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WEAK CONTINUITY OF A COMPOSITION MAP BETWEEN SPACES OF COMPACT OPERATORS AND BANACH VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. This paper characterizes the Banach space E for the sequential continuity and the continuity on bounded sets of the composition map m: $C(S,E)_{wk} \times K(E,F)_{wk} \longrightarrow C(S,F)_{wk}$. Here, $K(E,F)$ denotes the Banach space of compact linear operators on the Banach space E to the Banach space F with the usual operator norm, and for any Banach space E, E_{wk} denote the Banach space E with the weak topology. Also we denote by $C(S,E)$ the Banach space of E valued continuous functions on a nonvoid compact Hausdorff space S with sup norm.

A Banach space X has the Dunford-Pettis property [3] whenever given a weakly null sequence (x_n) in X and a weakly null sequence (x_n^*) in X^* it follows that $\lim_n x_n^* x_n = 0$. An immediate consequence of this condition is the following:

THEOREM. *The Banach space E has the Dunford-Pettis property if and only if the composition map m: $C(S,E)_{wk} \times K(E,F)_{wk} \longrightarrow C(S,F)_{wk}$ given by $(\phi, T) \longrightarrow To\phi$ (where $(To\phi)(s) = T(\phi(s))$ for all s in S) is sequentially continuous.*

PROOF. A quick observation that for non-void compact S and Banach spaces E, F the spaces E, E^*, and $\mathbf{K}$ can be embedded into $C(S,E)$, $K(E,F)$, and $C(S,F)$, respectively, in a fairly canonical manner implies that the evaluation map $e : E \times E^* \longrightarrow \mathbf{K}$ is the restriction of the composition mapping $C(S,E) \times K(E,F) \longrightarrow C(S,F)$. Thus the sequential continuity of m can be tested at its restriction e having weak sequential continuity, i.e., is equivalent to the Dunford-Pettis property of E.

So, it remains to be shown that if E has the Dunford-Pettis property then m is sequentially continuous. We use Kalton's and Lewis' characterisations of weak sequential convergence in $K(E,F)$ and $C(S,E)$, respectively.

Kalton's test [2, 4] for weak sequential convergence in $K(E,F)$ states that a bounded sequence (T_n) in $K(E,F)$ is weakly null if and only if $\lim_n e^{**} T_n^* f^* = 0$ for $e^{**} \epsilon E^{**}$, $f^* \epsilon F^*$. On the one hand, this implies that the bounded sequence (T_n) in $K(E,F)$ is weakly null if and only if for each $f^* \epsilon F^*$ the sequence $(T_n^* f^*)$ is a weakly null sequence in E^*. On the other hand, Lewis's test [2, 5] for weak sequential convergence in $C(S,E) = (C(S) \check{\otimes} E)$ shows that a bounded sequence (ϕ_n) in $C(S,E)$ is weakly null if and only if $(\phi_n(s))$ in E is weakly null for each $s \epsilon S$. Thus, suppose E has the Dunford-Pettis property, (T_n) is weakly null in $K(E,F)$ and (ϕ_n) is weakly null in $C(S,E)$. Then the sequence $(T_n o \phi_n)$ is bounded in $C(S,F)$; $(T_n^* f^*)$ is a weakly null sequence in E^* for each f^* in

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F^* and $(\phi_n(s))$ is weakly null in E for each s in S. Therefore for each s in S and $f^* \epsilon F^*$, $\lim_n(T_n^* f^*)(\phi_n(s)) = 0$ and hence $\lim_n((T_n o \phi_n)(s), f^*) = 0$. Now, from Lewis's test we can conclude that $(T_n o \phi_n)$ is a weakly null sequence in $C(S, F)$. ∎

REMARK. If the composition map m is continuous on bounded sets, then as observed earlier, the evaluation map e is also weakly continuous on bounded sets. Hence from Corollary 1 of [1], it follows that there exists no infinite dimensional Banach space E for which the composition map m is continuous on bounded sets.

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