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ANISOTROPIC PRINCIPAL SERIES AND GENERATORS OF A FREE GROUP

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In this paper we prove that the equivalence of anisotropic principal series of a free group Γ related to different generator sets induces a Γ-isomorphism between the related Cayley-graphs. As a consequence we obtain that a nontrivial change of generators for Γ leads to inequivalent anisotropic principal series.

1. INTRODUCTION

The subject of this paper is discrete noncommutative harmonic analysis, and more precisely harmonic analysis for anisotropic random walks of a finitely generated free group Γ on homogeneous trees. Various explicit constructions of irreducible unitary representations of Γ may be found in Cartier [3] Figà-Talamanca and Picardello [7], Pytlik [15], Mantero and Zappa [12, 13], Cowling and Steger [4], Kuhn and Steger [11] and Figà-Talamanca and Steger [8]. Almost of all the representations constructed may be realised as boundary representations of Γ.

We are particularly interested in the study of the representations belonging to the anisotropic principal series of Γ. This work arises from a generalisation of a result that was obtained analysing the isotropic case [14]. We fix two bases for Γ and construct the corresponding Cayley-graphs on which Γ acts by left multiplication. Now we select two representations π_1 and π_2, one in each of the anisotropic principal series of Γ related to the fixed bases. We prove that if π_1 and π_2 are equivalent as unitary representations, then there exists a Γ-isomorphism of trees between the Cayley-graphs considered. As a consequence of this result we obtain that a nontrivial change of generators in a free group leads to inequivalent families of anisotropic principal representations of Γ. Using a result of Culler and Morgan on ℝ-trees [5] this is equivalent to the claim that the length translation function related to the chosen generator sets are inequivalent.

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2. Definitions and general results

Let (Γ, A) be a noncommutative free group on finitely many generators, where $A = \{a_j^{\pm 1} \mid a_j \in \Gamma \text{ with } j = 1, \dots, q+1\}$ consists of the basis elements and their inverses. Let Φ_A be the Cayley-graph of (Γ, A), and Ω_A be its boundary. We identify the vertices of Φ_A with the group elements generated by $\{a_1, \dots, a_{q+1}\}$ and the boundary Ω_A of Φ_A with the set of all infinite reduced words $\omega = a_{j_1} a_{j_2} \cdots$. A boundary representation of (Γ, A) belonging to the anisotropic principal series of (Γ, A) is obtained from a particular pair (ν, P) where:

(1) ν is a Radon measure on Ω_A, and for every $\gamma \in (\Gamma, A)$ we have that $d\nu(\gamma^{-1}\omega)$ is absolutely continuous with respect to $d\nu(\omega)$.

(2) P mapping $(\Gamma, A) \times \Omega_A \to \mathbb{C}$ is ν-measurable in ω.

(3) $|P(\gamma, \omega)|^2 = d\nu(\gamma^{-1}\omega)/d\nu(\omega)$.

(4) $P(\gamma_1\gamma_2, \omega) = P(\gamma_1, \omega)P(\gamma_2, \gamma_1^{-1}\omega)$.

By going through this construction we obtain a boundary representation which naturally acts on $L^2(\Omega, d\nu)$. First of all we start characterising the pair (ν, P) as described, for example, in Figà-Talamanca-Steger [**8**].

On (Γ, A) we fix a real probability measure $\mu \in l^1(\Gamma)$, supported on A and symmetric, [**10**]. Observe that the operator R given by $Rf = f*\mu$ is bounded linear and self-adjoint on $l^2(\Gamma)$, with norm not greater then 1. Therefore its spectrum, denoted $sp(\mu)$, as an operator on $l^2(\Gamma)$ is a closed subset of the interval $[-1, 1]$. If z is a complex number, $z \notin sp(\mu)$, then $(z - R)^{-1}$ is a bounded operator on $l^2(\Gamma)$ and it is possible to see its action as the right convolution operation by some function $g_z = (z - \mu)^{-1}$ defined in the convolution algebra $l^1(\Gamma)$. This function g_z, which is called the Green function, is of a special type as proved in the following lemma basically due to Aomoto [**1**] and to Gerl and Woess [**9, 16**].

Lemma. *Let z be complex number not in the spectrum of μ. If $z \neq 0$ then there exist $w \in \mathbb{C}$, an appropriate choice of $\pm$, and a multiplicative function h_z on (Γ, A), that is: $h_z(\gamma_1\gamma_2) = h_z(\gamma_1)h_z(\gamma_2)$ when $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$, such that:*

(i) $g_z(\gamma) = (h_z(\gamma))/(2w)$

(ii) $h_z(a) = \xi_a$ *for every* $a \in A$

(iii) $z = -(q-1)w + \sum_{a \in A} \pm\sqrt{w^2 + \mu^2(a)}$

where $\xi_a = 1/\mu(a)\left(\pm\sqrt{w^2 + \mu^2(a)} - w\right)$.

The functions g_z, as functions of z are defined on $\mathbb{C} \setminus sp(\mu)$. But they may be analytically continued on a compact Riemann surface S containing $\mathbb{C} \setminus sp(\mu)$ as a subset. This can be used to prove that, for $\sigma \in sp(\mu)$, $\lim_{\varepsilon \to 0} g_{\sigma + i\varepsilon}(\gamma) = g_{\sigma + i0}(\gamma)$ and $\lim_{\varepsilon \to 0} g_{\sigma - i\varepsilon}(\gamma) = g_{\sigma - i0}(\gamma)$ exist, determine continuous functions of σ and they are distinct, unless σ is a branch point of $sp(\mu)$ [8]. Associated to the function h_z we have the Poisson kernel

$$P_z(\gamma,\omega) = \lim_{\psi\to\omega} \frac{h_z(\psi^{-1}\gamma)}{h_z(\psi^{-1})}$$

where $\gamma \in (\Gamma, A)$ and $\omega \in \Omega_A$. Suppose that γ and ω agree through their first s letters, but not further. Let $\gamma = a_{i_1}\dots a_{i_s}a_{i_{s+1}}\dots a_{i_n}$ and $\omega = a_{i_1}\dots a_{i_s}a_{j_1}a_{j_2}a_{j_3}\dots$. Choose $\psi = a_{i_1}\dots a_{i_s}a_{j_1}\dots a_{j_t}$. Then we have the following explicit expression for $P_z(\gamma,\omega)$:

$$\begin{aligned} P_z(\gamma,\omega) &= \lim_{t\to+\infty} \frac{h_z(\psi^{-1}\gamma)}{h_z(\psi^{-1})} \\ &= \lim_{t\to+\infty} \frac{h_z\left(a_{j_t}^{-1}\dots a_{j_1}^{-1}a_{i_{s+1}}\dots a_{i_n}\right)}{h_z\left(a_{j_t}^{-1}\dots a_{j_1}^{-1}a_{i_s}^{-1}\dots a_{i_q}^{-1}\right)} \\ &= \xi_{i_1}^{-1}\dots\xi_{i_s}^{-1}\xi_{i_{s+1}}\dots\xi_{i_n}. \end{aligned}$$

Let $\sigma \in sp(\mu)$ and suppose that σ is neither zero nor a branch point of $sp(\mu)$. In order to define a positive Radon measure on Ω_A it is possible to consider only the following sets $\Omega_A(x) = \{\omega \in \Omega_A \mid \omega \text{ starts with } x\}$, [8]. Let $O \in \Phi_A$ be the vertex corresponding to the identity element e of Γ. We define

$$\nu_{O,\sigma}(\Omega_A(xa)) = |h_\sigma(x)|^2 \frac{|\xi_a|^2}{1+|\xi_a|^2}$$

where $a \in A$ with $|xa| = |x|+1$. With $|x|$ we mean the usual length of the reduced word x, that is the distance of the vertex x from the origin O of Φ_A.

Let $\Gamma(x) = \{y \in \Gamma \mid y \text{ starts with } x\}$.

Proposition. *For every $\gamma \in (\Gamma, A)$ we have:*

(i) $\gamma\Omega_A(x) = \Omega_A(\gamma x)$, *if* $\gamma \notin \Gamma(x)$

(ii) $\nu_{\gamma O,\sigma}(\gamma E) = \nu_{O,\sigma}(E)$

for every borel set E.

Proof: (i) follows from the definition of $\Omega_A(x)$.

(ii) γ preserves the tree structure, so

$$\nu_{\gamma O,\sigma}(\gamma\Omega_A(x)) = \nu_{\gamma O,\sigma}(\Omega_A(\gamma x)) = \nu_{O,\sigma}(\Omega_A(x)).$$

In particular we get $\nu_{\gamma^{-1}O,\sigma}(E) = \nu_{O,\sigma}(\gamma E)$. ∎

Now we can define the anisotropic principal series of Γ. Given $\sigma \in sp(\mu)$, neither zero nor a branch point of $sp(\mu)$, the boundary representation constructed by (ν, P) acting on $L^2(\Omega, d\nu)$ is defined as follows:

$$[\pi_\sigma(\gamma)F](\omega) = P_{\sigma+i0}(\gamma, \omega)F(\gamma^{-1}\omega).$$

Conditions (1) and (2) guarantee that $\pi_\sigma(\gamma)$ takes ν-measurable functions to ν-measurable functions, condition (3) guarantees that $\pi_\sigma(\gamma)$ acts unitarily on $L^2(\Omega, d\nu)$ and condition (4) guarantees that $\pi_\sigma(\gamma_1\gamma_2) = \pi_\sigma(\gamma_1)\pi_\sigma(\gamma_2)$.

3. Main results

In the sequel, we need to assume the following conditions. We note all of them with $[\Diamond]$:

Let γ be a free group on finitely many generators. Fix two new bases for Γ and let A_1 and A_2 consist of the new basis elements and their inverses. Construct the Cayley-graphs Φ_{A_1} and Φ_{A_2} related to A_1 and A_2 on which Γ acts by left multiplication. Let π_1 be a representation in the anisotropic principal series of (Γ, A_1) and π_2 be a representation in the anisotropic principal series of (Γ, A_2).

Our goal consists of the following statement

Theorem 1. *We suppose that $[\Diamond]$ holds. If $\pi_1 \simeq \pi_2$ then there exists a tree-isomorphism $j\colon \Phi_{A_1} \to \Phi_{A_2}$ such that the following diagram is commutative for every $\gamma \in \Gamma$,*

$$\begin{array}{ccc} \Phi_{A_1} & \xrightarrow{\;j\;} & \Phi_{A_2} \\ {\scriptstyle \gamma(\cdot)}\downarrow & & \downarrow{\scriptstyle \gamma(\cdot)} \\ \Phi_{a_1} & \xrightarrow{\;j\;} & \Phi_{A_2} \end{array}$$

where $\gamma(\cdot)$ means the action of left multiplication by the word γ thought of an element in (Γ, A_1) and (Γ, A_2) respectively.

Corollary. *We suppose that $[\Diamond]$ holds. Then there exists an element $\gamma_0 \in \Gamma$ such that $A_2 = \gamma_0^{-1}A_1\gamma_0$.*

In what follows, when $z \in sp(\mu)$, it is convenient to denote the multiplicative function h_z by h only.

Observe that using techniques of Bishop-Steger [2], Figà-Talamanca and Nebbia [6] and Figà-Talamanca and Steger [8] we get

Theorem 2. *Suppose $[\Diamond]$ holds. Let h_1 and h_2 be the multiplicative functions related to the choice of the sets A_1 and A_2. If $\pi_1 \simeq \pi_2$ then*

$$\sum_{\gamma\in\Gamma} [|h_1(\gamma)|\,|h_2(\gamma)|]^{1/(1+\delta)} = +\infty$$

for every positive δ.

For the proof of this theorem we need the following results.

LEMMA A. *Let π_σ be a Γ-representation in the anisotropic principal series. Construct the self-adjoint operator $\pi_\sigma(\mu)$, acting on the Γ-representation space, obtained from the usual extension of π_σ to $l^1(\Gamma)$. Then for every $\varepsilon > 0$ we have*

$$[\sigma + i\varepsilon - \pi_\sigma(\mu)]^{-1} = \pi_\sigma[(\sigma + i\varepsilon - \mu)^{-1}].$$

PROOF: We have only to note that, for ε sufficiently large

$$g_{\sigma+i\varepsilon} = (\sigma + i\varepsilon - \mu)^{-1} \in l^1(\Gamma)$$

hence

$$[\sigma + i\varepsilon - \pi_\sigma(\mu)]^{-1} = \pi_\sigma[g_{\sigma+i\varepsilon}].$$

Then we extend this result to small ε, by analytic continuation. □

LEMMA B. *Let π_σ be a representation in the anisotropic principal series of Γ. For every v_1 and v_2, chosen from the dense set in the representation space H, of linear combinations of left translates of a cyclic vector, there exists a constant C such that*

$$|\langle \pi_\sigma(\gamma) v_1, v_2 \rangle| \leqslant C\, |h_\sigma(\gamma)|.$$

where h_σ is the related multiplicative function.

PROOF: Recall that π_σ has associated to it a positive definite function ϕ_σ defined as follows:

$$\phi_\sigma(\gamma) = \frac{g_{\sigma+i0}(\gamma) - g_{\sigma-i0}(\gamma)}{g_{\sigma+i0}(e) - g_{\sigma-i0}(e)}.$$

So for fixed $\sigma \in sp(\mu)$ and for a cyclic vector $\mathbf{1}$, in the above dense set, we have

$$|\langle \pi_\sigma(\gamma)\mathbf{1}, \mathbf{1} \rangle| = |\phi_\sigma(\gamma)| \leqslant C\, |h_\sigma(\gamma)|.$$

Then

$$\begin{aligned} |\langle \pi_\sigma(\gamma) v_1, v_2 \rangle| &= |\langle \pi_\sigma(\gamma)\pi_\sigma(\gamma_1)\mathbf{1}, \pi_\sigma(\gamma_2)\mathbf{1} \rangle| \\ &= \left|\langle \pi_\sigma(\gamma_2^{-1}\gamma\gamma_1)\mathbf{1}, \mathbf{1} \rangle\right| \\ &\leqslant C \left|h_\sigma(\gamma_2^{-1}\gamma\gamma_1)\right| \\ &\leqslant \frac{C}{\left|h_\sigma(\gamma_2^{-1})\right| |h_\sigma(\gamma_1)|}\, |h_\sigma(\gamma)|. \end{aligned}$$

So by taking $C = C(\gamma_1, \gamma_2) = C/\left(\left|h_\sigma(\gamma_2^{-1})\right| |h_\sigma(\gamma_1)|\right)$ we get the result. □

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LEMMA C. *For every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that*

$$|h_{\sigma+i\varepsilon}(\gamma)| \leqslant |h_\sigma(\gamma)|^{1/(1+\delta)}$$

for every $0 < \varepsilon < \varepsilon_0$ and γ in Γ.

PROOF: We have only to prove this on the set A of generators and their inverses. Fix $a \in A$. Since $|h_\sigma(a)| \leqslant 1$ we have $|h_\sigma(a)| < |h_\sigma(a)|^{1/(1+\delta)}$. Since

$$\lim_{\varepsilon \to 0+} h_{\sigma+i\varepsilon}(a) = h_\sigma(a)$$

it is possible to select $\varepsilon_0 > 0$ such that

$$\max_{0 \leqslant \varepsilon \leqslant \varepsilon_0} |h_{\sigma+i\varepsilon}(a)| \leqslant |h_\sigma(a)|^{1/(1+\delta)}.$$

By choosing some ε_0 which works for all $a \in A$, we get the result. ∎

PROOF OF THEOREM 2: We consider the self-adjoint operators

$$\pi_2(\mu_2)\colon H_2 \longrightarrow H_2 \text{ and } \pi_1(\mu_2)\colon H_1 \longrightarrow H_1.$$

There exists a vector $v_{0,2} \neq 0$ such that $\pi_2(\mu_2)v_{0,2} = \sigma_2 v_{0,2}$ [8]. Because $\pi_2 \simeq' \pi_1$ there exists a unitary map $J\colon H_2 \longrightarrow H_1$ such that

$$\begin{array}{ccc} H_2 & \xrightarrow{\pi_2(\mu_2)} & H_2 \\ J\downarrow & & \downarrow J \\ H_1 & \xrightarrow{\pi_1(\mu_2)} & H_1 \end{array}$$

is commutative. Then

$$\pi_1(\mu_2)Jv_{0,2} = J\pi_2(\mu_2)v_{0,2} = J\sigma_2 v_{0,2} = \sigma_2 J v_{0,2}.$$

As a consequence of Spectral Theorem

$$\text{weak} \lim_{\varepsilon \to 0+} i\varepsilon(\sigma_2 + i\varepsilon - \pi_1(\mu_2))^{-1}$$

exists and it is not zero.

Hence for all vectors v_1 and v_2 chosen from the dense set in H_1 of linear combination of left translates of a cyclic vector, let it be $\mathbf{1}$, we have

$$\lim_{\varepsilon \to 0+} \langle i\varepsilon[\sigma_2 + i\varepsilon - \pi_1(\mu_2)]^{-1} v_1, v_2 \rangle \neq 0.$$

Fix $\varepsilon > 0$. Then

$$\begin{aligned}
0 &\neq \left|\langle i\varepsilon[\sigma_2 + i\varepsilon - \pi_1(\mu_2)]^{-1}v_1,\, v_2\rangle\right| \qquad \text{(from Lemma A)}\\
&= \left|\langle i\varepsilon\pi_1[(\sigma_2 + i\varepsilon - \mu_2)^{-1}]v_1,\, v_2\rangle\right|\\
&= \frac{\varepsilon}{2\,|w_{\sigma_2+i\varepsilon}|}\left|\sum_{\gamma\in\Gamma} h_{\sigma_2+i\varepsilon}(\gamma)\langle\pi_1(\gamma)v_1,\, v_2\rangle\right|\\
&\leqslant \frac{\varepsilon}{2\,|w_{\sigma_2+i\varepsilon}|}\sum_{\gamma\in\Gamma}|h_{\sigma_2+i\varepsilon}(\gamma)|\,|\langle\pi_1(\gamma)v_1,\, v_2\rangle| \qquad \text{(from Lemma B)}\\
&\leqslant \frac{C\varepsilon}{2\,|w_{\sigma_2+i\varepsilon}|}\sum_{\gamma\in\Gamma}|h_{\sigma_2+i\varepsilon}(\gamma)|\,|h_1(\gamma)| \qquad \text{(from Lemma C)}\\
&\leqslant C\varepsilon\sum_{\gamma\in\Gamma}[|h_2(\gamma)|]^{1/(1+\delta)}[|h_1(\gamma)|]^{1/(1+\delta)} \qquad \text{(for } 0 < \varepsilon < \varepsilon_0\text{)}.
\end{aligned}$$

By taking the limit as ε goes to zero, we get the result. □

So we resolve our initial problem, proving the following theorem

THEOREM 3. *Let Γ be a free group on finitely many generators. Fix two new bases for Γ and let A_1 and A_2 consist of the new basis elements and their inverses. Construct the Cayley-graphs Φ_{A_1} and Φ_{A_2} related to A_1 and A_2 on which Γ acts by left multiplication. Let h_1 and h_2 be the multiplicative functions related to the choice of the sets A_1 and A_2. If for every positive δ*

$$\sum_{\gamma\in\Gamma}[|h_2(\gamma)|]^{1/(1+\delta)}[|h_1(\gamma)|]^{1/(1+\delta)} = +\infty$$

then there exists a tree-isomorphism $j\colon \Phi_{A_1} \longrightarrow \Phi_{A_2}$ such that the following diagram is commutative for every $\gamma \in \Gamma$

$$\begin{array}{ccc}
\Phi_{A_1} & \xrightarrow{\;j\;} & \Phi_{A_2}\\
{\scriptstyle\gamma(\cdot)}\big\downarrow & & \big\downarrow{\scriptstyle\gamma(\cdot)}\\
\Phi_{A_1} & \xrightarrow{\;j\;} & \Phi_{A_2}
\end{array}$$

where $\gamma(\cdot)$ means the action of left multiplication by the word γ thought of as an element in (Γ, A_1) and (Γ, A_2) respectively.

The proof of this result depends on a result of Culler-Morgan on $\mathbb{R}$-trees [5]. Select a Cayley-graph Φ_A related to (Γ, A) and assign the following distance on it:

$$d(x, y) = \log\frac{1}{|h(x^{-1}y)|^2}.$$

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Then we define the translation length function l of Γ as follows:

$$l : \Gamma \longrightarrow [0, +\infty)$$
$$\gamma \longrightarrow l(\gamma) = \inf_{x \in \Phi_A} d(x, \gamma x)$$

Now we can state the following result [5].

THEOREM 4. (Culler-Morgan) *Suppose that $T_1 \times G \longrightarrow T_1$ and $T_2 \times G \longrightarrow T_2$ are two minimal semisimple actions of a group G on $\mathbb{R}$-trees with the same translation length function. Then there exists an equivariant isometry from T_1 to T_2. If either action is not a shift then the equivariant isometry is unique.*

Remember that Γ acts by left multiplication on one of its Cayley-graphs, so in a minimal and semisimple way as required by Culler-Morgan.

In the next section technical results are described, in order to apply the previous theorem. The proof of Theorem 3 will be exposed in Section 5.

4. Technical results

The following useful lemmas are easily obtained, with small changes, from their analogues in the isotropic case, so we refer to [14] for the proof.

We want to point out, once and for all, some assumptions that repeatedly occur in the following.

Let Γ be a free group. We fix two bases for Γ and let A_0 and A consist of the basis elements and their inverses respectively. We shall use A_0 to define the set $L(\gamma)$ and A to construct a Cayley-graph Φ_A on which Γ acts by left multiplication.

DEFINITION 1: For every $\gamma, \gamma' \in \Gamma$ we define the following set:

$$L(\gamma) = \{\omega \in \Omega_A \mid \omega \text{ is a limit of points of type } \gamma\gamma' O \text{ where } |\gamma\gamma'| = |\gamma| + |\gamma'|\}.$$

(In particular $L(e) = \Omega_A$).

DEFINITION 2: Fix $\sigma \in sp(\mu)$ where μ is a probability measure on A. We define for every $\gamma \in \Gamma$ the following function:

$$B(\gamma) = B_\sigma(\gamma) = \nu_{O,\sigma}(L(\gamma)),$$

where $\nu_{O,\sigma}$ is the positive Radon measure previously defined on Ω_A.

LEMMA 1. *Fix $O \in \Phi_A$ and $\sigma \in sp(\mu)$. Then there exists a constant C such that*

$$|h(\gamma)|^2 \leqslant C B(\gamma)$$

for every $\gamma \in \Gamma$.

Lemma 2. *Fix* $\gamma_0 \in \Gamma$ *and* $\sigma \in sp(\mu)$ *(where* μ *is a probability measure on* A*). Then there exists a constant* M_0*, depending only on the last letter of* γ_0*, such that for* $\gamma_1 \in \Gamma$ *with* $|\gamma_1\gamma_0| = |\gamma_1| + |\gamma_0|$ *and* $B(\gamma_1) < \min_{a\in A} |\xi_a|^2 / \left(1 + |\xi_a|^2\right)$ *we have*

$$\frac{B(\gamma_1\gamma_0^m)}{B\left(\gamma_1\gamma_0^{m+1}\right)} = e^{\pm l(\gamma_0)}$$

for all $m \geqslant 1$ *except at most* M_0 *values.*

Lemma 3. *There exists a constant* $\eta_0 > 0$ *such that for every* γ_1, $a \in \Gamma$ *with* $|a| = 1$ *and* $|\gamma_1 a| = |\gamma_1| + 1$

$$\frac{B(\gamma_1 a)}{B(\gamma_1)} \geqslant \eta_0$$

holds.

Corollary. *For every* $\gamma \in \Gamma$ *with* $|\gamma| \geqslant 1$

$$B(\gamma) \leqslant [1 - (q-1)\eta_0]^{|\gamma|}$$

holds.

Let Γ be a free group and A_0, A_1, A_2 fixed generator sets, with their inverses. As usual we use A_0 to define the sets $L(\gamma)$ for every $\gamma \in \Gamma$, A_1 and A_2 to define the following objects:

two Cayley-graphs Φ_{A_1} and Φ_{A_2} on which Γ acts by left multiplication;
the multiplicative functions h_1 and h_2;
the boundary measures through which we construct the functions B_1 and B_2.

Lemma 4. *If for every* $\delta > 0$

$$\sum_{\gamma\in\Gamma} \{[B_1(\gamma)]^{1/2}[B_2(\gamma)]^{1/2}\}^{1/(1+\delta)} = +\infty$$

then for every natural numbers N *and* N' *and for every* $\varepsilon > 0$ *and* $c \in A_0$*, there exists* $\gamma_2 \in \Gamma$ *with* $|\gamma_2| \geqslant N$ *such that* γ_2 *ends with* c *and*

$$\left|\frac{B_1(\gamma_2\gamma_0')}{B_1(\gamma_2\gamma_0'')} - \frac{B_2(\gamma_2\gamma_0')}{B_2(\gamma_2\gamma_0'')}\right| < \varepsilon$$

where $|\gamma_0'|$ *and* $|\gamma_0''| \leqslant N'$ *and* $|\gamma_2\gamma_0'| = |\gamma_2| + |\gamma_0'|$ *and* $|\gamma_2\gamma_0''| = |\gamma_2| + |\gamma_0''|$.

5. Proof of Theorem 3

We divide it into two steps.

First of all from Lemma 1, there exists a positive constant C such that for every $\gamma \in \Gamma$

$$|h_1(\gamma)|^2 \leqslant C\, B_1(\gamma)$$

and

$$|h_2(\gamma)|^2 \leqslant C\, B_2(\gamma).$$

So for every $\delta > 0$

$$C \sum_{\gamma \in \Gamma} \{[B_1(\gamma)]^{1/2}[B_2(\gamma)]^{1/2}\}^{1/(1+\delta)} \geqslant \sum_{\gamma \in \Gamma} [|h_1(\gamma)|\,|h_2(\gamma)|]^{1/(1+\delta)} = +\infty.$$

Step one.

Claim:

If for every $\delta > 0$

$$\sum_{\gamma \in \Gamma} \{[B_1(\gamma)]^{1/2}[B_2(\gamma)]^{1/2}\}^{1/(1+\delta)} = +\infty$$

then

$$l_1(\gamma) = l_2(\gamma)$$

for every $\gamma \in \Gamma$, where l_1 and l_2 are the translation length functions related to h_1 and h_2.

We prove this by contradiction. Fix $\gamma_0 \in \Gamma$. We can choose a unique constant M_0 such that Lemma 2 holds for both the actions of Γ, one on Φ_{A_1} and the other on Φ_{A_2}. Fix $N' = (2 + 2M_0)\,|\gamma_0|$. From the corollary to Lemma 3, we can choose N such that $|\gamma_1| \geqslant N$ implies $B_1(\gamma_1) < \min_{a \in A_1} |\xi_a|^2 \big/ \left(1 + |\xi_a|^2\right)$ and $B_2(\gamma_1) < \min_{a \in A_2} |\xi_a|^2 \big/ \left(1 + |\xi_a|^2\right)$. Suppose that

$$e^{l_1(\gamma_0)} \neq e^{l_2(\gamma_0)}.$$

Let $0 < \varepsilon < 1$ be such that

$$\left| e^{-l_1(\gamma_0)} - e^{-l_2(\gamma_0)} \right| > \varepsilon.$$

Let $c \in \Gamma$ be a letter such that $|c| = 1$ and $|c\gamma_0| = 1 + |\gamma_0|$. By applying Lemma 4, with N, N', ε and c chosen as above, there exists $\gamma_2 \in \Gamma$ with $|\gamma_2| \geqslant N$ such that γ_2 ends with c and

$$\left| \frac{B_1(\gamma_2\gamma_0')}{B_1(\gamma_2\gamma_0'')} - \frac{B_2(\gamma_2\gamma_0')}{B_2(\gamma_2\gamma_0'')} \right| < \varepsilon$$

where $|\gamma_0'|$ and $|\gamma_0''| \leqslant N'$ and $|\gamma_2\gamma_0'| = |\gamma_2| + |\gamma_0'|$ and $|\gamma_2\gamma_0''| = |\gamma_2| + |\gamma_0''|$.

From this last result, it follows that

$$\left| \frac{B_1(\gamma_2\gamma_0^m)}{B_1\left(\gamma_2\gamma_0^{m+1}\right)} - \frac{B_2(\gamma_2\gamma_0^m)}{B_2\left(\gamma_2\gamma_0^{m+1}\right)} \right| < \varepsilon$$

for $1 \leqslant m \leqslant 2M_0 + 1$. Remember, $|\gamma_2| \geqslant N$ and $B_1(\gamma_1) < \min_{a\in A_1} |\xi_a|^2 / \left(1 + |\xi_a|^2\right)$ and $B_2(\gamma_1) < \min_{a\in A_2} |\xi_a|^2 / \left(1 + |\xi_a|^2\right)$ and that γ_2 ends with c where $|\gamma_2\gamma_0| = |\gamma_2| + |\gamma_0|$. Then from Lemma 2, there are at most M_0 values of m such that

$$\frac{B_1(\gamma_2\gamma_0^m)}{B_1\left(\gamma_2\gamma_0^{m+1}\right)} \notin \{e^{\pm l_1(\gamma_0)}\} \text{ and } \frac{B_2(\gamma_2\gamma_0^m)}{B_2\left(\gamma_2\gamma_0^{m+1}\right)} \notin \{e^{\pm l_1(\gamma_0)}\}.$$

Then there exists at least one value of m with $1 \leqslant m \leqslant 2M_0 + 1$ such that

$$\frac{B_1(\gamma_2\gamma_0^m)}{B_1\left(\gamma_2\gamma_0^{m+1}\right)} \in \{e^{\pm l_1(\gamma_0)}\} \text{ and } \frac{B_2(\gamma_2\gamma_0^m)}{B_2\left(\gamma_2\gamma_0^{m+1}\right)} \in \{e^{\pm l_1(\gamma_0)}\}.$$

So

$$\left| \frac{B_1(\gamma_2\gamma_0^m)}{B_1\left(\gamma_2\gamma_0^{m+1}\right)} - \frac{B_2(\gamma_2\gamma_0^m)}{B_2\left(\gamma_2\gamma_0^{m+1}\right)} \right| \geqslant \left| e^{-l_1(\gamma_0)} - e^{-l_2(\gamma_0)} \right| > \varepsilon$$

and this is a contradiction.

STEP TWO.

We think of Φ_{A_1} and Φ_{A_2} as $\mathbb{R}$-trees endowed with the canonical distance, the one which assigns distance 1 to two neighbouring vertices. Γ acts in a minimal and semisimple way. Besides the translation length functions of Γ on the two $\mathbb{R}$-trees are the same, so applying the result of Culler and Morgan stated in Theorem 4, we get the result. ∎

Now we prove the Corollary to Theorem 1.

PROOF OF COROLLARY: From Theorem 1, there exists a tree isomorphism

$$j \colon \Phi_{A_1} \longrightarrow \Phi_{A_2}$$

such that for every $\gamma \in \Gamma$ the following diagram is cummutative:

$$\begin{array}{ccc} \Phi_{A_1} & \xrightarrow{\;j\;} & \Phi_{A_2} \\ {\scriptstyle \gamma(\cdot)}\big\downarrow & & \big\downarrow{\scriptstyle \gamma(\cdot)} \\ \Phi_{A_1} & \xrightarrow{\;j\;} & \Phi_{A_2} \end{array}$$

where $\gamma(\cdot)$ means the action of left multiplication by the word γ thought of as an element of (Γ, A_1) and (Γ, A_2) respectively.

This is, $j \circ \gamma(\cdot) = \gamma(\cdot) \circ j$ for every $\gamma \in \Gamma$. As usual, it is better to identify the vertices with the corresponding group elements. Let e be the identity element of (Γ, A_1). Then for every γ thought of as an automorphism of Φ_{A_1} and Φ_{A_2} we have

$$j(\gamma) = \gamma j(e).$$

We define $j(e) = \gamma_0$. So $j(\gamma) = \gamma\gamma_0$. The edges of Φ_{A_1} and Φ_{A_2} are of $(\gamma, \gamma a)$-type where $a \in A_1$ and $a \in A_2$ respectively. The neighbours of γ in Φ_{A_1} and Φ_{A_2} are $\{\gamma a\}_{a\in A_1}$ and $\{\gamma a\}_{a\in A_2}$. Then we have

$$\begin{aligned} & \{j(\gamma a)\}_{a\in A_1} = \{j(\gamma)a\}_{a\in A_2} \\ \leftrightarrow \quad & \{\gamma a\gamma_0\}_{a\in A_1} = \{\gamma\gamma_0 a\}_{a\in A_2} \\ \leftrightarrow \quad & \{a\gamma_0\}_{a\in A_1} = \{\gamma_0 a\}_{a\in A_2} \\ \leftrightarrow \quad & \gamma_0 A_1 \gamma_0^{-1} = A_2. \end{aligned}$$

Hence we obtain an anisotropic principal series of Γ equivalent to the original one if and only if we interchange the generators of Γ and their related measure (in this case $\gamma_0 = e$) or we replace some of them with their inverses (again we have $\gamma_0 = e$) or we apply conjugation by $\gamma \in \Gamma$ (in this case $\gamma_0 = \gamma$) or we combine these kinds of operations. □

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