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THREE THEOREMS ON THE GROWTH OF ENTIRE TRANSCENDENTAL SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

SH. STRELITZ

1. G. Polya [4] has posed the problem as to whether there are entire transcendental functions of order zero satisfying an algebraic differential equation with rational coefficients. G. Polya himself showed that this is impossible for a first order algebraic differential equation. The general problem is now completely solved. G. Valiron demonstrated an example of a third order algebraic differential equation with an entire transcendental solution of order zero (Theorem 1); V. V. Zimogljad (Theorem 2) proved that every entire transcendental solution of a second order algebraic differential equation is of a positive order. It seems to us expedient to bring these results all together. We give here a proof of Theorem 2 different from and in our view simpler than that of V. V. Zimogljad. Theorem 3 refines the results of G. Polya (and of others, see for example [10]) and establishes an exact lower bound for the order of an arbitrary entire transcendental solution satisfying a first order algebraic differential equation. Our proof of Theorem 2 we state below and our results of Theorem 3 were published to our knowledge only in Russian in [6] and are hardly accessible to the English reader, so that this publication may be helpful to bridge the gap.

2. We formulate in this section the three theorems indicated in the title of the paper.

THEOREM 1. (G. Valiron [9], [10]). *There is an entire transcendental function of order zero which satisfies a third order algebraic differential equation with rational coefficients.*

THEOREM 2. (V. V. Zimogljad [11]). *Every entire transcendental solution of a second order algebraic differential equation with rational coefficients is of a positive order.*

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THEOREM 3. (Sh. Strelitz [**6**]). *Every entire transcendental solution of a first order algebraic differential equation with rational coefficients is of an order no less than* 1/2. *The number* 1/2 *is exact: there is a first order algebraic differential equation with rational coefficients which has an entire solution of order* 1/2.

3. Proof of theorem 1. It is known from the elliptic function theory that the Weierstrass function $\sigma(z)$ satisfies the differential equation

$$(3.1)\qquad \left(\frac{\sigma'}{\sigma}\right)''^{2} = 4\left(\frac{\sigma'}{\sigma}\right)'^{3} + A_0\left(\frac{\sigma'}{\sigma}\right)' + B_0$$

([**7**], Volume 2, p. 19) where A_0 and B_0 are certain constants. (All the constants here and in the sequel in this section depend on the periods of the corresponding elliptic Weierstrass function; we will not remind the reader of this fact below.) On the other hand ([**7**], Volume 1, p. 175, 178)

$$(3.2)\qquad \sigma(z) = C_0\theta_1(z)e^{az^2} \quad (C_0,\ a = \text{const}),\ \text{where}$$

$$\theta_1(z) = A'(e^{\beta z} - e^{-\beta z})\prod_{n=1}^{\infty}(1 - q^{2m}e^{2\beta z})(1 - q^{2m}e^{-2\beta z}),$$

A', β, q are constants and $|q| < 1$. Hence the function

$$\left(\frac{\sigma'}{\sigma}\right)' = \left(\frac{\theta_1'}{\theta_2}\right)' + 2a$$

satisfies equations (3.1) and so $\theta_1(z)$ is a solution of a third order algebraic differential equation with constant coefficients. Let $u = e^{\beta z}$. Then the function

$$(3.3)\qquad \theta(u) = \theta_1\left(\frac{1}{\beta}\ln u\right)$$

$$= A\left(u - \frac{1}{u}\right)\prod_{m=1}^{\infty}\left(1 - \frac{1}{(q^m + q^{-m})^2}\left(u - \frac{1}{u}\right)^2\right);$$

$$A = A'\prod_{m=1}^{\infty}(1 + q^{2m})^2$$

satisfies the equation

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$$(3.4)\qquad \beta^4u^2\left[u\left(u\frac{\theta'}{\theta}\right)'\right]'^2 = 4\left[\beta u\left(u\frac{\theta'}{\theta}\right)' + a\right]^3 + A_0\left[\beta u\left(u\frac{\theta'}{\theta}\right)' + a\right] + B_0.$$

Now

$$F(v) = \theta(u) = Av\prod_{m=1}^{\infty}(1-(q^m+q^{-m})^{-2}v^2);\quad v = u - \frac{1}{u}$$

is an entire transcendental solution of order zero of equation (3.4) if we replace there u by v according to the equality $u = v + \sqrt{v^2+4}$. After this substitution we rationalize (3.4). $F(v)$ is thus a solution of a third order algebraic differential equation with rational coefficients.

Theorem 1 is proven.

4. In order to prove Theorems 2 and 3 we need certain properties of analytic functions. We state them in this section without proofs.

Let $f(z)$; $z = re^{i\varphi}$ be an analytic function in the disc $|z| < R \leqq \infty$. Denote

$$(4.1)\qquad M(r) = \max_{|z|=r}|f(z)|,\quad r < R,$$

and by $\xi = re^{i\varphi(r)}$ the point on the circle $|z| = r$ where $M(r) = |f(\zeta)|$. We call a function $h(r)$ *algebroidal* in the segment $[r_1, r_2]$ if $h(r)$ is analytic at every point of the interval (r_1, r_2) and may have at the endpoints of the segment only algebraic singularities.

We have:

a) By a suitable choice of the points $\zeta = re^{i\varphi(r)}$ $\varphi(r)$ is a piecewise algebroidal function in every segment $0 \leqq r \leqq \rho < R$ (note that $\varphi(r)$ is not necessarily continuous in $[0, \rho]$: a continuous choice for $\varphi(r)$ is in general impossible). In the sequel $\varphi(r)$ will always mean the function described here (see [**1**], [**5**]).

b) The function $M(r)$ is continuous and piecewise algebroidal in every segment $0 \leqq r \leqq \rho < R$ ([**1**], [**5**]).

c) If $f(z) \not\equiv$ Const then always

$$(4.2)\qquad \frac{\zeta f'(\zeta)}{f(\zeta)} = \frac{rM'(r)}{M(r)} > 0,\quad r > 0$$

where at each point of discontinuity of $\varphi(r)$ the relation (4.2) is correct from the right and from the left of the point (the set of such points is as we saw in a) finite in each segment $[0, \rho]$, $\rho < R$).

d) We define

$$(4.3) \qquad K(r) = \frac{rM'(r+0)}{M(r)} = \frac{\zeta f'(\zeta)}{f(\zeta)}$$

with $\zeta = re^{i\varphi(r+0)}$. $K(r)$ is an increasing function in $[0, R)$ provided $f(z) \not\equiv Ax^m$, A, m = Const ([**5**]). At each point ζ we have by differentiation of (4.3)

$$(4.4) \qquad rK'(r) = \zeta\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)'(1 + ir\varphi'(r))$$

where at a singular point of $\varphi(r)$ we suppose $K'(r) = K'(r+0)$ and $\varphi'(r) = \varphi'(r+0)$ (here may be $K'(r) = \infty$ $\varphi'(r) = \infty$ the set of such point in each segment $0 \leqq r \leqq \rho < R$ is finite).

e) From (4.2) it follows ($r_0 > 0$)

$$(4.5) \qquad \ln M(r) - \ln M(r_0) = \int_{r_0}^{r} \frac{K(t)}{t}\, dt$$

and, since $K(t)$ is a non-decreasing function, then

$$(4.6) \qquad K(r_0) \ln \frac{r}{r_0} \leqq \ln M(r) - \ln M(r_0) \leqq K(r) \ln \frac{r}{r_0}.$$

In particular for $r_0 = r/2$ and $\ln M(r_0) \geqq 0$ we have

$$K\left(\frac{r}{2}\right) \ln 2 \leqq \ln M(r)$$

so that

$$(4.7) \qquad \varlimsup_{r\to\infty} \frac{\ln K(r)}{\ln r} \leqq \varlimsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r}.$$

Remark that for an entire transcendental function

$$K(r) \underset{r\to\infty}{\longrightarrow} \infty.$$

f) E. Borel's Theorem [**2**]. Let $h(r)$:$h(r) > 0$ be a non-decreasing continuous function from the right on $[0, \infty)$ with $h(r) \uparrow \infty$. Then outside a certain sequence of intervals E_0 of finite measure

$$h\left(r + \frac{1}{h^{1+\alpha}(r)}\right) - h(r) < 1; \quad 0 < \alpha = \text{an arbitrary constant.}$$

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The sequence $E_0 = E_0(\alpha)$ has the following property: each segment $[0, R]$ has common points no more than with a finite number of intervals of E_0.

Applying this theorem to the function $\ln h(e^t)$ we obtain the inequality

$$(4.8) \quad h(re^{\tau}) - h(r) < 2h(r)$$

with

$$(4.9) \quad \tau \leqq \ln^{-(1+\alpha)} h(r),$$

which is correct outside a sequence E having the same property described in Borel's theorem for the set E_0 and is of bounded logarithmic measure: $\int_E dr/r < \infty$.

g) An algebraic equation $F(z, w) = 0$, where $F(z, w)$ is a polynomial in z and w and of degree n with respect to w has in the neighborhood of $z = \infty$ n solutions of the form

$$(4.10) \quad w = (1 + o(1))A_k z^{m_k}, \quad k = 1, 2, \dots, n$$

with complex constants A_k and rational constants m_k ([**3**], Chapter 12). Some solutions may coincide.

h) Let $A = \{(p_j, n_j)\}_{j=0}^{m}, p_k < p_{k+1}, k = 1, 2, \dots, m - 1$ be a set of points on the real plane $\mathbf{R}^2$. Denote by A^* the convex cover of A. A^* is a closed polygon and has two vertices whose abscissae are the endpoints of the orthogonal projection of A^* on the abscissae axis. These two vertices divide A^* into two parts. Let A° be the upper part of A^* and λ_q, $q = 1, 2, \dots, m_0$ the slopes of the sides of A° (A° is a version of the Puisseu-Newton-Hadamard polygon; see for example [**3**], [**10**]). Fix now the number q. Suppose that the points $(p_{j_{t_k}}, n_{j_{t_k}})$, $k \equiv 1, 2, \dots, s_q$ and only they lie on the side of A° with the slope λ_q. Then

$$(4.11) \quad (m - j_{t_i})\lambda_q + n_{j_{t_i}} = (m - j_{t_k})\lambda_q + n_{j_{j_k}}, k, i = 1, 2, \dots, s_q$$

and

$$(4.12) \quad (m - j_{t_u})\lambda_q + n_{j_{t_i}} > (m - j)\lambda_q + n_j;$$
$$j \neq t_i, i = 1, 2, \dots, s_q, j = 1, 2, \dots, m_0$$

(see for example [**3**], [**6**]).

5. Proof of theorem 2 [**6**]. Let $w(z)$ be an entire transcendental solution of order zero of a second order algebraic differential equation.

$$(5.1) \quad F(z, w, w', w'') = 0$$

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where $F(z, w, u, v)$ is a polynomial in all its variables and of degree $s \geqq 1$ with respect to w, u and v. Denote by H the set of points on the complex plane where the following two conditions are satisfied:

i. $|w(z)| > [M(r)]^{1-1/4s}$, $|z| = r$, $M(r) = \max_{|z|=r}|w(z)|$

and

ii. $K^{1/3}(r - 0) < \left|\dfrac{zw'(z)}{w(z)}\right| < K^{5/3}(r)$; $K(r) = \dfrac{rM'(r + 0)}{M(r)}$

(see Section 4).

Since $w(z)$ and $zw'(z)/w(z)$ are continuous functions outside the set of zeroes of $w(z)$ and $K(r)$ is piecewise continuous and monotonous in each segment $0 \leqq r \leqq R < \infty$, then the set H has no isolated points and consists of connected components only. From (4.3) we immediately conclude that on each circle $|z| = r$ there are points belonging to H. From i) and ii) we additionally obtain that

$$\lim_{\substack{|z|\to 0 \\ z \in H}} \frac{|z|^p \left|\dfrac{zw'(z)}{w(z)}\right|^q}{|w(z)|} = 0 \tag{5.2}$$

for arbitrary fixed real constants p and q, because according to (4.7)

$$\lim_{r\to\infty} \frac{\ln K(r)}{\ln r} = 0 \tag{5.3}$$

for an entire function of order zero (see (4.7)). We have

$$\left| z\left(\frac{zw'(z)}{w(z)}\right)' \right| < C_0 \, |z|^2 \, M^{1/4s}(\,|z|\,), \quad C_0 = \text{Const} \tag{5.4}$$

for $z \in H$. In order to prove this note first that

$$z\left(\frac{zw'}{w}\right)' = \frac{z^2w''}{w} - \left(\frac{zw'}{w}\right)^2 + \frac{zw'}{w}. \tag{5.5}$$

Let $z \in H$. Then

$$w''(z) = \frac{1}{\pi i} \int_{|z-y|=1} \frac{w(y)}{(y - z)^2}\, dy.$$

Consequently

$$|w''(z)| \leqq 2M(r + 1); \quad |z| = r. \tag{5.6}$$

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In view of (4.6)

$$\ln M(r+1) - \ln M(r) \leqq K(r+1) \ln \left(1 + \frac{1}{r}\right)$$

and according to (5.3)

$$K(r+1) \ln \left(1 + \frac{1}{r}\right) \underset{r \to \infty}{\to} 0$$

so that $M(r+1) < C_1 M(r)$, $C_1 = \text{Const}$ and (5.6) gives us: $|w''(z)| < 2C_1 M(r)$. Thus for $z \in H$

$$(5.7) \quad \left|\frac{z^2 w''(z)}{w(z)}\right| < 2C_1 \frac{r^2 M(r)}{[M(r)]^{1-1/4s}} = 2C_1 r^2 M^{1/4s}(r).$$

From (ii), (5.3) and (5.7) from (5.5) we obtain (5.4).

We substitute now w for $w(z)$ in equation (5.1) and bring it to the following form

$$(5.8) \quad \sum_{j=0}^{n} P_j\left(z, \frac{zw'}{w}, z\left(\frac{zw'}{w}\right)'\right) w^{-j}(z) = 0.$$

Recall now that all the polynomials $P_j(z, u, v)$ are no more than of degree $2s$ in u, v. Then for $z \in H$ in view of i), (5.2) and (5.4) we have

$$(5.9) \quad P_0\left(z, \frac{zw'}{w}, z\left(\frac{zw'}{w}\right)'\right) = \omega_0(z)$$

with

$$(5.10) \quad \lim_{\substack{z \to \infty \\ z \in H}} z^N \omega_0(z) = 0$$

for every real N.

If P_0 does not depend on $z(zw'/w)'$ then by the usual Wiman-Valiron method [**10**], [**6**], it follows from (5.9) that Theorem 2 is correct.

Suppose therefore in the sequel that P_0 from (5.9) depends on $z(zw'/w)'$. We rewrite now (5.9) in the form:

$$(5.11) \quad \sum_{j=0}^{n} Q_j\left(z, \frac{zw'}{w}\right) \left[z\left(\frac{zw'}{w}\right)'\right]^{m-j} = \omega_0(z).$$

Denote

$$(5.12)\quad \frac{zw'(z)}{w(z)} = e^{i\theta}z^{\omega} = \eta; \quad \theta = \theta(z),\ \omega = \omega(z).$$

For $z \in H$ according to i) and ii), (5.3) and e), Section 4

$$\omega(z) > 0; \quad \omega(z) \underset{z\to\infty}{\to} 0,\ z^{\omega(z)} \underset{z\to\infty}{\to} \infty;$$

besides $\omega(z)$ is a continuous function. Put

$$\theta = \theta(z),\ \omega = \omega(z) \quad \text{and} \quad \frac{zw'(z)}{w(z)} = \eta.$$

Then

$$(5.13)\quad Q_j(z, \eta) = \sum_{p=0}^{n_j} Q_{jp}(z)\eta^p = \sum_{p=0}^{n_j} \sum_{q=0}^{s_p} a_{jp}^q\, z^{s_p - q} e^{ip\theta} z^{p\omega}$$

$$= \sum_{p=0}^{n_j} (1 + o(1))a_{jp}^0\, e^{ip\theta} z^{s_p + p\omega},\ o(1) \underset{\substack{z\to\infty\\ z\in H}}{\to} 0 .$$

Let

$$n_j^0 = \max_{0 \leqq p \leqq n_j} (s_p + \omega(z)p).$$

For $|z| > 0$, $z \in H$ with $|z| > r_0$ sufficiently large this maximum is achieved by only one value of p, say $p = \rho^*$. Indeed this is obvious if all the s_p are different numbers, since

$$\omega(z) \underset{z\to\infty}{\to} 0, \quad z \in H.$$

If $\max_{0 \leqq p \leqq n_j} s_p = s_{p_1} = s_{p_2} = \ldots = s_{p_l}$, $p_i < p_{i+1}$, then

$$n_j^\circ = s_{p_l} + \omega(z)p_l$$

since $\omega(z) > 0$. Denote $s_{p*} = \sigma_j$; $p^* = t_j$ and $a_{jp*}^\circ = A_j$. From (5.13) it follows

$$(5.14)\quad Q_j(z, \eta) = (1 + o(1))\, A_j e^{i\theta_j} z^{\sigma_j + \omega t_j}$$

(we use the fact that $z^{\omega(z)} \to \infty$). Put $z(zw'/w) = \mu$. Equality (5.11) obtains now the form

$$(5.15)\quad \sum_{j=0}^{m} (1 + o(1))A_j e^{i\theta t_j} z^{\sigma_j + \omega t_j} \mu^{m-j} = \omega_0(z).$$

Our first step is now to find solutions of the equation

$$(5.16)\quad \sum_{j=0}^{m} (1 + o(1))\, A_j e^{i\theta t_j} z^{\sigma_j + \omega t_j} \mu^{m-j} = 0$$

in the neighborhood of $z = \infty$, $z \in H$. For this purpose we consider the set points of $A = \{ (j, \sigma_j + t_j\omega) \}_{j=0}^{m}$, $\omega = \text{Const}$ on the plane $\mathbf{R}^2$. Let A° be the polygon described in h) Section 4 and let

$$(5.16')\quad \lambda_q = \nu_q(z) + \beta_q(z)\omega_q(z), \quad q = 1, 2, \ldots, m_0$$

be the slopes of the sides of A° for various z. It is important that for $z \in H$, $|z| > r_0$ with r_0 great enough $\beta_q(z) \equiv \beta_q = \text{Const}$, $\nu_q(z) \equiv \nu_q = \text{Const}$, $q = 1, 2, \ldots, m_0$. To see it note that λ_q equals a certain ratio

$$\frac{\sigma_k - \sigma_l + (t_k - t_l)\omega}{l - k}$$

and that those equalities

$$(5.17)\quad \frac{\sigma_k - \sigma_l + (t_k - t_l)\omega}{l - k} = \frac{\sigma_k - \sigma_j(t_k - t_j)}{j - k}, \quad k \neq l,\ k \neq j$$

different from identities define only a finite number of ω values. Thus for $\omega > 0$ small enough ($\omega(z) \to 0$, see above) the relation (5.17) can be true only if it is an identity in ω. Fix now the number q and find all the points of A which lie on the side of A° with the slope $\nu_q + \beta_q\omega$. Let these be the points

$$(j_p, \sigma_{j_p} + t_{j_p}\omega), \quad p = 1, 2, \ldots, s_0, j_p > j_{p+1}.$$

Then according to (4.11) and (4.12)

$$(5.18)\quad (m - j_{p_k})\lambda_q + n_{j_k} = (m - j_l)\lambda_q + n_{j_l}; \quad k, l = 1, 2, \ldots, s_0$$

and

$$(5.19)\quad (m - j_k)\lambda_q + n_{j_k} > (m - j)\lambda_q + n_j;$$
$$j \neq j_k;\ k = l, j = 0, 1, \ldots, m.$$

By the transformation

$$(5.20)\quad \mu = e^{i\beta_q\theta} z^{\nu_q + \beta_q\omega} \nu$$

we obtain from (5.16)

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$$\sum_{j=0}^{m} (1 + o(1))A_j e^{i(t_j+(m-j)\beta_q)\theta} z^{(m-j)\nu_q+\sigma_j+[(m-j)\beta_q+t_j]\omega} v^{m-j} = 0$$

whence in view of (5.18) and (5.19)

$$(5.21)\quad \sum_{k=1}^{s_0} (1 + o(1))A_{j_k} v^{j_{s_0}-j_k} + \sum_{\substack{l\neq k\\ k=1,2,\ldots,s_0}} A_{j_l}(1 + O(1))z^{-\sigma^\circ_{j_l}-\beta^\circ_{j_l}\omega} e^{-i\beta_{j_l}\theta} v^{m-j} = 0.$$

But

$$\sigma^\circ_{j_l} + \beta^\circ_{j_l}\omega = (j_1 - j_l)\nu_q + \sigma_{j_l} - \sigma_{j_1} + [(j_1 - j_l)\beta_q + t_{j_l} - t_{j_1}]\omega \geqq a\omega$$

with a certain constant $a > 0$, so that

$$(5.22)\quad \sum_{k=1}^{s_0} (1 + o(1))A_{j_k} v^{j_{s_0}-j_k} = O\left(\frac{1}{z^{a\omega}}\right), \quad a\omega(z) \underset{z\to\infty}{\to} \infty,\ z \in H.$$

Since the solutions of an algebraic equation are continuous functions of its coefficients we get from (5.22) s_0 roots:

$$v = (1 + o(1))a_i;\ 0 \neq a_i = \text{Const},\ i = 1, 2, \ldots, s_0.$$

Coming back to the unknown μ we obtain s_0 solutions of (5.16) corresponding to the considered side of A°:

$$(5.23)\quad \mu_k = (1 + o(1))a_k e^{i\beta_q\theta} z^{\nu_q+\beta_q\omega},\quad k = 1, 2, \ldots, s_0.$$

Since under the conditions c) and ii) in Section 5

$$\left|Q_o\left(z, \frac{zw'}{w}\right)\right| \geqq C_o > 0,$$

equation (5.15) yields now:

$$(5.24)\quad \prod_{k=1}^{m} \left[z\left(\frac{zw'}{w}\right)' - \mu_k(z)\right] = \omega_0(z);\quad z \in H;\ |z| > r_0$$

where

$$\omega_0(z)z^N \underset{z\to\infty}{\to} \infty \quad \text{for every fixed real } N.$$

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(5.24) shows that at each point $z \in H$, $|z| > r_0$ one of the equalities

$$(5.25)\quad z\left(\frac{zw'}{z}\right)' - \mu_k(z) = \omega_k(z), \quad k = 1, 2, \dots, m,$$

where $\mu_k(z)$ are defined in (5.23) with

$$\omega_k(z)z^N \underset{z\to\infty}{\longrightarrow} 0, \quad k = 1, 2, \dots, m,$$

has to be satisfied. Remark additionally that if $A_m = 0$ in (5.15) then among the equalities (5.24) there is

$$(5.26)\quad z\left(\frac{zw'}{w}\right)' = \omega_*(z).$$

6. We go on with the proof of Theorem 2. According to (5.12) in view of (5.23) from (5.25) we obtain:

$$(6.1)\qquad z\left(\frac{zw'}{w}\right) = a_k(1 + o(1))z^{\nu_k}\left(\frac{zw'}{w}\right)^{\beta_k}, \quad z \in H, k = 1, 2, \dots, m.$$

Suppose that at some point $z \in H$ is the equality

$$(6.2)\qquad z\left(\frac{zw'}{w}\right)' = (1 + o(1))\, az^{\nu}\left(\frac{zw'}{w}\right)^{\beta}$$

with $a = a_k$, $\nu = \nu_k$, $\beta = \beta_k$ and a certain k holds, $|z| > r_0$. The way we found the equalities (6.1) shows that in the connected component $G_0 \in H$, $|z| > r_0$, to which the considered z belongs, (6.1) is satisfied everywhere with the same constants, a, ν and β.

Suppose now that the maximum point ζ of $|w(z)|$, $|\zeta| = r > r_0$ belongs to G_0, so that at this point (6.2) is true. Let first $\nu < 0$. In this case there is an annular region $r_0 < r' < |z| < r''$ belonging to G_0. Our first step in proving this statement is to show that the circle $C:|z| = |\zeta| = r$ belongs to G_0. Suppose we are wrong and that there are points on this circle where at least one of the conditions i) or ii) of Section 5 does not take place. Let z^* be a point on this circle where condition i) is not satisfied, that is

$$|w(z^*)| \leqq [M(r)]^{1-1/4s}.$$

The function $w(z)$ is continuous on C and therefore there is an arcus $l = (\zeta, z_0)$ on C such that

$$|w(z)| > [M(r)]^{1-1/4s},$$

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$z \neq z_0$ on it and

$$|w(z_0)| = [M(r)]^{1-1/4s}.$$

We have to consider two cases:

I. On l condition ii) is satisfied, then $l \in G_0$ and

II. At some point $\tilde{z} \in l$ ii) is violated.

The case II can take place only if either

$$\left|\frac{\tilde{z}w'(\tilde{z})}{w(\tilde{z})}\right| \leqq K^{1/3}(r-0) \tag{6.3}$$

or

$$\left|\frac{\tilde{z}w'(\tilde{w})}{w(\tilde{z})}\right| \geqq K^{5/3}(r). \tag{6.4}$$

Suppose now (6.3) to be correct. Then there exists a point $z' \in l$ such that

$$\left|\frac{z'w'(z)}{w(z)}\right| = K^{1/2}(r) \tag{6.5}$$

and $l_0 = \{z:z \in (\tilde{\zeta}, z')\} \subset G_0$. On l_0

$$\left|\frac{z'w'(z)}{w(z)}\right| < K^{5/3}(r)$$

and therefore

$$\left|\frac{zw'(z)}{w(z)}\right| < r^{5/3\omega(r)}.$$

Consequently on l_0

$$az^{\nu}\left(\frac{zw'(z)}{w(z)}\right) = o(1),\ o(1) \underset{|z|\to\infty}{\to} 0$$

and we can rewrite equation (6.2) as

$$\left(\frac{zw'(z)}{w(z)}\right)' = o(1). \tag{6.6}$$

By integration along l_0

$$\frac{zw'}{w} = K(r) + o(1). \tag{6.7}$$

This equality is contradictory to (6.3). Similarly we show that (6.4) is impossible too. Thus case II cannot take place and $l_0 = l$. By further integration of (6.7) along l putting $\zeta = re^{i\varphi(r)}$ and $z_0 = re^{i\varphi_0}$ we find

$$(6.8) \qquad \ln w(z_0) = \ln w(\zeta) + iK(r)(\varphi_0 - \varphi(r)) + o(1)$$
$$\Rightarrow |w(z_0)| = (1 + o(r))\, M(r).$$

This equality is not compatible with (6.5). Hence $l = C \subset G_0$. Isolated maximum points do not exist and therefore there is an annular region $r_0 < |z| < r''$ belonging to G_0. Moreover the whole region $r_0 < |z| < \infty$ belongs to G_0. Indeed suppose the region $r_0 < |z| < R$ belongs to G_0. Then the circle $\{ |z| = R\} \subset G_0$ too, because on $|z| = R$ there is a maximum point of $|w(z)|$. Thus on $G_0' = G_0 \cap \{ |z| > r_0\} = \{ |z| > r_0\}$ (6.2) takes place with constant a, ν and β. Consequently (6.9) is satisfied in G_0' whence it follows that the function $w'(z)/w(z)$ has no poles in G_0'. But this is impossible for an entire transcendental function of order zero. Thus (6.2) cannot take place if $\nu < 0$. The same is obviously true in case of equation (5.26) too.

Suppose now that in G_0, a connected component of H, the equality (6.1) is satisfied with $\nu = 0$, that is

$$(6.9) \qquad z\left(\frac{zw'(z)}{w(z)}\right)' = (1 + o(1))\, a\left(\frac{zw'(z)}{w(z)}\right)^{\beta}$$

with $0 \neq a = \text{Const.}$

Let first $\beta < 0$ and as above let $l = (\zeta, z_0) \subset G_0$. Suppose $l \neq C$. Then we come to the alternative I – II. But

$$\left|\frac{zw'}{w}\right| \geqq K^{1/3}(r) \underset{r\to\infty}{\longrightarrow} \infty$$

as above in G_0 so that (6.9) is reduced to

$$\left(\frac{zw'}{w}\right)' = o(1)$$

and we return to (6.6). Thus equation (6.8) with $\beta < 0$ is also impossible.

7. Completion of the proof of theorem 2. We have to consider now equation

(7.1) $$z\left(\frac{zw'}{w}\right) = (1 + o(1))\, a\left(\frac{zw'}{w}\right)^{\beta}$$

in G_0 while $\beta \geqq 0$.

Suppose first that $0 \leqq \beta < 1$. Let ζ, z_0, z, $l = (\zeta, z_0) \subset G_0$ and $l' = (\zeta, z') \subseteq l$ have the same meanings as in the previous section so that at z' either (6.3) or (6.4) is satisfied. By integration of (7.1)

(7.2) $$\left(\frac{zw'}{w}\right)^{1-\beta} - K^{1-\beta}(r) + i(1 + o(1))(1 - \beta)\, a(\varphi - \varphi(r))$$

$$= K^{1-\beta}(r) \cdot \left[1 + \frac{i(1 + o(1))(1 - \beta)a(\varphi - \varphi(r))}{K^{1-\beta}(r)}\right]$$

$$\Rightarrow \frac{zw'}{w} = K(r) + i(1 + o(1))aK^{\beta}(r)(\varphi - \varphi(r))$$

so that II is wrong. Thus

$$l_0 = l \subset G_0 \quad \text{and} \quad |w(z_0)| = M^{1-1/4s}(r).$$

By further integration of (7.2)

(7.3) $$\ln \frac{w(z)}{w(\zeta)} = iK(r)(\varphi - \varphi(r)) - \frac{1}{2}(1 + o(1))aK^{\beta}(r)(\varphi - \varphi(r))^2$$

$$\Rightarrow \ln w(z) = \ln M(r) - \frac{1}{2}(1 + o(1))\{\text{Re}\, a\}K^{\beta}(r)(\varphi - \varphi(r))^2.$$

According to (4.6)

(7.4) $$K(r)\tau \leqq \ln M(re^{\tau}) - \ln M(r)$$

and in view of (4.8) and (4.9)

(7.5) $$\ln M(re^{\tau}) - \ln M(r) < 2 \ln M(r)$$

for

$$\tau = \ln^{1+\alpha} \ln M(r)$$

with an arbitrary constant $\alpha > 0$ outside a set of intervals $E(\alpha)$ of finite logarithmic measure. Now Equation (7.5) shows that

$$K(r) < 2 \ln M(r)\{\ln \ln M(r)\}^{1+\alpha}, \quad r \notin E(\alpha).$$

Hence for $r \notin E(\alpha)$ since $0 \leqq \beta < 1$

$$\frac{K^{\beta}(r)}{\ln M(r)} < \frac{2^{\beta}\ln^{\beta} M(r)\{\ln \ln M(r)\}^{\beta(1+\alpha)}}{\ln M(r)} \underset{r\to\infty}{\to} 0.$$

Now (7.3) gives us

$$\ln |w(z)| = (1 + o(1)) \ln M(r)$$
$$\Rightarrow |w(z)| = [M(r)]^{1+o(1)}, \ o(1) \underset{r\to\infty}{\to} 0, \ r \notin E(\alpha).$$

But this equality contradicts our supposition I at z_0. Thus for $r = |z| \notin E(\alpha)$ we have $l = C \subset G_0$. But, as it follows from (7.2), on C $zw'(z)/w(z)$ is a multivalued function, whence we conclude that $w(z)$ can satisfy (7.1) if at all only at a sequence of annular regions $r_j' < |z| < r_j''$, $j = 1, 2, 3\dots$, such that

$$\sum_{j=1}^{\infty} \ln \frac{r_j''}{r_j'} = \int_{E(\alpha)} \frac{dr}{r} < \infty.$$

So there remains the following possibility: either $\beta \geqq 1$ in (7.1) or $\nu > 0$ in (6.2), when $r = |z| \notin E(\alpha)$. If $\nu > 0$ then

$$(7.6) \qquad r^{\omega(\zeta)} = K(r) = \frac{\zeta w'(\zeta)}{w(\zeta)}, \ \omega(\zeta) \underset{r\to\infty}{\to} 0 \quad \text{and}$$
$$r = K^{\zeta/\omega(\zeta)}(r), \ \omega(1) \underset{r\to\infty}{\to} 0.$$

Let

$$\min_{a_k \neq 0} \{ |a_k| \} = A_0 > 0,$$

where a_k are the coefficients in (6.1).

From (6.1) with respect to (7.5) we obtain in both of the here described cases at the maximum point ζ of $w(z)$:

$$\left|\zeta\left(\frac{\zeta w'(\zeta)}{w(\zeta)}\right)'\right| \geqq (1 + o(1))A_0K(r); \quad |\zeta| = r, \ r \notin E(\alpha)$$

whence in view of (4.4) for $r \notin E(\alpha)$

$$rK'(r) \geqq (1 + o(1))A_0K \Rightarrow d \ln K(r) \geqq (1 + o(1))A_0 \frac{dr}{r}$$
$$\Rightarrow \ln K(r) \geqq A_0(1 + o(1))\left[\int_{r_0}^{r} \frac{dr}{r} - \int_{E(\alpha)} \frac{dr}{r}\right]$$

because $K(r)$ is an increasing function. But

$$\int_{E(\alpha)} \frac{dr}{r} < \infty$$

so that

$$\ln K(r) \geqq (A_0 + o(1)) \ln r.$$

This means in accordance with (4.7) that $w(z)$ is of a positive order.

Thus we see that if $w(z)$ is an entire transcendental solution of order zero of equation (5.1) no one equality of (5.25) at the point ζ with $|z| \notin E(\alpha)$ can be satisfied. But this contradicts the fact proven in Section 4.

Hence (5.1) has no entire transcendental solutions of order zero.

Theorem 2 is completely proven.

8. Proof of theorem 3. Let $w(z)$ be an entire transcendental solution of order less than 1/2 of the algebraic first order differential equation

$$(8.1) \quad \sum_{j=0}^{n} P_j\left(z, \frac{zw'}{w}\right) w^{n-j} = 0,$$

where $P_j(z, n)$ are polynomials with respect to both the variables. We denote as usual in this paper by ζ the maximum point of the function $|w(z)|$ on the circle $|z| = r$:$|w(\zeta)| = \max_{|z|=r}|w(z)| = M(r)$; $|\zeta| = r$. Put

$$(8.2) \quad \frac{rM'(r+0)}{M(r)} = K(r).$$

At the maximum points ζ we have from (8.1) according to (4.3):

$$(8.3) \quad P_0(\zeta, K(r)) = -\sum_{j=1} P_j(\zeta, K(r)) w^{-j}(\zeta).$$

But for an entire transcendental function $w(z)$ of order less than 1/2 $K(r) < Cr^{1/2}$, $C =$ Const (see (4.7)). Besides

$$\frac{\ln M(r)}{\ln r} \underset{r\to\infty}{\to} \infty.$$

So

$$\left|\frac{P_j(\zeta, K(r))}{w(\zeta)}\right| \underset{r\to\infty}{\to} 0, \quad j = 1, 2, \dots, n$$

and (8.3) gives us

$$(8.4) \quad P_0(\zeta, K(r)) = o(1).$$

Suppose $P_0(z, \eta)$ to be of degree m with respect to η. The equation $P_0(z, \eta) = 0$ has in the neighbourhood of $z = \infty$ m solutions of the form (4.10):

$$\eta = (1 + o(1))A_j z^{p_j}, \quad j = 1, 2, \dots, m.$$

Thus we can rewrite (8.4) for $|z| > R_0$ as

$$\prod_{j=1}^{m} (K(r) - (1 + o(1))A_j z^{p_j}) = o(1).$$

Hence $K(r)$ satisfies one of the equalities

$$(8.5) \quad K(r) - (1 + o(1))A_j \zeta^{p_j} = o(1), \quad j = 1, 2, \dots, m.$$

But $K(r)$ is an increasing function, and therefore for $r > r_0$ where r_0 is large enough

$$(8.5) \quad K(r) = (1 + o(1))A_j \zeta^{p_j} = (1 + o(1))Br^{\rho}$$

with a constant j, $|A_j| = B$ and $p_j = \rho > 0$, $\rho < 1/2$, where ρ is the order of the solution $w(z)$ of (1.2). Consider on the complex plane the set H of points where the inequality

$$(8.6) \quad |w(z)| > M^{\alpha_0}(r) \; ; \quad |z| = r, \; \alpha_0 = \frac{1}{2} \cos \pi\rho$$

holds. H is obviously an open set. Evidently every maximum point ζ belongs to H. We will now show that on the connected arcus $l \subset H$ of the circle $|z| = r$ containing ζ

$$(8.7) \quad \left| \frac{zw'(z)}{w(z)} \right| < K^2(r), \quad |z| = r = |\zeta|$$

if $r > r_0$ with a sufficiently large r_0. In order to prove (8.7) suppose we are wrong, so that at a certain point $z_0 \in l$

$$(8.8) \quad \left| \frac{z_0 w'(z_0)}{w(z_0)} \right| = K^2(r) \quad \text{and} \quad \left| \frac{zw'(z)}{w(z)} \right| < K^2(r), \quad z \neq z_0$$

for $|z| = r$, $z \in l_0 = (\zeta, z_0) \subset l$. Hence for $z \in l_0$

$$\frac{|z|^p \left| \frac{zw'(z)}{w(z)} \right|}{|w(z)|} \leqq \frac{r^p K^{2q}(r)}{M^{\alpha_0}(r)} \underset{r \to \infty}{\longrightarrow} 0$$

in view of (8.5).

Equation (8.3) shows now that on l_0

$$P_0\left(z, \frac{zw'(z)}{w(z)}\right) = o(1),$$

whence it follows that $zw'(z)/w(z)$ satisfies on l_0 one of the equalities

$$(8.9) \quad \frac{zw'(z)}{w(z)} = (1 + o1))A_j z^{p_j}, \quad j = 1, 2, 3, \ldots, m$$

(see (4.10) and (8.5)). Note now that $zw'(z)/w(z)$ is a continuous function on $l_0 \subset l$ and that $\zeta \in l_0$. At $\zeta \in l_0$ (8.5) is correct with $p_j = \rho$, so that (8.3) is right with $p_j = \rho$ and

$$\left|\frac{z_0 w'(z_0)}{w(z_0)}\right| = (1 + o(1))Br^\rho = (1 + o(1))K(r)$$

in contradiction with (8.8). Thus on l (8.7) is true for $z \subset l \subset H$ and

$$(8.10) \quad \frac{zw'(z)}{w(z)} = (1 + o(1))A_j z^\rho.$$

Our intention is now to show that l coincides with the whole circle $C:|z| = r$. In order to prove it suppose that there is a point $z_0 \in C$ with $|w(z_0)| < M^{\alpha_0}(r)$. By integration of (8.10) along the arcus $(\zeta, z_0) = l_0 \subset l$ assuming $z = re^{i\varphi}$, $\zeta = re^{i\varphi(r)}$, $A_j = Be^{i\alpha}$ and $|\varphi(r) - \varphi| \leqq \pi$ (such an arcus l_0 obviously exists) we obtain:

$$\ln \frac{w(z)}{w(\zeta)} = (1 + o(1)) \frac{Br^\rho e^{i\alpha}}{\rho} (e^{i\rho\varphi} - e^{i\rho\varphi(r)}).$$

Consequently

$$\ln |w(z)| = \ln M(r) + (1 + o(1))[\cos(\rho\varphi + \alpha) - \cos(\rho\varphi(r) + \alpha)].$$

By integration of (8.5) from r_0 to r (in view of (8.2)) we get

$$\ln M(r) = (1 + o(1)) \frac{Br^\rho}{\rho}$$

so that on l_0

$$(8.11) \quad \ln w(z) = \frac{Br^\rho}{\rho}\left[1 + \cos(\rho\varphi + \alpha) - \cos(\rho\varphi(r) + \alpha)\right] + o(1)r^\rho.$$

The equality (8.5) shows now that at every point ζ we have

$$e^{i(\alpha + \varphi(r)\rho)} = 1 + o(1).$$

Consequently

$$\cos(\rho\varphi(r) + \alpha) = 1 + o(1) \quad \text{and}$$
$$\cos(\rho\varphi + \alpha) = (1 + o(1)) \cos \rho(\varphi - \varphi(r)).$$

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We rewrite now (8.11) in the form

$$\ln|w(z)| = (1 + o(1))\frac{Br^{\rho}}{\rho}\cos\rho(\varphi - \varphi(r)) + o(1)r^{\rho}.$$

But $\rho < \frac{1}{2}$ and $|\varphi - \varphi(r)| \leqq \pi$. Then

$$\begin{aligned}\ln|w(z)| &\geqq \frac{Br^{\rho}}{\rho}(\cos\pi\rho + o(1))\\ &= (1 + o(1))\cos\pi\rho\ln M(r)\\ &= (1 + o(1))\ln[M(r)]^{2\alpha_0}.\end{aligned}$$

Whence tending $z \rightarrow z_0$ on l_0

$$|w(z_0)| \geqq M^{(2+o(1))\alpha_0}(r)$$

in contradiction with our supposition

$$|w(z_0)| < M^{\alpha_0}(r).$$

Thus $l = C$ and $|w(z)| \geqq M^{\alpha_0}(r) > 0$ on C for $r = |z| > r_0$. But this is impossible for an entire transcendental function of order $\rho < 1/2$. So $\rho \geqq 1/2$.

To complete the proof of Theorem 3 we note that the entire function $\cos\sqrt{z}$ of order $\rho = 1/2$ satisfies the algebraic differential equation

$$y^2 + 4zy'^2 = 1.$$

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University of Haifa,
Haifa, Israel