M. Oka
Nagoya Math. J.
Vol. 64 (1976), 1–15

EXPANSIVE FLOWS AND THEIR CENTRALIZERS

MASATOSHI OKA

1. Introduction and preliminaries

R. Bowen and P. Walters [2] have defined expansive flows on metric spaces which generalized the similar notion by D. Anosov [1]. On the other hand, P. Walters [4] investigated continuous transformations of metric spaces with discrete centralizers and unstable centralizers and proved that expansive homeomorphisms have unstable centralizers.

M will denote a compact connected C^∞ manifold without boundary. We assume that we have some fixed Riemannian metric $|\cdot|$ on M. We denote by $d(x, y)$ the distance between $x, y \in M$ given by this Riemannian metric. $C^1(M)$ (resp. $C^0(M)$) will denote the set of all C^1 (resp. continuous) functions on M.

X will denote a compact connected metric space with metric function $d(x, y)$ which denotes the distance between $x, y \in X$.

$\boldsymbol{R}$ denote the additive group of real numbers.

A map $F: \boldsymbol{R} \times X \to X$ is called a continuous flow on X if F is continuous and $F(t + s, x) = F(t, F(s, x)), F(0, x) = x$ for every $t, s \in \boldsymbol{R}$ and $x \in X$. We shall sometimes use the notation $f_t(x) = F(t, x)$ and write $\{f_t\}$ for the flow instead of F.

DEFINITION 1. A continuous flow F on X is called an expansive flow if it has the following property $(*)$;

$(*)$ For any $\varepsilon > 0$, there exists $\delta > 0$ with the property that if there exist a pair of points $x, y \in X$ and a continuous map $s: \boldsymbol{R} \to \boldsymbol{R}$ with $s(0) = 0$ such that $d(f_t(x), f_{s(t)}(y)) < \delta$ for every $t \in \boldsymbol{R}$, then $y = f_t(x)$ for some $|t| < \varepsilon$.

Let v be a C^1-vector field on M and $\{f_t\}$ be the one-parameter group of C^1-diffeomorphisms f_t of M generated by v. We shall sometimes use the notation $f(t, x) = f_t(x)$ for every $t \in \boldsymbol{R}$ and $x \in M$. A C^1-vector field

Received February 24, 1976.

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

v(or $\{f_t\}$) on M is sometimes called a C^1-flow on M.

v(or $\{f_t\}$) is called a C^1-expansive flow on M if $\{f_t\}$ satisfies the property $(*)$.

DEFINITION 2. Let v be a C^1-vector field on M. A C^1-vector field w on M is called C^1-commutative with v if $[v\ w] = 0$, where [ ] is Lie bracket. Let $\{f_t\}$ be a continuous flow on X, then a continuous flow $\{g_s\}$ on X is called commutative with $\{f_t\}$ if $f_t \circ g_s = g_s \circ f_t$ for every t, $s \in \boldsymbol{R}$.

Cent(F) (resp. Cent(v)) will denote the centralizer of F (resp. v), i.e. the set of all continuous flows (resp. C^1-vector fields) (C^1)-commutative with F (resp. v).

DEFINITION 3. Let v be a C^1-flow on M. v is said to have an unstable centralizer if it satisfies the property that $w \in$ Cent(v) if and only if $w = h \cdot v$ with $h \in C^1(M)$, $v(h) = 0$.

DEFINITION 4. A continuous flow F on X is said to have an unstable centralizer if it satisfies the property that G is in Cent(F) if and only if there exists a continuous function A on X such that

$$G(t, x) = F(A(x)t, x)\ , \qquad A(x) = A(F(t, x))$$

for every $t \in \boldsymbol{R}$ and $x \in X$.

For a continuous flow F on X, we put

$$\varepsilon_0(F) = \inf\{t > 0\,; F(t, x) = x \text{ for some } x \in X\}$$

in the case when there exists a periodic (or fixed) point of F. When there is no periodic point we put $\varepsilon_0(F) = +\infty$.

In this paper, as an analogue of the case of expansive homeomorphisms, we shall prove that expansive flows have unstable centralizers. K. Kato and A. Morimoto [3] proved the above fact for the case of Anosov flows by using the topological stability.

Next, we shall prove that the set of all expansive flows in Cent(F), where F is an expansive flow on X, is an open subset of Cent(F) with respect to C^0-topology.

In the section 4, we shall prove that a flow commutative with an Anosov flow is an Anosov flow if it is a C^1-expansive flow on M.

The idea of the proof of Lemma 3 was inspired by that of Theorem B [3].

The author wishes to express his sincere gratitude to Prof. A. Morimoto for his many valuable advices.

2. Unstable centralizers

In this section, we shall prove the following Theorem 1.

THEOREM 1. *Expansive flows on X have unstable centralizers.*

To prove Theorem 1, first, we shall prove the following Lemma 1 and Lemma 2.

Remark. An expansive flow F on a connected metric space X has no fixed points (cf. [2]). Further, if a continuous flow F on X has no fixed points, then $\varepsilon_0(F) > 0$.

LEMMA 1. *Let F be an expansive flow on X and $G \in \mathrm{Cent}(F)$. Then for any $0 < \varepsilon < \varepsilon_0(F)/3$, we can find $\mu > 0$ such that there exists uniquely a function z on $[-\mu, \mu] \times X$ satisfying*

$$G(s, x) = F(z(s, x), x)\ , \qquad |z(s, x)| < \varepsilon\ ,$$

for any $(s, x) \in [-\mu, u] \times X$.

Proof. For any $\varepsilon > 0$, we have $\delta > 0$ such that if $d(f_t(x), f_t(y)) < \delta$ for every $t \in \boldsymbol{R}$, then $y = f_t(x)$ for some $|t| < \varepsilon$. Since M is a compact manifold, there exists sufficiently small $\mu > 0$ with

$$\max \{d(g_0(x), g_t(x))\, ;\, t \in [-\mu, \mu], x \in M\} < \delta\ .$$

Since $f_t \circ g_s = g_s \circ f_t$ for every $t, s \in \boldsymbol{R}$, we get

$$\begin{aligned} d(f_t(x), f_t(g_s(x))) &= d(f_t \circ g_0(x), f_t \circ g_s(x)) \\ &= d(g_0(f_t(x)), g_s(f_t(x))) < \delta \end{aligned}$$

for any $x \in M$ and every $t, s \in \boldsymbol{R}$ with $|s| \leqq \mu$.
Therefore we get

$$g_s(x) = F(z(s, x), x)\ , \qquad |z(s, x)| < \varepsilon\ .$$

Let $\varepsilon > 0$ be sufficiently small and $\varepsilon < \varepsilon_0(F)/3$, then

$$F(z_1(s, x), x) = F(z_2(s, x), x)$$

implies $z_1(s, x) = z_2(s, x)$. Hence, we know that $z(s, x)$ is unique.

Q.E.D.

LEMMA 2. *Let F be a continuous flow on X without fixed points. Let G be a continuous flow on X such that for fixed $\mu > 0$, there exists a function z on $[-\mu, \mu] \times X$, and*

$$G(s, x) = F(z(s, x), x)\ , \qquad |z(s, x)| < \varepsilon\ ,$$

where $0 < \varepsilon < \varepsilon_0(F)/3$. Then we get (i), (ii);

(i) *z is continuous on $[-\mu, \mu] \times X$,*

(ii) *if $t, s, t + s \in [-\mu, \mu]$, then*

$$z(t + s, x) = z(t, x) + z(s, G(t, x))$$

for any $x \in X$.

Proof. We shall prove (i). If z is not continuous, there exist $(s, x) \in [-\mu, \mu] \times X$ and $\{(s_n, x_n)\}_{n=1}^{\infty} \subset [-\mu, \mu] \times X$ such that $(s_n, x_n) \to (s, x)$ and $\{z(s_n, x_n)\}_{n=1}^{\infty}$ does not converge to $z(s, x)$ as $n \to \infty$. Therefore, we have $\delta_0 > 0$ and subsequence $\{(s_m, x_m)\}$ with $|z(s_m, x_m) - z(s, x)| \geqq \delta_0$ for any positive integer m. $\{z(s_m, x_m)\}$ is bounded, hence there exists $\delta_1 > 0$ such that

$$\delta_1 \leqq d(F(z(s_m, x_m), x), F(z(s, x), x))$$

for any m.
Whence we get

$$\begin{aligned} \delta_1 \leqq\ & d(F(z(s_m, x_m), x_m), F(z(s_m, x_m), x)) \\ & + d(F(z(s_m, x_m), x_m), F(z(s, x), x))\ . \end{aligned} \tag{1}$$

From $G(s_m, x_m) \to G(s, x)$ as $m \to \infty$, we get

$$d(F(z(s_m, x_m), x_m), F(z(s, x), x)) \to 0 \quad \text{as} \quad m \to \infty\ .$$

Since $\{z(s_m, x_m)\}$ is bounded, there exist $t_0 \in \boldsymbol{R}$ and subsequence $\{z(s_k, x_k)\}$ with $\lim z(s_k, x_k) = t_0$. Therefore

$$d(F(z(s_k, x_k), x_k), F(z(s_k, x_k), x)) \to 0 \quad \text{as} \quad k \to \infty\ .$$

This contradicts to (1).

Next, we prove (ii). We can calculate as follows.

$$\begin{aligned} G(t + s, x) &= G(t, G(s, x)) \\ &= F(z(t, g_s(x)), g_s(x)) \end{aligned}$$

$$= F(z(t, g_s(x)), F(z(s, x), x))$$
$$= F(z(t, g_s(x)) + z(s, x), x) ,$$

on the other hand

$$G(t + s, x) = F(z(t + s, x), x) .$$

Hence we have

$$z(t + s, x) = z(t, g_s(x)) + z(s, x)$$

for $t, s, t + s \in [-\mu, \mu]$ and $x \in X$.

Q.E.D.

Using Lemma 2, we shall prove the following:

LEMMA 3. *Let F be a continuous flow on X without fixed points. Let G be a continuous flow on X such that for fixed $\mu > 0$, there exists a continuous function z on $[-\mu, \mu] \times X$, and*

$$G(s, x) = F(z(s, x), x) , \qquad |z(s, x)| < \varepsilon ,$$

for every $s \in [-\mu, \mu]$ and $x \in X$, where $0 < \varepsilon < \varepsilon_0(F)/3$. Then there exists a unique continuous function $p: \boldsymbol{R} \times X \to \boldsymbol{R}$ such that

$$G(t, x) = F(p(t, x), x)$$

for $(t, x) \in \boldsymbol{R} \times X$, and $p = z$ on $[-\mu, \mu] \times X$.

Proof. Take a positive integer N so large as $1/2^N \leqq \mu$.

First, we define a continuous function z_1 on $[1/2^N, 2/2^N] \times X$ by

$$z_1(t, x) = z(t - 1/2^N, x) + z(1/2^N, G(t - 1/2^N, x))$$

for $(t, x) \in [1/2^N, 2/2^N] \times X$.

We shall prove the following equalities;

(a) $z_1(1/2^N, x) = z(1/2^N, x)$ for any $x \in X$,

(b) $G(t, x) = F(z_1(t, x), x)$

for any $(t, x) \in [1/2^N, 2/2^N] \times X$.

(a) is clear from the definition of z_1.

To prove (b), first, we can calculate from Lemma 2 as follows;

$$O = z(1/2^N - 1/2^N, g_t(x))$$
$$= z(1/2^N, G(t - 1/2^N, x)) + z(-1/2^N, g_t(x)) .$$

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

Therefore we get

$$z(-1/2^N, g_t(x)) = -z(1/2^N, G(t - 1/2^N, x))$$

for $t \in [1/2^N, 2/2^N]$ and $x \in X$. Hence,

$$\begin{aligned} F(-z(1/2^N, G(t - 1/2^N, x)), F(z_1(t, x), x)) \\ = F(z(t - 1/2^N, x), x) \\ = G(t - 1/2^N, x) \\ = G(-1/2^N, g_t(x)) \\ = F(z(-1/2^N, g_t(x)), g_t(x)) . \end{aligned}$$

Whence we get

$$G(t, x) = F(z_1(t, x), x)$$

for $(t, x) \in [1/2^N, 2/2^N] \times X$.

Next, we define a continuous function $\bar{z}_1$ on $[-2/2^N, -1/2^N] \times X$ by

$$\bar{z}_1(t, x) = z(t + 1/2^N, x) + z(-1/2^N, G(t + 1/2^N, x))$$

for $(t, x) \in [-2/2^N, -1/2^N] \times X$.

We shall prove the following equalities;

(c) $\bar{z}_1(-1/2^N, x) = z(-1/2^N, x)$ for $x \in X$,

(d) $G(t, x) = F(\bar{z}_1(t, x), x)$

for $(t, x) \in [-2/2^N, -1/2^N] \times X$.

(c) is clear from the definition of $\bar{z}_1$.

To prove (d), we can obtain the following equality from Lemma 2,

$$z(1/2^N, g_t(x)) = -z(-1/2^N, G(t + 1/2^N, x))$$

for $t \in [-2/2^N, -1/2^N]$ and $x \in X$. Hence,

$$\begin{aligned} F(-z(-1/2^N, G(t + 1/2^N, x)), F(\bar{z}_1(t, x), x)) \\ = F(z(t + 1/2^N, x), x) \\ = G(t + 1/2^N, x) \\ = G(1/2^N, g_t(x)) \\ = F(z(1/2^N, g_t(x)), g_t(x)) . \end{aligned}$$

Therefore, we get

$$G(t, x) = F(\bar{z}_1(t, x), x)$$

for $(t, x) \in [-2/2^N, -1/2^N] \times X$.

Now, for any positive integer k we define a continuous function z_k on $[k/2^N, (k+1)/2^N] \times X$ by

$$z_k(t, x) = z(t - k/2^N, x) + \sum_{i=1}^{k} z(1/2^N, G(t - i/2^N, x)) .$$

We shall prove the following equalities;

(e) $z_k((k+1)/2^N, x) = z_{k+1}((k+1)/2^N, x)$ for $x \in X$,

(f) $z_k(t, x) = z_{k-1}(t - 1/2^N, x) + z(1/2^N, G(t - 1/2^N, x))$

for $(t, x) \in [k/2^N, (k+1)/2^N] \times X$,

(g) $G(t, x) = F(z_k(t, x), x)$

for $t \in [k/2^N, (k+1)/2^N]$ and $x \in X$.

In fact, the right-hand side of (e) is $\sum_{i=0}^{k} z(1/2^N, G(i/2^N, x))$, while left-hand side of (e) is

$$\begin{aligned} z_k((k+1)/2^N, x) &= z(1/2^N, x) + \sum_{i=1}^{k} z(1/2^N, G((k+1-i)/2^N, x)) \\ &= \sum_{i=0}^{k} z(1/2^N, G(i/2^N, x)) . \end{aligned}$$

Next, we shall prove (f) as follows;

$$\begin{aligned} z_k(t, x) &= z(t - 1/2^N - (k-1)/2^N, x) \\ &\quad + \sum_{i=1}^{k} z(1/2^N, G(t - 1/2^N - (i-1)/2^N, x)) \\ &= z(1/2^N, G(t - 1/2^N, x)) + z(t - 1/2^N - (k-1)/2^N, x) \\ &\quad + \sum_{i=1}^{k-1} z(1/2^N, G(t - 1/2^N - i/2^N, x)) \\ &= z(1/2^N, G(t - 1/2^N, x)) + z_{k-1}(t - 1/2^N, x) . \end{aligned}$$

We shall show (g) by induction. We have already proved (g) for $k = 1$. From (f),

$$\begin{aligned} F(-z(1/2^N, G(t - 1/2^N, x)), F(z_k(t, x), x)) & \\ = F(z_{k-1}(t - 1/2^N, x), x) & \\ = G(t - 1/2^N, x) & \\ = G(-1/2^N, g_t(x)) & \\ = F(z(-1/2^N, g_t(x)), g_t(x)) & \\ = F(-z(1/2^N, G(t - 1/2^N, x)), g_t(x)) . & \end{aligned}$$

Whence we get

$$G(t, x) = F(z_k(t, x), x)$$

for $t \in [k/2^N, (k+1)/2^N]$ and $x \in X$.

Next, we define a continuous function $\bar{z}_k$, where k is a positive integer, on $[-(k+1)/2^N, -k/2^N] \times X$ by

$$\bar{z}_k(t, x) = z(t + k/2^N, x) + \sum_{i=1}^{k} z(-1/2^N, G(t + i/2^N, x)) .$$

We can verify the following (h), (i), (j) in the same way as the proof of (e), (f), (g).

(h) $\bar{z}_k(-(k+1)/2^N, x) = \bar{z}_{k+1}(-(k+1)/2^N, x)$ for $x \in X$,

(i) $\bar{z}_k(t, x) = \bar{z}_{k-1}(t + 1/2^N, x) + z(-1/2^N, G(t + 1/2^N, x))$

for $(t, x) \in [-(k+1)/2^N, -k/2^N] \times X$

(j) $G(t, x) = F(\bar{z}_k(t, x), x)$

for $t \in [-(k+1)/2^N, -k/2^N]$ and $x \in X$.

Consequently, we can define the function $p(t, x)$ on $\boldsymbol{R} \times X$ by

$$p(t, x) = \begin{cases} z_k(t, x) & \text{if } t \in [k/2^N, (k+1)/2^N] \\ \bar{z}_k(t, x) & \text{if } t \in [-(k+1)/2^N, -k/2^N] \end{cases}$$

for $k = 0, 1, 2, \cdots$, where $z_0 = z$. Using (e), (h), we see that p is a continuous function on $R \times X$ and from (g), (j), we know

$$G(t, x) = F(p(t, x), x)$$

for every $t \in \boldsymbol{R}$ and $x \in X$.

It is clear that $p(t, x) = z(t, x)$ for every $t \in [-\mu, \mu]$ and $x \in X$.

Finally, we shall prove the uniqueness. We assume that there exist two functions p_1, p_2 such that

$$G(t, x) = F(p_i(t, x), x)$$

for $(t, x) \in \boldsymbol{R} \times X$ and $p_i = z$ on $[-\mu, \mu] \times X$ $(i = 1, 2)$. Put $\alpha(t, x) = p_1(t, x) - p_2(t, x)$ and $T_x = \{t \in \boldsymbol{R} ; \alpha(t, x) = 0\}$ for fixed $x \in X$. Then, since $F(\alpha(t, x), x) = x$ holds for $(t, x) \in \boldsymbol{R} \times X$, we see that T_x is a non-empty, open and closed subset of $\boldsymbol{R}$. Therefore we get $T_x = \boldsymbol{R}$ for any $x \in X$ which implies $p_1 = p_2$. Q.E.D.

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

Proof of Theorem 1. Let F be an expansive flow on X and let $G \in \mathrm{Cent}(F)$. By Lemma 3, we can calculate as follows;

$$\begin{aligned} G(t, F(s, x)) &= g_t \circ f_s(x) = f_s \circ g_t(x) \\ &= F(s, G(t, x)) = F(s + p(t, x), x) , \end{aligned}$$

on the other hand

$$G(t, F(s, x)) = F(p(t, f_s(x)), f_s(x)) = F(p(t, f_s(x)) + s, x) .$$

Therefore, for sufficiently small $\mu > 0$, we have

$$p(t, x) = p(t, f_s(x))$$

for every $t \in [-\mu, \mu]$ and $s \in \boldsymbol{R}$. From the uniqueness of the function p and using (f), (i) in the proof of Lemma 3, we can prove by induction that

$$p(t, x) = p(t, f_s(x))$$

for every $t, s \in \boldsymbol{R}$.

Now, we get

$$G(t + s, x) = G(t, G(s, x)) = F(p(t, g_s(x)) + p(s, x), x) ,$$

on the other hand

$$G(t + s, x) = F(p(t + s, x), x) .$$

Hence, for sufficiently small $\mu > 0$, we get

$$p(t + s, x) = p(t, x) + p(s, x)$$

for every $t, s, t + s \in [-\mu, \mu]$ and $x \in X$. From the uniqueness of the function p and using (f), (i), we can prove that

$$p(t + s, x) = p(t, x) + p(s, x)$$

for every $t, s \in \boldsymbol{R}$ and $x \in X$. Therefore, we can write

$$p(t, x) = A(x) \cdot t$$

for $t \in \boldsymbol{R}$ and $x \in X$, where $A(x)$ is a continuous function on X. Since $p(t, x) = p(t, f_s(x))$ for any $s \in \boldsymbol{R}$, we get $A(x) = A(F(s, x))$ for every $s \in \boldsymbol{R}$.

Conversely, if G is a continuous flow on X and there exists a continuous function $A(x)$ on X such that

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

$$G(t, x) = F(A(x)t, x)\ , \qquad A(x) = A(F(s, x))$$

for every $t, s \in \boldsymbol{R}$, then it is clear that G is in Cent(F). Q.E.D.

COROLLARY 2. *C^1-expansive flows on M have unstable centralizers.*

Proof is omitted.

COROLLARY 3 (K. Kato and A. Morimoto). *Anosov flows on M have unstable centralizers.*

3. Expansive flows in Cent (F)

For continuous maps $f, g: X \to X$, $d_0(f, g)$ is defined by

$$d_0(f, g) = \max\{d(f(x), g(x))\,;\, x \in X\}\ .$$

First, we state the following:

LEMMA 4 (R. Bowen and P. Walters). *Let F be an expansive flow on X and let G be a continuous flow on X. If there exists a continuous function $p: \boldsymbol{R} \times X \to \boldsymbol{R}$ such that $G(t, x) = F(p(t, x), x)$ for every $t \in \boldsymbol{R}$ and $x \in X$, and $p_x: \boldsymbol{R} \to \boldsymbol{R}$ is a homeomorphism of $\boldsymbol{R}$ with $p_x(0) = 0$ for any $x \in X$, where $p_x(t) = p(t, x)$. Then G is an expansive flow on X.*

For the proof, see [2] Corollay 4.

THEOREM 4. *Let F be an expansive flow on X, and let $G \in$ Cent(F). Then there exist $\varepsilon > 0$ and $\delta > 0$ such that if $d_0(f_t, g_t) < \varepsilon$ for any $t \in [0, \delta]$, then G is an expansive flow on X.*

Proof. Put $\delta = \varepsilon_0(F)/2$, and

$$Q(x) = \max\{d(x, f_t(x))\,;\, t \in [0, \delta]\}$$

for any $x \in X$. Since F has no fixed points, $Q(x) > 0$ for every $x \in X$. Moreover, for any $x \in X$, there exist $\delta_1(x) > 0$ and a neighborhood U of x with $Q(y) \geqq \delta_1(x)$ for $y \in U$. Therefore, we get $\varepsilon = \inf\{Q(x)\,;\, x \in X\} > 0$.

Now, if $d_0(f_t, g_t) < \varepsilon$ for every $t \in [0, \delta]$, then G is an expansive flow. In fact, if G is not an expansive flow, then by Lemma 4 there exists $x_0 \in X$ with $A(x_0) = 0$, where A is a continuous funtin on X such that $G(t, x) = F(A(x)t, x)$ for every $t \in \boldsymbol{R}$ and $x \in X$. Hence, $G(t, x_0) = x_0$ for $t \in \boldsymbol{R}$. We can estimate as follows;

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

$$\varepsilon \leqq Q(x_0) = d(x_0, f_s(x_0)) = d(g_s(x_0), f_s(x_0)) \leqq d_0(g_s, f_s) < \varepsilon$$

for some $s \in [0, \delta]$. This is a contradiction. Therefore, G is an expansive flow on X. Q.E.D.

THEOREM 5. *Let F be an expansive flow on X. In* Cent(F), *the set of all expansive flows on X is open with respect to C^0-topology.*

Proof. Let G be an expansive flow on X such that $G \in$ Cent(F). Let H (or $\{h_t\}$) be a continuous flow in Cent(F) and $d_0(g_t, h_t) < \varepsilon(G)$ for every $t \in [0, \delta(G)]$, where $\varepsilon(G)$ and $\delta(G)$ are positive numbers which are obtained in Theorem 4. By Theorem 1, there exists a continuous function B on X such that

$$H(t, x) = F(B(x)t, x)\ , \qquad B(x) = B(F(t, x))$$

for $t \in \boldsymbol{R}$ and $x \in X$. On the other hand, we can write

$$G(t, x) = F(A(x)t, x)\ , \qquad A(x) = A(F(t, x))$$

for $t \in \boldsymbol{R}$ and $x \in X$, where A is a continuous function on X. Therefore, we get

$$\begin{aligned} h_t \circ g_s(x) &= h_t(F(A(x)s, x)) \\ &= F(B(x)t, F(A(x)s, x)) \\ &= F(A(x)s, F(B(x)t, x)) \\ &= g_s(F(B(x)t, x)) \\ &= g_s \circ h_t(x) \end{aligned}$$

for every $t, s \in \boldsymbol{R}$ and $x \in X$. Hence, by Theorem 4 H is an expansive flow on X. So we see that in Cent(F) the set of all expansive flows on X is an open set with respect to C^0-topology. Q.E.D.

EXAMPLE. Let S^1 be the unit circle. We consider S^1 as a compact connected C^∞ manifold by polar coordinates. Let a Riemannian metric $d(e^{it}, e^{is})$ on S^1, where $i = \sqrt{-1}$, be defined by

$$d(e^{it}, e^{is}) = |t - s|\ , \qquad -\pi < t - s < \pi \pmod{2n\pi}\ .$$

A continuous flow $F(t, e^{is}) = e^{i(t+s)}$ is an expansive flow on S^1.

By Theorem 1, we get that for any continuous flow on S^1 $G \in$ Cent(F), there exists a unique constant $a \in \boldsymbol{R}$ with $G(t, x) = F(a \cdot t, x)$ for

$t \in \boldsymbol{R}$ and $x \in S^1$. Consequently, we know that $G \in \mathrm{Cent}(F)$ is not an expansive flow on S^1 if and only if $G(t, x) = x$ for every $t \in \boldsymbol{R}$ and $x \in S^1$.

4. Anosov flows

TM will denote the tangent bundle of M and $V^1(M)$ (resp. $V^0(M)$) the vector space of all C^1 (resp. continuous) vector fields on M. A diffeomorphism f of M induces a linear automorphism $F = F(f)$ of $V^0(M)$ defined by $F(v) = df \circ v \circ f^{-1}$ for $v \in V^0(M)$, where df denotes the differential of f.

DEFINITION 5. A vector field $v \in V^1(M)$ or the flow $\{f_t\}$ generated by v is called an Anosov flow on M if $v(x) \neq 0$ for $x \in M$ and if there exist a Riemannian metric $|\cdot|$ on M, constants $C > 0$, $0 < \lambda < 1$ and a decomposition of the tangent space $T_x M = E_x^0 \oplus E_x^s \oplus E_x^u$ into three subspaces, which vary continuously with x on M satisfying the following conditions;

(0) $E_x^0 = \boldsymbol{R} \cdot v(x)$,

(i) df_t leaves invariant the subbundles E^s and E^u respectively, where $E^\alpha = \bigcup_{x \in M} E_x^\alpha$, $\alpha = s, u$,

(ii) $|df_t w| \leqq C\lambda^t |w|$ for $w \in E^s$, $t \geqq 0$
$|df_t w| \leqq C\lambda^{-t} |w|$ for $w \in E^u$, $t \leqq 0$.

The splitting $TM = E^0 \oplus E^s \oplus E^u$, $E^0 = \bigcup_{x \in M} E_x^0$, is the continuous Whitney sum.

Now, the vector space $V^0(M)$ becomes a Banach space with the norm

$$\|v\| = \sup \{|v(x)|\,;\, x \in M\}$$

for $v \in V^0(M)$. An equivalent way of defining an Anosov flow is as follows; v or $\{f_t\}$ is an Anosov flow if there exist a Riemannian metric $|\cdot|$ on M and constants $C > 0$, $0 < \lambda < 1$, such that $V^0(M) = V^0 \oplus V^s \oplus V^u$ (vector space direct sum), where we have put $V^0 = \{h \cdot v\,;\, h \in C^0(M)\}$, $F^t V^\alpha = V^\alpha$ $(t \in \boldsymbol{R})$, $\alpha = s, u$, and the restriction $F_\alpha^t = F^t|_{V^\alpha}$, $\alpha = s, u$, satisfies

$$\|F_s^t\| \leqq C\lambda^t \qquad t \geqq 0$$
$$\|F_u^t\| \leqq C\lambda^{-t} \qquad t \leqq 0\,,$$

where we define $F^t(w) = df_t \circ w \circ f_{-t}$ for $w \in V^0(M)$ and the norm

$$\|F_\alpha^t\| = \sup \{\|F^t(w)\|\,;\, w \in V^\alpha, \|w\| \leqq 1\}\,, \qquad \alpha = s, u\,.$$

LEMMA 5. *Let v (or $\{f_t\}$) be an Anosov flow on M and let $\{g_t\}$ be a C^1-flow on M such that $\{g_t\} \in \mathrm{Cent}(v)$. Let $V^0(M) = V^0 \oplus V^s \oplus V^u$ be the decomposition of $V^0(M)$ with respect to v. We decompose the operator $G^t(w) = dg_t \circ w \circ g_{-t}$ into*

$$G^t = \begin{pmatrix} G^t_{00} & G^t_{0s} & G^t_{0u} \\ G^t_{s0} & G^t_{ss} & G^t_{su} \\ G^t_{u0} & G^t_{us} & G^t_{uu} \end{pmatrix}$$

according to its components in V^0, V^s and V^u. Then

$$G^t_{\alpha\beta} = 0 \qquad \text{if} \quad \alpha \neq \beta .$$

Proof. Since $f_r \circ g_t = g_t \circ f_r$ for $t, r \in \boldsymbol{R}$, we get

$$\begin{pmatrix} G^t_{00} & G^t_{0s} & G^t_{0u} \\ G^t_{s0} & G^t_{ss} & G^t_{su} \\ G^t_{u0} & G^t_{us} & G^t_{uu} \end{pmatrix} \begin{pmatrix} F^r_0 & 0 & 0 \\ 0 & F^r_s & 0 \\ 0 & 0 & F^r_u \end{pmatrix} = \begin{pmatrix} F^r_0 & 0 & 0 \\ 0 & F^r_s & 0 \\ 0 & 0 & F^r_u \end{pmatrix} \begin{pmatrix} G^t_{00} & G^t_{0s} & G^t_{0u} \\ G^t_{s0} & G^t_{ss} & G^t_{su} \\ G^t_{u0} & G^t_{us} & G^t_{uu} \end{pmatrix}$$

for every $t, r \in \boldsymbol{R}$. Hence we have

$$G^t_{s0} \circ F^r_0 = F^r_s \circ G^t_{s0} \tag{1}$$
$$G^t_{u0} \circ F^r_0 = F^r_u \circ G^t_{u0} \tag{2}$$
$$G^t_{0s} \circ F^r_s = F^r_0 \circ G^t_{0s} \tag{3}$$
$$G^t_{0u} \circ F^r_u = F^r_0 \circ G^t_{0u} \tag{4}$$
$$G^t_{us} \circ F^r_s = F^r_u \circ G^t_{us} \tag{5}$$
$$G^t_{su} \circ F^r_u = F^r_s \circ G^t_{su} \tag{6}$$

By (5), we can estimate as follows;

$$\|G^t_{us}\| \leqq \|F^{-r}_u\| \cdot \|G^t_{us}\| \cdot \|F^r_s\| \leqq C^2\lambda^{2r} \|G^t_{us}\|$$

for every $t, r \in \boldsymbol{R}$. Since $C^2\lambda^{2r} < 1$ for sufficiently large $r > 0$, we get $G^t_{us} = 0$ for every $t \in \boldsymbol{R}$. Similarly from (6), we get $G^t_{su} = 0$ for $t \in \boldsymbol{R}$.

Now, put $M_1 = \max\{|v(x)|;\, x \in M\}$, $m_1 = \min\{|v(x)|;\, x \in M\}$. Then since $F^r(v) = v$ and $F^r(h \cdot v) = h \circ f_{-r} \cdot v$ for $h \in C^0(M)$ and $r \in \boldsymbol{R}$, we get $\|F^r_0\| \leqq M_1/m_1$. Therefore, from (1), (2), (3), (4), we have $G^t_{0\alpha} = 0$ and $G^t_{\alpha 0} = 0$, $\alpha = s, u$, for $t \in \boldsymbol{R}$. Q.E.D.

THEOREM 6. *Let v (or $\{f_t\}$) be an Anosov flow on M and let $\{g_t\}$ be a C^1-flow on M such that $\{g_t\} \in \mathrm{Cent}(v)$. Then $\{g_t\}$ is a C^1-expansive flow on M if and only if $\{g_t\}$ is an Anosov flow on M.*

https://doi.org/10.1017/S0027763000017517 Published online by Cambridge University Press

Proof. Since "if" has been proved by D. Anosov [1] (cf. [2]), we shall prove "only if". From Theorem 1 and $v(x) \neq 0$ for any $x \in M$, there exists a C^1-function A on M such that

$$g(t, x) = f(A(x)t, x) , \qquad A(x) = A(f(t, x))$$

for $t \in R$ and $x \in M$. Since M is connected and $\{g_t\}$ has no fixed points, $A(x) > 0$ for any $x \in M$ or $A(x) < 0$ for any $x \in M$.

We assume that $A(x) > 0$ for $x \in M$ and put $M_2 = \max\{A(x); x \in M\}$ and $m_2 = \min\{A(x); x \in M\}$.

Now, let $V^0(M) = V^0 \oplus V^s \oplus V^u$ be the decomposition of $V^0(M)$ with respect to v. To get the norm of G^t_{ss}, G^t_{uu}, we calculate as follows. Fix $x_0 \in M, w \in V^s, t \in \boldsymbol{R}$ and $h \in C^1(M)$, then we get

$$(G^t(w)h)(x_0) = dg_t(w(g_{-t}(x_0)))h = w(g_{-t}(x_0))(h \circ g_t) .$$

Take a neighborhood of $g_{-t}(x_0)$ with local coordinate system $\{y_1, \cdots, y_n\}$, and put

$$w(g_{-t}(x_0)) = \sum_{i=1}^{n} a_i(\partial/\partial y_i)_{g_{-t}(x_0)} ,$$

where $n = \dim M$ and $a_1, \cdots, a_n$ are C^1-functions defined on the neighborhood of $g_{-t}(x_0)$. We put $y_0 = g_{-t}(x_0)$.

$$\begin{aligned}
w(y_0)(h \circ g_t) &= \sum_{i=1}^{n} a_i(\partial/\partial y_i)_{y_0}(h \circ f(A(y)t, y)) \\
&= \sum_{i=1}^{n} a_i \left[\frac{dh \circ f(s, y_0)}{ds}\right]_{s=A(y_0)t} \cdot \left[\frac{\partial A(y)t}{\partial y_i}\right]_{y=y_0} \\
&\quad + \sum_{i=1}^{n} a_i \left[\frac{\partial h \circ f(A(y_0)t, y)}{\partial y_i}\right]_{y=y_0} \\
&= t \cdot w(A)(y_0) \cdot v(f(A(y_0)t, y_0))h + w(y_0)(h \circ f(A(y_0)t, y)) \\
&= t \cdot w(A)(y_0) \cdot v(f(A(y_0)t, y_0))h + df_{A(y_0)t}(w(y_0))h \\
&= t \cdot (w(A) \circ g_{-t})(x_0) \cdot v(x_0)h + df_{A(y_0)t} \circ w \circ f(A(x_0)(-t), x_0)h \\
&= [(t \cdot w(A) \circ g_{-t}) \cdot v + F^{A(x_0)t}(w)](x_0)h .
\end{aligned}$$

Therefore we get

$$G^t_{ss}w(x) = F^{A(x)t}w(x) , \qquad w(A) = 0$$

for $x \in M$ and $w \in V^s$. For any $w \in V^s$, $\|w\| \leqq 1$, $t \geqq 0$, we have

$$\begin{aligned} \|G^t_{ss}w\| &= \sup\{|G^t_{ss}w(x)|\,;\, x \in M\} \\ &= \sup\{|F^{A(x)t}w(x)|\,;\, x \in M\} \\ &\leqq \sup\{C\lambda^{A(x)t}\,|w(x)|\,;\, x \in M\} \\ &\leqq \sup\{C\lambda^{m_2 t}\,|w(x)|\,;\, x + M\} \\ &\leqq C(\lambda^{m_2})^t\,. \end{aligned}$$

Hence we get

$$\|G^t_{ss}\| \leqq C(\lambda^{m_2})^t \qquad t \geqq 0\,.$$

Similarly we get

$$G^t_{uu}w(x) = F^{A(x)t}w(x)\,, \qquad w(A) = 0$$

for $x \in M$ and $w \in V^u$. Whence we have

$$\|G^t_{uu}\| \leqq C(\lambda^{m_2})^{-t} \qquad t \leqq 0\,.$$

In the case of $A < 0$, we get

$$\begin{aligned} \|G^t_{ss}\| &\leqq C(\lambda^{-M_2})^{-t} \qquad t \leqq 0\,, \\ \|G^t_{uu}\| &\leqq C(\lambda^{-M_2})^{t} \qquad t \geqq 0\,. \end{aligned}$$

Consequently, in either case, using Lemma 5 we see that $\{g_t\}$ is an Anosov flow on M. Q.E.D.

REFERENCES

[1] D. V. Anosov: Geodesic flows on closed Riemannian manifolds with negative curvature, Proc. Steklov Inst. Math. 90 (1967).

[2] R. Bowen and P. Walters: Expansive One-parameter Flows, J. Differential Equations **12** (1972), 180–193.

[3] K. Kato and A. Morimoto: Topological stability of Anosov flows and their centralizers, Topology **12** (1973), 255–273.

[4] P. Walters: Homeomorphisms with Discrete Centralizers and Ergodic Properties, Math. System Theory, Vol. **4**, No. **4** (1970), 322–326.

Department of Mathematics,
Faculty of Science and Technology,
Science University of Tokyo