_{j=1}^{N}(\vartheta + \alpha_j), \tag{4}$$

$$\sum_{j=1}^{N} b_j\vartheta^j = \vartheta\prod_{j=1}^{N-1}(\vartheta + \beta_j - 1), \tag{5}$$

we have another form of generalised hypergeometric differential operator.

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3. Transformation matrix. Define

$$c_i = (r(N, i-1, x) - s(N, i-1))/(-N)^{N-i+1}, \quad f_i = -s(N, i-1)/(-N)^{N-i+1},$$

where $r(N, i, x)$ is defined by

$$\sum_{i=0}^{N} r(N, i, x)t^i = \prod_{j=1}^{N} (t + x - j + 1)$$

and $s(N, i)$ denotes the Stirling number of the first kind.

We set

$$p_{ij} = (-1)^{N+j} \sum_{s=0}^{j-1} \binom{N-L-1}{s} \binom{L}{j-1-s} (L+s)^{i-1}/(-N)^{i-1}$$

and consider the square matrix P of order N with its (i, j) entries p_{ij}.

PROPOSITION 1. *The matrix P satisfies*

$$\frac{L}{N} P \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ c_1 & \cdots & c_N \end{pmatrix} P,$$

$$\frac{1}{N} P \begin{pmatrix} 0 & 0 & \cdots & 0 \\ L & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ L & \cdots & L & -N+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 0 \\ 0 & & \cdots & 0 & 1 \\ 0 & f_2 & \cdots & f_{N-1} & f_N \end{pmatrix} P.$$

4. Inverse matrix. Define q_{ij} by the relation

$$\sum_{j=0}^{N-1} q_{i,j+1} t^j = \prod_{k=0}^{i-2} (t - L - k) \prod_{k=i}^{N-1} (t - k).$$

The matrix Q with entries $(-N)^{j-1} q_{ij}$ satisfies the relation

$$QP = \prod_{k=1}^{N} (-L + k) I_N,$$

where I_N is the identity matrix of order N.

5. Scalar differential operator. Assume $L > N$. Then using Proposition 1 we see that the entries in (1), (2) are given by

$$b_j = -(-N)^{-N+j-1} s(N, j-1),$$
$$a_j = (-N)^{-N+j-1} N^{-N+j-1} r(N, j-1, -L).$$

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The corresponding Nth-order differential operator (3) is expressed as

$$s\prod_{k=1}^{N}\left(\vartheta+\frac{L+k-1}{N}\right)-\prod_{k=1}^{N}\left(\vartheta+\frac{k-1}{N}\right).$$

Defining $H = PG$, we see that the first component H_1 is annihilated by the above operator.

Further using the inverse matrix Q components of G are given as

$$\begin{aligned}G_i &= \sum_{j=1}^{N}(-N)^{j-1}q_{ij}H_j/\prod_{k=1}^{N}(k-L)\\ &= (-1)^{N-1}\prod_{k=0}^{i-2}(N\vartheta+L+k)\prod_{k=i}^{n-1}(N\vartheta+k)\,H_1.\end{aligned}$$

Defining

$$L_i = s\prod_{k=1}^{n}(N\vartheta+L+i+k-2)-\prod_{k=1}^{n}(N\vartheta+i-k)$$

and using

$$\vartheta s = s(\vartheta+1),$$

we have the following.

THEOREM 1.

$$L_iG_i = 0, \quad i = 1, \dots, N.$$

Rewriting these differential equations those for F_j and assuming that L is a positive integer, we proved the conjecture of von Gehlen and Roan.

6. Power series solutions at $s = 0$. Here we assume that L is a positive integer. Let us consider generalised hypergeometric series

$$\begin{aligned}&F\left(\begin{matrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_n & \\ \beta_1, & \beta_2, & \cdots, & \beta_{n-1}, & 1\end{matrix}\Big|s\right)\\ &= \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_n)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_{n-1})_k k!}s^k,\\ &(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1),\end{aligned}$$

where $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_{n-1}$ are parameters. The symbols $(\alpha)_k$ are sometimes called Pochhammer symbol.

As is known solutions of generalised hypergeometric differential equation (3) around $s = 0$ are given by

$$F\left(\begin{matrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_n & \\ \beta_1, & \beta_2, & \cdots, & \beta_{n-1}, & 1\end{matrix}\Big|s\right),$$

with $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_{N-1}$ defined by the relations (4), (5) and also

$$s^{1-\beta_j} F\left(\begin{matrix} 1+\alpha_1-\beta_j, & \cdots, & 1+\alpha_{j-1}-\beta_j, \\ 1+\beta_1-\beta_j, & \cdots, & 1+\beta_{j-1}-\beta_j \end{matrix}\right. \left.\begin{matrix} 1+\alpha_j-\beta_j, & 1+\alpha_{j+1}-\beta_j, & \cdots, & 1+\alpha_N-\beta_j & \\ 2-\beta_j, & 1+\beta_{j+1}-\beta_j, & \cdots, & 1+\beta_{N-1}-\beta_j & 1 \end{matrix} \;\middle|\; s\right),$$

for $j = 1, \dots, N-1$.

Therefore in our case the power series solutions of $L_i f = 0$ are given by the following generalised hypergeometric series:

$$F\left(\begin{matrix} \frac{L+i-1}{N}, & \frac{L+i}{N}, & \dots, & \frac{L+N-1}{N}, & \dots, & \frac{L+i+N-2}{N} \\ \frac{i}{N}, & \frac{i+1}{N}, & \dots, & 1, & \dots, & \frac{i+N-1}{N} \end{matrix} \;\middle|\; s\right).$$

In our case since the parameters are special, the product of Pochhammer symbols in the coefficients are simplified. As a result we have the following series:

$$\sum_{k=0}^{\infty} \frac{(L+i-1)_{kN}}{(i)_{kN}} s^k.$$

We see that these are essentially a sum of binominal series in $s^{1/N}$:

$$\frac{1}{N} \sum_{j=0}^{N} f_i(\omega^j s^{1/N}), \qquad \omega = \exp(2\pi i/N),$$

$$f_i(x) = \sum_{n=0}^{\infty} \frac{(L+i-1)_n}{(i)_n} x^n$$
$$= \frac{1}{\binom{-L}{i-1}} \left(x^{1-i}(1-x)^{-L} - x^{1-i} \sum_{k=0}^{i-2} \binom{-L}{k} (-x)^k \right).$$

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