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ORDERED DESARGUESIAN AFFINE HJELMSLEV PLANES

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1. **Introduction.** A Desarguesian affine Hjelmslev plane (D.A.H. plane) may be coordinatized by an affine Hjelmslev ring (A.H. ring), which is a local ring whose radical is equal to the set of two-sided zero divisors and whose principal right ideals are totally ordered (cf. [3]). In his paper on ordered geometries [4], P. Scherk discussed the equivalence of an ordering of a Desarguesian affine plane with an ordering of its coordinatizing division ring. We shall define an ordered D.A.H. plane and follow Scherk's methods to extend his results to D.A.H. planes and their A.H. rings i.e., we shall show that a D.A.H. plane is ordered if and only if its A.H. ring is ordered. We shall also give an example of an ordered A.H. ring. Finally, we shall discuss some infinitesimal aspects of the radical of an ordered A.H. ring.

2. **Prerequisites.** The definition and elementary properties of an A.H. plane $\aleph$ and its A.H. ring H may be found in [3]. The line through a point P parallel to a line l will be denoted by $L(P, l)$. If the set T of translations of $\aleph$ forms a transitive group, $\aleph$ is called a *translation plane* and T is abelian. The set N of *neighbour translations* consists of those translations $\tau \in T$ such that every point is a neighbour ($\sim$) of its image under τ. If D_τ is the set of traces of a translation τ and H is the ring of trace-preserving endomorphisms of T, then an A.H. translation plane $\aleph$ is *Desarguesian* if for every $\tau_1 \in T\backslash N$, $\tau_2 \in T$ such that $D_{\tau_1} \subseteq D_{\tau_2}$, there exists an $a \in H$ such that $\tau_1^a = \tau_2$. With this restriction, H is an A.H. ring.

2.1. LEMMA. *Let $\aleph$ be an* A.H. *plane with* A.H. *ring H. Let $a \in H\backslash\{0\}$ and let σ, A, B be non-collinear points of $\aleph$. If A, B I l,* then *$\tau_{OB}^a(\sigma)\, I\, L(\tau_{OA}^a(\sigma), l)$.*

Proof. $\tau_{AB} = \tau_{OB}\,\tau_{AO} = \tau_{OB}\,\tau_{OA}^{-1}$. Hence $\tau_{AB}^a = \tau_{OB}^a\,\tau_{OA}^{-a}$ and $\tau_{AB}^a(\tau_{OA}^a(\sigma)) = \tau_{OB}^a\,\tau_{OA}^{-a}\,\tau_{OA}^a(0) = \tau_{OB}^a(0)$. Since l is a τ_{AB}-trace, it is a τ_{AB}^a-trace. Let $m = L(\tau_{OA}^a(\sigma), l)$. Then m is a τ_{AB}^a-trace. Thus $\tau_{AB}^a(\tau_{OA}^a(\sigma))\ I\ m$.

2.2. REMARK. It is possible that $\tau_{OB}^a(\sigma) = B$, even if $\tau_{OA}^a(\sigma) \neq A$. This is avoided if we confine our discussion to the situation where $l \not\sim s$; O, A I s; O, B I t.

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3. Ordered A.H. rings and ordered D.A.H. planes

3.1. Definition. An A.H. ring H with radical η will be called *ordered* if there exists a subset H^+ of H (the *positive cone* of H) such that:

(i) Every $a \in H$ satisfies exactly one of $a \in H^+$, $-a \in H^+$, $a = 0$.
(ii) If $a, b \in H^+$, then $a + b \in H^+$.
(iii) If $a, b \in H^+$ and $b \notin \eta$, then $ab \in H^+$.

(In no ordered A.H. ring with nontrivial radical is H^+ closed under multiplication. In Section 6, we present an example to show that even $H^+ \cup \{0\}$ need not be closed under multiplication. In [2], Klingenberg introduced the concept of an ordered P.H. ring. His definition differs from the one presented here, however, as his positive cone contains the zero element and in addition is closed under multiplication.)

3.2. Definition. An ordering of an affine Hjelmslev plane $\aleph$ is a ternary relation on the set $\mathbb{P}$ of points of $\aleph$, which satisfies the following Hilbert axioms, as in [4].

01. (A, B, C) implies that A, B and C are mutually distinct and collinear. (We say that B lies between A and C.)
02. (A, B, C) implies (C, B, A).
03. If A, B and C are mutually distinct and collinear then exactly one of (A, B, C), (B, C, A) and (C, A, B) holds.
04. If A, B, C and D are mutually distinct and collinear then (A, B, C) and (B, C, D) imply (A, B, D) and (A, C, D).
05. If A, B, C and D are as in 04, then (A, B, C) and (A, C, D) imply (A, B, D) and (B, C, D).
06. If A, B, C and D are as in 04, then any two of (B, A, C), (C, A, D) and (D, A, B) exclude the third.
07. Non-degenerate parallel projections preserve order.

In the context of A.H. planes we define a *parallel projection* from a line l to a line m in the direction of a parallel pencil π, where $\pi \not\sim \pi_m$, to be the map

$$K_\pi : l \to m(X \rightsquigarrow L(X, \pi) \wedge m).$$

K_p is called *non-degenerate* if and only if K_p is a bijection. It is easy to see that K_π is non-degenerate if and only if $\pi \not\sim \pi_l$.

These axioms are not independent. It can easily be verified that 01, 02, 03, 04 and 06 imply 05.

In [1], Hjelmslev considered ordered H-planes, but did not mention his order axioms specifically.

3.3. If A, B and C are collinear then B and C are said to *lie on the same* *side of* A if exactly one of (A, B, C), $(A, C, B,)$ and $B = C \neq A$ holds. This will be denoted $B, C \mid A$. If l is any line through A, then the property of lying on the same side of A, on l, is an equivalence relation on $(\mathbb{P} \setminus \{A\}) \wedge l$. Also (B, A, C) and (B, A, D) imply $C, D \mid A$ (cf. [4]). Clearly for the parallel projection of 3.2. if $A, B, C \; I \; l$ and $BC \mid A$, then $B'C' \mid A'$.

3.4. Lemma. *Translations preserve order.*

Proof. Let $A, B, C \; I \; l$ and (A, B, C) and $\tau \in T$. Then τB, $\tau C \; I \; L(\tau A, l) = m$. First we assume that $l \not\sim m$. Then $A \not\sim X$ for any $X \; I \; m$, so $A\tau A \not\sim m$ and $\tau A \not\sim Y$ for any $Y \; I \; l$, so $A\tau A \not\sim l$. Thus for any $Y \; I \; l$, $L(Y, A\tau A) \not\sim l$, m. Since τ is a translation $\tau Y \; I \; L(Y, A\tau A) \wedge m$ so τ can be considered to be generated by a non-degenerate parallel projection having a pencil of lines parallel to $A\tau A$. Thus 07 applies and so (A, B, C) implies $(\tau A, \tau B, \tau C)$.

If $l \sim m$ then there exists a point X with $X \not\sim Y$ for any $Y \; I \; l$. Thus $n = L(X, l) \not\sim l$, m. Then $\tau = \tau_2 \tau_1$ where $\tau_1 = \tau_{AX}$ and $\tau_2 = \tau_{X\tau A}$. Then (A, B, C) implies $(\tau_1 A, \tau_1 B, \tau_1 C)$ which implies $(\tau_2\tau_1 A, \tau_2\tau_1 B, \tau_2\tau_1 C) = (\tau A, \tau B, \tau C)$.

4. **An ordering of the ring of a D.A.H. plane.** Let $O \not\sim A$ and define $H^+(O, A) = \{a \in H \mid \tau_{OA}^a(O).\ A \mid O\}$. It can easily be verified, using 2.1, that our definition is independent of the choices of O and A (cf. [4]). Thus we may write $H^+(O, A) = H^+$. In the rest of this section we shall prove the following result.

Theorem 1. *The* A.H. *ring of an ordered* D.A.H. *plane is ordered.*

4.1. We may assume that every line of $\aleph$ is incident with at least three points, otherwise 01 to 07 are satisfied trivially and H has only two elements with no ordering. In fact, for every proper A.H. plane, H has at least four elements and each line is incident with at least four points. Also the characteristic of H cannot be equal to two (cf. [4], p. 31).

4.2. Lemma. *If* $Q, C, O, B, P \; I \; l$ *and* (P, B, C) *and* O *is the midpoint on* l *of both* Q *and* P *and* C *and* B (*i.e.* $\tau_{QO} = \tau_{OP}$ *and* $\tau_{CO} = \tau_{OB}$), *then* (P, B, Q).

Proof. This is very similar to ([4], p. 31).

4.3. Lemma. *If* a, $b \in H^+$ *and* $a \neq b$, *then* $a + b \in H^+$.

Proof. Choose any points O, $A \; I \; l$ such that $O \not\sim A$ and let $\tau = \tau_{OA}$. Then a, $b \in H^+$ imply $\tau^a(O)$, $\tau^b(O) \mid O$ by transitivity. Since $a \neq b$, $\tau^a(O) \neq \tau^b(O)$, thus $(O, \tau^a(O), \tau^b(O))$ or $(O, \tau^b(O), \tau^a(O))$.

Suppose $(O, \tau^a(O), \tau^b(O))$. Then if we let E be the midpoint on l of $\tau^a(O)$ and $\tau^b(O)$, E is also the midpoint on l of O and $\tau_{OE}(E) = Q$. However $\tau_{OQ} = \tau^{a+b}$. Hence, by 4.2, $(O, \tau^a(O), \tau^b(O))$ implies $(O, \tau^a(O), \tau^{a+b}(O))$. Therefore, $\tau^a(O)$, $\tau^{a+b}(O) \mid O$ and since A, $\tau^a(O) \mid O$ we have A, $\tau^{a+b}(O) \mid O$, i.e. $a + b \in H^+$.

Symmetrically, if $(O, \tau^b(O), \tau^a(O))$ we also have $a+b \in H^+$.

4.4. Let $H^- = H \backslash (H^+ \cup \{0\})$. Clearly $a \in H^-$ iff$(\tau^a_{OA}(O), O, A)$, where $O \not\sim A$.

4.5. LEMMA $a \in H^+$ *implies* $-a \in H^-$.

Proof. Since the characteristic of H is not two, $a \neq -a$ and clearly $-a \neq 0$. If $-a \in H^+$, then $a+(-a) = 0 \in H^+$; a contradiction.

4.6. As in 4.3 and 4.5, one can show that $a, b \in H^-$, $a \neq b$ implies $a+b \in H^-$, and hence $a \in H^-$ implies $-a \in H^+$. Thus for all $a \in H$, exactly one of $a \in H^+$, $a = 0$, $-a \in H^+$ holds.

4.7. LEMMA. *If* $a \in H^+$, *then* $2a \in H^+$.

Proof. Choose $O \not\sim A$ and put $\tau = \tau_{OA}$. We show that $\tau^a(O), \tau^{2a}(O) \mid O$ and complete the proof as in 4.3. Since $O \not\sim A$, $\tau^a(O) \neq O$ and since the characteristic of H is not equal to two, $\tau^{2a}(O) \neq O$. Thus, either $\tau^a(O), \tau^{2a}(O) \mid O$ or $(\tau^{2a}(O), O, \tau^{2a}(O))$. However if we apply τ^{-a} to $(\tau^{2a}(O), O, \tau^a(O))$ we get $(\tau^a(O), \tau^{-a}(O), O)$. But, since $-a \in H^-$, we have $(\tau^{-a}(O), O, A)$. By 04, this implies $(\tau^a(O), O, A)$; a contradiction.

4.8. LEMMA. *If* $a, b \in H^+$ *and* $b \notin \eta$, *then* $ab \in H^+$.

Proof. Choose $O, A \in \mathbb{P}$; $O \not\sim A$ and put $\tau = \tau_{OA}$. Then $b \in H^+$ implies $\tau^b(O), A \mid O$ and $b \notin \eta$ implies $O \not\sim \tau^b(O)$ (cf. [3]). Therefore, $a \in H^+$ implies $\tau^b(O), (\tau^b)^a(O) \mid O$; i.e., $\tau^b(O), \tau^{ab}(O) \mid O$. Hence $A, \tau^{ab}(O) \mid O$ and so $ab \in H^+$.

4.9. From 4.3, 4.6, 4.7 and 4.8, H is an ordered A.H. ring with positive cone H^+.

5. **The construction of an ordered D.A.H. plane from an ordered A.H. ring.** We wish to prove the following result.

THEOREM 2. *A* D.A.H. *plane is ordered if its ring H is ordered.*

Lorimer and Lane [3] have constructed a D.A.H. plane $\aleph(H)$ from an A.H. ring H. A given D.A.H. plane $\aleph$ with A.H. ring H is isomorphic to $\aleph(H)$ and H is isomorphic to the A.H. ring of $\aleph(H)$. It remains to prove the following result.

5.1. LEMMA. *An ordering of an* A.H. *ring H induces an ordering of* $\aleph(H)$.

We may identify the ordered ring H and the ordered A.H. ring of $\aleph(H)$.

5.2. If A, B, C are mutually distinct and collinear with a line l, then there exist points O and E on l such that $O \not\sim E$. We define $(A, B, C)_{O,E}$ if $a < b < c$ or $c < b < a$, where $A = \tau^a_{OE}(O)$, $B = \tau^b_{OE}(O)$, $C = \tau^c_{OE}(O)$, and we say that B *lies between A and C* with respect to O and E.

5.3. LEMMA. *Order on a line l is imdependent of the choice of O, E on l, where $O \not\sim E$.*

Proof. Let $O \not\sim E$ and $(A, B, C)_{O,E}$. Take any two non-neighbour points O', E' on l. Then $A = \tau^a_{OE}(O) = \tau^{a'}_{O'E'}(O')$, etc. Since $\tau_{OE} \notin N$, $\tau_{O'E'} = \tau^x_{OE}$ where $x \notin \eta$ and $\tau_{OO'} = \tau^y_{OE}$.
$A = \tau^a_{OE}(O) = \tau^{a'}_{O'E'}(O') = \tau^{a'}_{O'E'}\ \tau_{OO'}(O) = \tau^{a'x+y}_{OE}(O)$. Thus $a = a'x + y$. If $a' < b' < c'$ and $x \in H^+$, then clearly $a < b < c$. If $x \in H^-$ then $c < b < a$. Similarly, one can deal with the case where $a' > b' > c'$. Thus, $(A, B, C)_{O,E}$ implies $(A, B, C)_{O',E'}$ for any $O' \not\sim E'$ on l.

5.4. From 5.2 and 5.3, the order axioms 01 to 06 follow easily.

5.5. LEMMA. *Every non-degenerate parallel projection preserves order.*

Proof. Let K_π be a non-degenerate parallel projection from l to m. For any $X\ I\ l$, put $X' = m \wedge L(X, \pi)$. Assume A, B, $C\ I\ l$ and (A, B, C).

CASE 1. $l \wedge m \neq \phi$. Say $O\ I\ l$, m. Take $E\ I\ l$ such that $O \not\sim E$. Then $A = \tau^a_{OE}(O)$ etc. and $a < b < c <$ or $c < b < a$. By 2.1, $\tau^a_{OE'}(O)\ I\ L(\tau^a_{OE}(O), L(E, \pi))$. Hence, $A' = \tau^a_{OE'}(O)$ and $L(O, \pi) \parallel L(E, \pi)$ so $O \not\sim E'$. Since $a < b < c$ or $c < b < a$, 5.2 implies that (A', B', C').

CASE 2. $l \wedge m = \phi$, $l \parallel m$. The proof is straightforward and will be omitted.

CASE 3. $l \wedge m = \phi$, $l \parallel m$. There exists a line s such that $s \parallel l$ and $s \wedge m \neq \phi$. The desired conclusion then follows from cases 1 and 2.

5.6. By 5.4 and 5.5, $\aleph$ is an ordered D.A.H. plane. Thus proves 5.1 and Theorem 2.

6. **An example of an ordered A.H. ring.** A projective Hjelmslev ring (P.H. ring) is an A.H. ring whose principal left ideals are totally ordered. A.H. planes which are coordinatized by P.H. rings can be embedded in projective Hjelmslev planes (cf. [3]). The following example of an A.H. ring which is not a P.H. ring is originally due to R. Baer, and is examined in [3].

Let $Q(x)$ be a simple transcendental extension of the rational numbers Q. Let ϕ be the isomorphism from $Q(x)$ into $Q(x^2)$ which takes x into x^2. Then $Q(x)$ can be regarded as the field of real-valued rational functions with rational coefficients and can be made into an ordered field by defining

$$Q(x)^+ = \left\{\frac{f(x)}{g(x)} \in Q(x) \mid \exists\ x_0 \in Q, \frac{f(z)}{g(z)} > 0, \forall\ z < x_0\right\}.$$

If we let $H = Q(x) \times Q(x)$ and define addition and multiplication by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, \phi(a)d + bc)$, then H is clearly an A.H. ring, which is not a P.H. ring, and its radical is $\eta = \{(0, a, \mid a \in Q(x)\}$. H can be ordered lexicographically by setting $H^+ = \{(a, b) \in H \mid a > 0\} \cup \{(0, b) \in H \mid b > 0\}$. Then H is an ordered A.H. ring. However $\alpha \in H^+$ and

$\beta \in H^+ \cap \eta$ does not imply $\alpha\beta \in H^+$. For example, take $\alpha = (-x, g(x))$ for any $g(x)$, and $\beta = (0, 1)$. Then $\alpha\beta = (0, \phi(-x)) = (0, -x^2)$, which is not in H^+.

7. **Remarks on the archimedian axiom and infinitesimals.** One can readily verify that if H is an Archimedian ordered A.H. ring, then H is a division ring; i.e. $\eta = \{0\}$. Thus any Archimedian ordered D.A.H. plane is an ordinary affine plane. Hjelmslev mentioned in [1] that the Archimedian axiom was not possible in ordered H-planes, but Klingenberg actually proved in [2] that an Archimedian ordered P.H. ring (using his definition of order; cf. 3.1) is an ordered division ring.

Let H be an ordered A.H. ring and η the radical of H. If $r \in \eta$, then $-1 < r < 1$ and if $b \in H$ and $c \in \eta$, then $-c < b < c$ implies $b \in \eta$. Geometrically, this means that if O is any point on a line in an ordered D.A.H. plane, then all the neighbours of O lie between E and E', where E is any non-neighbour of O and $E' = \tau_{OE}^{-1}(O)$. If A is any neighbour of O then all points between O and A are also neighbours of O. Thus the neighbouring points of O can be regarded as the infinitesimals clustered around O.

The following generalization was suggested by Dr. D. A. Drake and proved by C. A. Baker.

THEOREM. *Let H be an ordered* A.H. *ring with radical η and positive cone H^+.*

(1) *If $a \in \eta^m$ and $b \in H^+ \cap (\eta^{m-1} \setminus \eta^m)$ for m, $m-1 \in \mathbb{Z}^+$, then $-b < a < b$.*

(2) *If $a \in H^+ \cap \eta^m$ and $-a < b < a$, then $b \in \eta^m$.*

By definition,

$$\eta^m = \left\{ \sum_{i=1}^{n} a_{i1} \ldots a_{im} \mid a_{ik} \in \eta (k = 1, \ldots, m) \text{ and } n \in \mathbb{Z}^+ \right\}, \text{ for } m \in \mathbb{Z}^+.$$

clearly, these sets from a decreasing chain of ideals $\eta \supseteq \eta^2 \supseteq \eta^3 \supseteq \cdots \supseteq \eta^m \supseteq \ldots$. We first prove an auxiliary result. Set $H^- = H \setminus (H^+ \cup \{0\})$.

LEMMA. *If a, $b \in H^+$, $a \in bH$ and $b \notin aH$, then $0 < a < b$.*

Proof. Assume $b \le a$. Since $a \in bH$ and $b \notin aH$, there exists $c \in \eta$ with $a = bc$. As $c \in \eta$, $c - 1 \in H^- \setminus \eta$; hence $b(c-1) \in H^-$. However $0 < b \le a$ implies that

$$b(c-1) = bc - b = a - b \in H^+ \cup \{0\}; \text{ a contradiction.}$$

Proof of the theorem. Assertion (1) is a corollary of (2) and so it remains to prove (2).

Let $a \in H^+ \cap \eta^m$ and $-a < b < a$. The result (2) is clear if $b = 0$, so without loss of generality, we may assume that $b \in H^+$; therefore $0 < b < a$. By the Lemma, $b \in aH$; i.e. there exists $c \in H$ such that $b = ac$. However, $a \in \eta^m$, which implies

$$a = \sum_{i=1}^{n} a_{i1}\, a_{i2} \cdots a_{im} \text{ for some } a_{ik} \in \eta \text{ and some } n \in \mathbb{Z}^{+}.$$

Therefore

$$b = ac = \sum_{i=1}^{n} a_{i1}\, a_{i2} \cdots a_{im}\, c \in \eta^{m}.$$

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